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# $\beta^c$-CLOSURE OPERATOR IN FUZZY SETTING

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**Abstract:** Fuzzy $\beta$-open set is introduced in [6]. Using this concept as a basic tool, in [2] we have introduced and studied fuzzy $\beta^*$-closure operator and fuzzy $\beta^*$-closed set. Here we introduce fuzzy $\beta^c$-closure operator and fuzzy $\beta^c$-closed set. This newly defined operator is coarser than fuzzy $\beta$-closure operator [6] and fuzzy $\beta^*$-closure operator. Also fuzzy $\beta^c$-closure operator is an idempotent operator. Then some mutual relationship of this operator with the operators defined in [2, 3, 4, 5, 6, 7, 8] are established. With the help of this operator a new type of fuzzy separation axiom is introduced. Lastly we characterize this operator via fuzzy net.

**Keywords and Phrases:** Fuzzy $\beta$-open set, fuzzy preopen set, fuzzy $\beta^c$-closed set, fuzzy $\beta^c$-regular space, $\beta^c$-convergence of a fuzzy net.

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## 1. Introduction

Many mathematicians have engaged themselves to introduce and study different types of fuzzy closure-like operators in fuzzy setting. In this context we have to mention [2, 3, 4, 5, 7, 8]. Using fuzzy $\beta$-open set, here we introduce and study fuzzy $\beta^c$-closed set and show that for any fuzzy set, fuzzy $\beta$-closure is weaker than fuzzy $\beta^c$-closure of this set and for a fuzzy open set these two operators coincide.

## 2. Preliminaries

Throughout the paper, by $(X, \tau)$ or simply by $X$ we mean a fuzzy topological space (fts, for short) in the sense of Chang [5]. A fuzzy set $A$ is a function from a non-empty set $X$ into a closed interval $I = [0, 1]$, i.e., $A \in I^X$ [11]. The support of a fuzzy set $A$ in $X$ will be denoted by $suppA$ [11] and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. A fuzzy point [10] with the singleton support $x \in X$ and the value $t$ $(0 < t \leq 1)$ at $x$ will be denoted by $x_t$. $0_X$ and $1_X$ are the constant fuzzy sets taking values 0 and 1 in $X$ respectively. The complement of a fuzzy set $A$ in $X$ will be denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for all $x \in X$ [11]. For two fuzzy sets $A$ and $B$ in $X$, we write $A \leq B$ if and only if $A(x) \leq B(x)$, for each $x \in X$, and $AqB$ means $A$ is quasi-coincident (q-coincident, for short) with $B$ if $A(x) + B(x) > 1$, for some $x \in X$ [10]. The negation of these two statements will be denoted by $A \not\leq B$ and $A \not{q} B$ respectively. $clA$ and $intA$ of a fuzzy set $A$ in $X$ respectively stand for the fuzzy closure [5] and fuzzy interior [5] of $A$ in $X$. A fuzzy set $A$ in $X$ is called fuzzy regular open [1] (resp., fuzzy preopen [9], fuzzy $\beta$-open [6]) if $A = intclA$ (resp., $A \leq intclA$, $A \leq clintclA$). The complement of a fuzzy preopen (resp., fuzzy $\beta$-open) set is called a fuzzy preclosed [9] (resp., fuzzy $\beta$-closed [6]) set. The smallest fuzzy preclosed (resp., fuzzy $\beta$-closed) set containing a fuzzy set $A$ is called fuzzy preclosure [9] (resp., fuzzy $\beta$-closure [6]) of $A$ and is denoted by $pclA$ (resp., $\beta clA$). The collection of all fuzzy regular open (resp., fuzzy preopen, fuzzy $\beta$-open) sets in an fts $X$ is denoted by $FRO(X)$ (resp., $FPO(X)$, $F\beta O(X)$) and that of fuzzy preclosed (resp., fuzzy $\beta$-closed) sets is denoted by $FPC(X)$ (resp., $F\beta C(X)$).

## 3. Fuzzy $\beta^c$-Closure Operator: Some Properties

In this section we first introduce fuzzy $\beta^c$-closure operator which is coarser than fuzzy $\beta$-closure operator. Then we characterize this operator via fuzzy open set and show that this operator is distributed over union but not on intersection.

**Definition 3.1.** *A fuzzy point $x_t$ in an fts $(X, \tau)$ is called a fuzzy $\beta^c$-cluster point of a fuzzy set $A$ in an fts $X$ if $clUqA$ for every $U \in F\beta O(X)$ with $x_tqU$.*
The union of all fuzzy $\beta^c$-cluster points of $A$ is called fuzzy $\beta^c$-closure of $A$, to be denoted by $[A]^c_\beta$. $A$ is called fuzzy $\beta^c$-closed set if $A = [A]^c_\beta$ and the complement of a fuzzy $\beta^c$-closed set in an fts $X$ is called fuzzy $\beta^c$-open set in $X$.

**Note 3.2.** *It is clear from definition that for any $A \in I^X$, $\beta clA \leq [A]^c_\beta$. But the converse is not necessarily true, follows from the following example.*

**Example 3.3.** Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.4$. Then $(X, \tau)$ is an fts. Here every fuzzy set is fuzzy $\beta$-open as well as fuzzy $\beta$-closed. Consider the fuzzy set $B$ defined by $B(a) = B(b) = 0.45$ and the fuzzy point $a_{0.5}$. Then $a_{0.5}qU \in F\beta O(X)$ with $U(a) > 0.5, U(b) = 0$. then $clU = 1_XqB \Rightarrow a_{0.5} \in [B]^c_\beta$. But $U \not{q} B$ where $U(a) = 0.51, U(b) = 0 \Rightarrow a_{0.5} \not\in \beta clB$.

The following theorem shows that under which condition fuzzy $\beta$-closure and fuzzy $\beta^c$-closure operators coincide.

**Theorem 3.4.** *For a fuzzy open set $A$ in an fts $X$, $\beta clA = [A]^c_\beta$.*
**Proof.** By Note 3.2, it suffices to show that $[A]^c_\beta \leq \beta clA$ for every fuzzy open set $A$ in $X$. Let $x_t \notin \beta clA$. Then there exists $V \in F\beta O(X)$, $x_t qV, V \not{q} A \Rightarrow V \leq 1_X \setminus A$ where $1_X \setminus A$ is fuzzy closed set in $X$. Therefore, $clV \leq cl(1_X \setminus A) = 1_X \setminus A \Rightarrow clV \not{q} A \Rightarrow x_t \notin [A]^c_\beta$. Hence the proof.

The next theorem characterizes fuzzy $\beta^c$-closure operator of a fuzzy set in an fts $X$.

**Theorem 3.5.** *For any fuzzy set $A$ in an fts $(X, \tau)$,*
*$[A]^c_\beta = \bigcap\{[U]^c_\beta : U$ is fuzzy open set in $X$ with $A \leq U\}$.*
**Proof.** Clearly L.H.S. $\leq$ R.H.S.
If possible, let $x_t \in$ R.H.S, but $x_t \notin$ L.H.S. Then there exists $V \in F\beta O(X)$ with $x_t qV$ and $clV \not{q} A \Rightarrow A \leq 1_X \setminus clV (\in \tau)$. By hypothesis, $x_t \in [1_X \setminus clV]^c_\beta$. But as $clV \not{q}(1_X \setminus clV)$, $x_t \notin [1_X \setminus clV]^c_\beta$, a contradiction.

**Note 3.6.** *By Theorem 3.4 and Theorem 3.5, we conclude that $[A]^c_\beta$ is fuzzy $\beta$-closed set in $X$ for any $A \in I^X$.*

**Theorem 3.7.** *In an fts $(X, \tau)$, the following statements are true :*
*(a) $0_X$ and $1_X$ are fuzzy $\beta^c$-closed sets in $X$,*
*(b) for any two fuzzy sets $A, B \in X$, $A \leq B \Rightarrow [A]^c_\beta \leq [B]^c_\beta$,*
*(c) for any two $A, B \in I^X$, $[A \bigcup B]^c_\beta = [A]^c_\beta \bigcup [B]^c_\beta$,*
*(d) for any two $A, B \in I^X$, $[A \bigcap B]^c_\beta \leq [A]^c_\beta \bigcap [B]^c_\beta$, the equality does not hold, in general, follows from the next example,*
*(e) union of any two fuzzy $\beta^c$-closed sets in $X$ is also so,*
*(f) intersection of any two fuzzy $\beta^c$-closed sets in $X$ is also so.*
**Proof.** (a) and (b) are obvious.
(c) By (b), we can write, $[A]^c_\beta \bigcup [B]^c_\beta \leq [A \bigcup B]^c_\beta$.
To prove the converse, let $x_t \in [A \bigcup B]^c_\beta$. Then for any $U \in F\beta O(X)$ with $x_t qU$, $clUq(A \bigcup B)$. Then there exists $y \in X$ such that $(clU)(y) + max\{A(y), B(y)\} > 1 \Rightarrow$ either $(clU)(y) + A(y) > 1$ or $(clU)(y) + B(y) > 1 \Rightarrow$ either $clUqA$ or $clUqB \Rightarrow$ either $x_t \in [A]^c_\beta$ or $x_t \in [B]^c_\beta \Rightarrow x_t \in [A]^c \bigcup [B]^c_\beta$.
(d) Follows from (b).
(e) Follows from (c).
(f) From (d), we have $[A \bigcap B]^c_\beta \leq [A]^c_\beta \bigcap [B]^c_\beta$ for any two fuzzy sets $A, B \in X$.
Conversely, let $A, B$ be two fuzzy $\beta^c$-closed sets in $X$. Then $[A]^c_\beta = A, [B]^c_\beta = B$.
Let $x_t \in [A]^c_\beta \bigcap [B]^c_\beta = A \bigcap B \leq [A \bigcap B]^c_\beta \Rightarrow [A]^c_\beta \bigcap [B]^c_\beta \leq [A \bigcap B]^c_\beta$.

**Example 3.8.** Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.6, B(a) = 0.3, B(b) = 0.5$. Then $(X, \tau)$ is an fts. Here $F\beta O(X) = \{0_X, 1_X, U, V, W\}$ where $U \geq A$, $0.3 \leq V(a) \leq 0.7, 0.4 < V(b) \leq 0.5, W \not\leq 1_X \setminus A$. Consider two fuzzy sets $C$ and $D$ defined by $C(a) = 0.4, C(b) = 0.1, D(a) = 0.1, D(b) = 0.55$ and the fuzzy point $a_{0.6}$. We claim that $a_{0.6} \in [C]^c_s \bigcap [D]^c_s$, but $a_{0.6} \notin [C \bigcap D]^c_s$. The fuzzy $\beta$-open sets $q$-coincident with $a_{0.6}$ are of the form $U, V_1, V_2, V_3$ where $0.4 < V_1(a) \leq 0.5, 0.4 < V_1(b) \leq 0.5, V_2(a) > 0.5, V_2(b) \leq 0.5, V_3(a) > 0.4, V_3(b) > 0.5$. Then $clU = clV_3 = 1_X$ and so $clU = clV_3 qC$ and $clU = clV_3 qD$. Also $clV_1 = clV_2 = 1_X \setminus B$ and so $clV_1 = clV_2 qC$ and $clV_1 = clV_2 qD$. As a result $a_{0.6} \in [C]^c_\beta$ and $a_{0.6} \in [D]^c_\beta \Rightarrow a_{0.6} \in [C]^c_\beta \bigcap [D]^c_\beta$. Let $E = C \bigcap D$. Then $E(a) = E(b) = 0.1$. Then $clV_1 = (1_X \setminus B) \not{q} E \Rightarrow a_{0.6} \notin [E]^c_\beta$.

**Note 3.9.** *So we can conclude that fuzzy $\beta^c$-open sets in an fts $(X, \tau)$ form a fuzzy topology $\tau_{\beta^c}$ (say) which is coarser than fuzzy topology $\tau$ of $(X, \tau)$.*

**Result 3.10.** *We conclude that $x_t \in [y_{t'}]^c_\beta$ does not imply $y_{t'} \in [x_t]^c_\beta$ where $x_t, y_{t'}$ $(0 < t, t' \leq 1)$ are fuzzy points in $X$ as shown from the following example.*

**Example 3.11.** Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.2, B(a) = 0.7, B(b) = 0.2$. Then $(X, \tau)$ is an fts. Here $F\beta O(X) = \{0_X, 1_X, U\}$ where $U \not\leq 1_X \setminus B$. Now consider two fuzzy points $a_{0.2}$ and $b_{0.61}$. We claim that $a_{0.2} \in [b_{0.61}]^c_\beta$, but $b_{0.61} \notin [a_{0.2}]^c_\beta$. Now any fuzzy $\beta$-open set $q$-coincident with $a_{0.2}$ is of the form $U_1$ where $U_1(a) > 0.8, U_1(b) \geq 0$. So $clU_1 = 1_X q b_{0.61} \Rightarrow a_{0.2} \in [b_{0.61}]^c_\beta$. Now $b_{0.61} q U_2 \in F\beta O(X)$ where $U_2(a) = 0.31, U_2(b) = 0.4$, but $clU_2 = (1_X \setminus A) \not{q} a_{0.2} \Rightarrow b_{0.61} \notin [a_{0.2}]^c_\beta$.

The next theorem shows that fuzzy $\beta^c$-closure operator is an idempotent operator.

**Theorem 3.12.** *For any fuzzy set $A$ in an fts $(X, \tau)$, $[A]^c_\beta = [[A]^c_\beta]^c_\beta$.*
**Proof.** we first show that $A \subseteq [A]^c_\beta$. Let $x_t \in A$ be arbitrary. If possible, let $x_t \notin [A]^c_\beta$. Then there exists $U \in F\beta O(X)$ with $x_t qU$, $clU \not{q} A \Rightarrow A \leq 1_X \setminus clU$. Since $x_t \in A$, $x_t \in 1_X \setminus clU \Rightarrow 1 - (clU)(x) \geq t \Rightarrow x_t \not{q} clU$ which contradicts the fact that $x_t qU$. So $A \subseteq [A]^c_\beta$. Then by Theorem 3.7(b), $[A]^c_\beta \subseteq [[A]^c_\beta]^c_\beta$....(1).
Conversely, let $x_t \in [[A]^c_\beta]^c_\beta$. We have to show that $x_t \in [A]^c_\beta$. Let $U \in F\beta O(X)$ with $x_t qU$. By hypothesis, $clU qB$ where $B = [A]^c_\beta$. Then there exists $y \in X$ such that $(clU)(y) + B(y) > 1$. Let $B(y) = k$. Then $y_k \in B = [A]^c_\beta$ and $y_k q clU$. Since $U \in F\beta O(X) \Rightarrow clU \in F\beta O(X)$, then for $y_k \in [A]^c_\beta$, we have $cl(clU) = clU qA \Rightarrow x_t \in [A]^c_\beta \Rightarrow [[A]^c_\beta]^c_\beta \subseteq [A]^c_\beta$ .... (2). Combining (1) and (2), we have $[A]^c_\beta = [[A]^c_\beta]^c_\beta$.

## 4. Mutual Relationship and Fuzzy $\beta^c$-Regular Space

In this section we first recall several types of fuzzy closure-like operators from [2, 3, 4, 7, 8] and then establish the mutual relationship between these closure operators with fuzzy $\beta^c$-closure operator. Next we introduce a new separation axiom in which fuzzy $\beta$-closure operator and fuzzy $\beta^c$-closure operator coincide.

**Definition 4.1.** *A fuzzy point $x_t$ in an fts $(X, \tau)$ is called fuzzy $p^*$-cluster point [3] (resp., fuzzy $\beta^*$-cluster point [2]) of a fuzzy set $A$ in $X$ if for every $U \in FPO(X)$ (resp., $U \in F\beta O(X)$) with $x_t qU$, $pclUqA$ (resp., $\beta clUqA$).*
The union of all fuzzy $p^*$-cluster (resp., fuzzy $\beta^*$-cluster) points of a fuzzy set $A$ is called fuzzy $p^*$-closure [4] (resp., fuzzy $\beta^*$-closure [2]) of $A$, denoted by $[A]_p$ (resp., $[A]_\beta$).

**Definition 4.2.** *A fuzzy point $x_t$ in an fts $(X, \tau)$ is called a fuzzy $\theta$-cluster point [8] (resp., fuzzy $\delta$-cluster point [7], fuzzy $\delta^*$-cluster point [4]) of a fuzzy set $A$ in $X$ if $clUqA$ (resp., $UqA$, $clUqA$) for every fuzzy open (resp., fuzzy regular open, fuzzy regular open) set $U$ in $X$ with $x_t qU$.*
The union of all fuzzy $\theta$-cluster (resp., fuzzy $\delta$-cluster, fuzzy $\delta^*$-cluster) points of a fuzzy set $A$ in an fts $X$ is called fuzzy $\theta$-closure [8] (resp., fuzzy $\delta$-closure [7], fuzzy $\delta^*$-closure [4]) of $A$, denoted by $[A]_\theta$ (resp., $[A]_\delta$, $[A]_{\delta^*}$).

**Note 4.3.** *It is clear from above discussion that for any fuzzy set $A$ in an fts $(X, \tau)$,*
*(i) $[A]_\beta \subseteq [A]^c_\beta \subseteq [A]_\theta, [A]_{\delta^*}$. But the reverse implications are not true, in general, follow from the following examples.*
*(ii) $[A]^c_\beta$ is an independent concept of $[A]_p, [A]_\delta, clA$ follow from the following examples.*

**Example 4.4.** $x_t \in [A]^c_\beta$, but $x_t \notin [A]_\beta, [A]_p, [A]_{\delta*}$
Consider Example 3.3. Here $\beta clU = U \not qB \Rightarrow a_{0.5} \notin [B]_\beta$. Now $FPO(X) = \{0_X, 1_X, W, T\}$ where $W \leq A$, $T \not\leq 1_X \setminus A$. Now consider the fuzzy set $S$ defined by $S(a) = 0.51, S(b) = 0$. Then $a_{0.5}qS \in FPO(X)$. But $pclS = S \not qB \Rightarrow a_{0.5} \notin [B]_p$. Again consider the fuzzy set $C$ defined by $C(a) = C(b) = 0.3$ and the fuzzy point $b_{0.5}$. We claim that $b_{0.5} \in [C]_{\delta*}$, but $b_{0.5} \notin [C]^c_\beta$. As $1_X \in FRO(X)$ only with $b_{0.5}q1_X$, clearly $b_{0.5} \in [C]_{\delta*}$. Now $b_{0.5}qS \in F\beta O(X)$ where $S(a) = 0.5$, $S(b) = 0.51$. But $clS = (1_X \setminus A)$ which is not $q$-coincident with $C \Rightarrow b_{0.5} \notin [C]^c_\beta$.

**Example 4.5.** $x_t \in [A]_\theta, [A]_\delta, [A]_p, clA$, but $x_t \notin [A]^c_\beta$
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.2, B(a) = 0.7, B(b) = 0.2$. Then $(X, \tau)$ is an fts. Here $F\beta O(X) = \{0_X, 1_X, U\}$ where $U \not\leq 1_X \setminus B$. Consider the fuzzy set $C$ defined by $C(a) = C(b) = 0.2$ and the

fuzzy point $b_{0.61}$. We claim that $b_{0.61} \in clC, [C]_\theta, [C]_\delta$, but $b_{0.61} \notin [C]^c_\beta$. Now $clC = 1_X \setminus B \ni b_{.061}$. Again $1_X$ is the only fuzzy open (resp., fuzzy regular open) set in $X$ with $b_{0.61} q 1_X$ and so $b_{0.61} \in [C]_\theta$ (resp., $b_{0.61} \in [C]_\delta$). Now $b_{0.61} qV \in F\beta O(X)$ where $V(a) = 0.31, V(b) = 0.4$. But $clV = (1_X \setminus A) \not{q} C \Rightarrow b_{0.61} \notin [C]^c_\beta$. Again $FPO(X) = \{0_X, 1_X, S, T\}$ where $0.3 < S(a) \leq 0.5, S(b) \leq 0.2, T \not\leq 1_X \setminus A$. So $FPC(X) = \{0_X, 1_X, 1_X \setminus S, 1_X \setminus T\}$ where $0.5 \leq (1_X \setminus S)(a) < 0.7, (1_X \setminus S)(b) \geq 0.8, 1_X \setminus T \not\geq A$. Now $b_{0.61} qT_1 \in FPO(X)$ with $T_1(a) > 0.5, T_1(b) > 0.39$. Then $pclT = S_1$ or $1_X$ according as $S_1(a) > 0.5, S_1(b) = 0.8$ and so $pclTqD$ where $D(a) = 0.5, D(b) = 0 \Rightarrow b_{0.61} \in [D]_p$. But as $clV = (1_X \setminus A) \not{q} D$, we write $b_{0.61} \notin [D]^c_\beta$.

**Example 4.6.** $x_t \in [A]^c_\beta$, but $x_t \notin clA, [A]_p, [A]_\delta$
Consider example 3.3. Consider the fuzzy set $B$ defined by $B(a) = 0.4, B(b) = 0.5$ and the fuzzy point $a_{0.6}$. Here $FRO(X) = \tau$. Here $a_{0.6} qA \in FRO(X)$, but $A \not{q} B \Rightarrow a_{0.6} \notin [B]_\delta$. Here every fuzzy set is fuzzy $\beta$-open set in $X$. Now $a_{0.6} qU$ where $U(a) > 0.4, U(b) \geq 0$. Then $clU = (1_X \setminus A) or 1_X$ and so $clUqB \Rightarrow a_{0.6} \in [B]^c_\beta$. Now $a_{0.6} qV \in FPO(X)$ where $V(a) = 0.6, V(b) = 0$. Then $pclV = V \not{q} B \Rightarrow a_{0.6} \notin [B]_p$. Also $clB = 1_X \setminus A \not\ni a_{0.6}$.
Let us now introduce the following separation axiom.

**Definition 4.7.** *An fts $(X, \tau)$ is called fuzzy $\beta^c$-regular space if for each fuzzy point $x_t$ and each fuzzy $\beta$-open set $U$ in $X$ with $x_t qU$, then there exists $V \in \tau$ such that $x_t qV \leq clV \leq U$.*

**Theorem 4.8.** *For an fts $(X, \tau)$, the following statements are equivalent :*
*(a) $X$ is fuzzy $\beta^c$-regular space,*
*(b) for any $A \in I^X$, $\beta clA = [A]^c_\beta$,*
*(c) for each fuzzy point $x_t$ and each $U \in F\beta C(X)$ with $x_t \notin U$, there exists $V \in \tau$ such that $x_t \notin clV$ and $U \leq V$,*
*(d) for each fuzzy point $x_t$ and each $U \in F\beta C(X)$ with $x_t \notin U$, there exist $V, W \in \tau$ such that $x_t qV$, $U \leq W$ and $V \not{q} W$,*
*(e) for any $A \in I^X$ and any $U \in F\beta C(X)$ with $A \not\leq U$, there exist $V, W \in \tau$ such that $AqV, U \leq W$ and $V \not{q} W$,*
*(f) for any $A \in I^X$ and any $U \in F\beta O(X)$ with $AqU$, there exists $V \in \tau$ such that $AqV \leq clV \leq U$.*

**Proof**. (a) $\Rightarrow$ (b) By Note 3.2, it suffices to show that $[A]^c_\beta \subseteq sclA$, for any $A \in I^X$. Let $x_t \in [A]^c_\beta$ be arbitrary and $V \in F\beta O(X)$ with $x_t qV$. By (a), there exists $U \in \tau$ such that $x_t qU \leq clU \leq V$. Since $U \in \tau \Rightarrow U \in F\beta O(X)$, by hypothesis, $clUqA \Rightarrow VqA \Rightarrow x_t \in \beta clA \Rightarrow [A]^c_\beta \subseteq \beta clA$.
(b) $\Rightarrow$ (a) Let $x_t$ be a fuzzy point in $X$ and $U \in F\beta O(X)$ with $x_t qU$. Then

$U(x)+t>1 \Rightarrow x_t \notin 1_X \setminus U(\in F\beta C(X)) = \beta cl(1_X \setminus U) = [1_X \setminus U]^c_\beta$ (by (b)). Then there exists $V \in F\beta O(X)$ with $x_t qV$, $clV \not{q}(1_X \setminus U) \Rightarrow clV \leq U$. Therefore, $x_t qV \leq clV \leq U \Rightarrow X$ is fuzzy $\beta^c$-regular space.

(a) $\Rightarrow$ (c) Let $x_t$ be a fuzzy point in $X$ and $U \in F\beta C(X)$ with $x_t \notin U$. Then $x_t q(1_X \setminus U) \in F\beta O(X)$. By (a), there exists $V \in \tau$ such that $x_t qV \leq clV \leq 1_X \setminus U$. Therefore, $U \leq 1_X \setminus clV$ $(= W$, say$)$. Then $W \in \tau$. Now $x_t qV = intV \Rightarrow x_t q intV \leq V \leq intclV \Rightarrow x_t q(intclV) \Rightarrow (intclV)(x)+t>1 \Rightarrow 1-(intclV)(x)<t \Rightarrow x_t \notin 1_X \setminus intclV = cl(1_X \setminus clV) = clW$.

(c) $\Rightarrow$ (d) Let $x_t$ be a fuzzy point in $X$ and $U \in F\beta C(X)$ with $x_t \notin U$. By (c), there exists $V \in \tau$ such that $U \leq V$ and $x_t \notin clV \Rightarrow$ there exists $W \in \tau$ such that $x_t qW$, $W \not{q} V$.

(d) $\Rightarrow$ (e) Let $A \in I^X$ and $U \in F\beta C(X)$ with $A \not\leq U$. Then there exists $x \in X$ such that $A(x) > U(x)$. Let $A(x)=t$. Then $x_t \notin U$. By (d), there exist $V, W \in \tau$ such that $x_t qV, U \leq W$ and $V \not{q} W$. Again $V(x)+t>1 \Rightarrow V(x)+A(x) > 1-t+t=1 \Rightarrow AqV$.

(e) $\Rightarrow$ (f) Let $A \in I^X$ and $U \in F\beta O(X)$ with $AqU$. Then $A \not\leq 1_X \setminus U \in F\beta C(X)$. By (e), there exist $V, W \in \tau$ such that $A \leq V, 1_X \setminus U \leq W$ and $V \not{q} W \Rightarrow V \leq 1_X \setminus W \in \tau^c \Rightarrow clV \leq cl(1_X \setminus W) = 1_X \setminus W \leq U$. Therefore, $A \leq V \leq clV \leq U$.

(f) $\Rightarrow$ (a) Obvious.

**Corollary 4.9.** *An fts $(X,\tau)$ is fuzzy $\beta^c$-regular if and only if fuzzy $\beta$-closed set in $X$ is fuzzy $\beta^c$-closed set in $X$.*

**Proof.** Let $(X,\tau)$ be fuzzy $\beta^c$-regular space and $A \in F\beta C(X)$. Then by Theorem 4.8 (a)$\Rightarrow$ (b), $A = \beta clA = [A]^c_\beta \Rightarrow A$ is fuzzy $\beta^c$-closed set in $X$.

Conversely, let $A = [A]^c_\beta$ for any $A \in F\beta C(X)$. Let $B \in I^X$. Then $\beta clB \in F\beta C(X)$ and so by hypothesis, $\beta clB = [\beta clB]^c_\beta$. Then $[B]^c_\beta \leq [\beta clB]^c_\beta = \beta clB$. By Note 3.2, $\beta clB \leq [B]^c_\beta$. Combining these two, we get $[B]^c_\beta = \beta clB$ for any $B \in I^X$. Then by Theorem 4.8 (b)$\Rightarrow$ (a), $X$ is fuzzy $\beta^c$-regular space.

## 5. Fuzzy $\beta^c$-Closure Operator : More characterizations Via Fuzzy Net

In this section we first introduce fuzzy $\beta^c$-cluster point and fuzzy $\beta^c$-convergence of a fuzzy net and then fuzzy $\beta^c$-closure operator of a fuzzy set is characterized in terms of these concepts.

**Definition 5.1.** *A fuzzy point $x_t$ in an fts $(X,\tau)$ is called a fuzzy $\beta^c$-cluster point of a fuzzy net $\{S_t : n \in (D, \geq)\}$ if for every fuzzy $\beta$-open set $U$ in $X$ with $x_t qU$ and for any $n \in D$, there exists $m \in D$ with $m \geq n$ such that $S_m q clU$.*

**Definition 5.2.** *A fuzzy net $\{S_n : n \in (D, \geq)\}$ in an fts $(X,\tau)$ is said to $\beta^c$-converge to a fuzzy point $x_t$ if for every fuzzy $\beta$-open set $U$ in $X$, $x_t qU$, there exists $m \in D$ such that $S_n q clU$, for all $n \geq m$ $(n \in D)$. This is denoted by $S_n \overrightarrow{\beta^c} x_t$.*

**Theorem 5.3.** *A fuzzy point $x_t$ is a fuzzy $\beta^c$-cluster point of a fuzzy net $\{S_n : n \in (D, \geq)\}$ in an fts $(X, \tau)$ iff there exists a fuzzy subset of $\{S_n : n \in (D, \geq)\}$ which $\beta^c$-converges to $x_t$.*

**Proof**. Let $x_t$ be a fuzzy $\beta^c$-cluster point of the fuzzy net $\{S_n : n \in (D, \geq)\}$. Let $C(Q_{x_t})$ denote the set of fuzzy closures of all fuzzy $\beta$-open sets of $X$ $q$-coincident with $x_t$. Then for any $A \in C(Q_{x_t})$, there exits $n \in D$ such that $S_n q A$. Let $E$ denote the set of all ordered pairs $(n, A)$ such that $n \in D$, $A \in C(Q_{x_t})$ and $S_n q A$. Then $(E, \gg)$ is a directed set, where $(m, A) \gg (n, B)$ $((m, A), (n, B) \in E)$ iff $m \geq n$ in $D$ and $A \leq B$. Then $T : (E, \gg) \to (X, \tau)$ given by $T(m, A) = S_m$ is clearly a fuzzy subnet of $\{S_n : n \in (D, \geq)\}$. We claim that $T \overrightarrow{\beta^c} x_t$. Let $V$ be any fuzzy $\beta$-open set in $X$ with $x_t q V$. Then there exists $n \in D$ such that $(n, clV) \in E$ and so $S_n q clV$. Now for any $(m, A) \gg (n, clV)$, $T(m, A) = S_m q A \leq clV \Rightarrow T(m, A) q clV$. Consequently, $T \overrightarrow{\beta^c} x_t$.

Conversely, let $x_t$ be not a fuzzy $\beta^c$-cluster point of the fuzzy net $\{S_n : n \in (D, \geq)\}$. Then there exists $U \in F\beta O(X)$ with $x_t q U$ and an $n \in D$ such that $S_m \not{q} clU$, for all $m \geq n$. Then clearly no fuzzy subnet of the net $\{S_n : n \in (D, \geq)\}$ can $\beta^c$-converge to $x_t$.

**Theorem 5.4.** *Let $A$ be a fuzzy set in an fts $(X, \tau)$. A fuzzy point $x_t \in [A]^c_\beta$ iff a fuzzy net $\{S_n : n \in (D, \geq)\}$ in $A$, which $\beta^c$-converges to $x_t$.*

**Proof.** Let $x_t \in [A]^c_\beta$. Then for any $U \in F\beta O(X)$ with $x_t q U$, $clU q A$, i.e., there exists $y^U \in suppA$ and a real number $s_U$ with $0 < s_U \leq A(y^U)$ such that the fuzzy point $y^U_{s_U}$ with support $y^U$ and the value $s_U$ belong to $A$ and $y^U_{s_U} q clU$. We choose and fix one such $y^U_{s_U}$ for each $U$. Let $\mathcal{D}$ denote the set of all fuzzy $\beta$-open set in $X$ $q$-coincident with $x_t$. Then $(\mathcal{D}, \succeq)$ is a directed set under inclusion relation, i.e., $B, C \in \mathcal{D}$, $B \succeq C$ iff $B \leq C$. Then $\{y^U_{s_U} \in A : y^U_{s_U} q clU, U \in \mathcal{D}\}$ is a fuzzy net in $A$ such that it $\beta^c$-converges to $x_t$. Indeed, for any fuzzy $\beta$-open set $U$ in $X$ with $x_t q U$, if $V \in \mathcal{D}$ and $V \succeq U$ (i.e., $V \leq U$) then $y^V_{s_V} q clV \leq clU \Rightarrow y^V_{s_V} q clU$.

Conversely, let $\{S_n : n \in (D, \geq)\}$ be a fuzzy net in $A$ such that $S_n \overrightarrow{\beta^c} x_t$. Then for any $U \in F\beta O(X)$ with $x_t q U$, there exists $m \in D$ such that $n \geq m \Rightarrow S_n q clU \Rightarrow A q clU$ (since $S_n \in A$). Hence $x_t \in [A]^c_\beta$.

**Remark 5.5.** *It is clear that an improved version of the converse of the last theorem can be written as "$x_t \in [A]^c_\beta$ if there exists a fuzzy net in $A$ with $x_t$ as a fuzzy $\beta^c$-cluster point".*

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