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ON A NEW GENERALIZED LOGISTIC DISTRIBUTION

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Abstract: A new statistical distribution, which was talked about between Swamee and Rathie in 2020, is given in this paper along with several statistical properties, such as approximations, inversion result, moments, order statistics, and hazard rate function. In addition, new distributions such as Marshall-Olkin-Swamee-Rathie, generalized gamma and beta generated and Swamee-Rathie skew distributions are also obtained. Approximations to mathematical and statistical distributions are indicated. Reliability probability $P(X < Y)$, for X and Y independent Swamee-Rathie distributions, is derived.

Keywords and Phrases: Swamee-Rathie distribution, H- function, Reliability $P(X < Y)$.

2020 Mathematics Subject Classification: 60E05, 62E17, 62N05, 33C99.

1. Introduction

The various email correspondence between Swamee and Rathie gave rise to a new statistical distribution called in Section 2 as the Swamee-Rathie distribution and its skew version is given in 50 Section 13.

In this Section, definitions of H and G functions, Student's t-distribution and some results are given.

1.1. The H-function

The H-function (see Mathai et al. (2009) [3]) is defined by:

$$H_{p,q}^{m,n}\left[z\Big|^{(a_1,A_1),...,(a_n,A_n),(a_{n+1},A_{n+1}),(a_p,A_p)}_{(b_1,B_1),...,(b_m,B_m),(b_{m+1},B_{m+1}),(b_q,B_q)}\right] = \frac{1}{2\pi i}\int_L \lambda(s) z^s\,ds, \tag{1.1}$$

where

$$\lambda(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j - B_j s)\prod_{j=1}^{n}\Gamma(1-a_j+A_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+B_j s)\prod_{j=n+1}^{p}\Gamma(a_j - A_j s)}, \tag{1.2}$$

and

a) $i = \sqrt{-1}$;

b) z $(\neq 0)$ is a complex variable;

c) $z^s = \exp(s(\ln|z| + i\,\mathrm{Arg}\,z))$;

d) An empty product is interpreted as unity;

e) m, n, q and p are non-negative integers, satisfying $0 \leq n \leq p$, $0 \leq m \leq q$ (both m and n are not zeros simultaneously);

f) A_j $(j = 1, ..., p)$ and B_j $(j = 1, ..., q)$ are assumed to be positive quantities;

g) a_j $(j = 1, ..., p)$ and b_j $(j = 1, ..., q)$ are complex numbers such that none of the poles of $\Gamma(b_j - B_j s), (j = 1, ..., m)$ coincide with the poles of $\Gamma(1-a_j+A_j s), (j = 1, ..., n)$;

h) The contour L runs from $c - i\infty$ to $c + i\infty$ such that the poles of $\Gamma(b_j - B_j s), (j = 1, ..., m)$ lie to the right of L and the poles of $\Gamma(1-a_j+A_j s), (j = 1, ..., n)$ lie to the left of L. Such a contour is possible on account of (g) with suitable indentations, if required.

The G-function is a special case of the H-function when $A_j = 1 = B_k$, for $j = 1, ..., p$ and $k = 1, ..., q$.

Symbolically,

$$H_{p,q}^{m,n}\left[z\Big|^{(a_1,1),...,(a_n,1),(a_{n+1},1),(a_p,1)}_{(b_1,1),...,(b_m,1),(b_{m+1},1),(b_q,1)}\right] = G_{p,q}^{m,n}\left[z\Big|^{a_1,...,a_n,a_{n+1},...,a_p}_{b_1,...,b_m,b_{m+1},...,b_q}\right] \tag{1.3}$$

1.2. Results

A few results, which will be used later on, are given below:

$$(1+x)^{-2} = \sum_{r=0}^{\infty}(-1)^r(1+r)x^r, \quad |x| < 1; \tag{1.4}$$

$$(1+x)^{-1} = \sum_{r=0}^{\infty}(-x)^r, \quad |x| < 1; \tag{1.5}$$

$$\int_0^{\infty} x^{\alpha-1} e^{-\beta x^\gamma} dx = \frac{\Gamma(\alpha/\gamma)}{\gamma\beta^{\alpha/\gamma}}, \tag{1.6}$$

for α, β, $\gamma > 0$;

$$\int_0^{\infty} x^{s-1} e^{-\eta x - z x^\sigma} dx = \eta^{-s} H_{1,1}^{1,1}\left[z\eta^{-\sigma} \Big|_{(0,1)}^{(1-s,\sigma)} \right], \tag{1.7}$$

for Re(s)>0, Re(η)>0, Re(z)>0 and $\sigma > 0$.

1.3. Known statistical distributions

The density and distribution functions for the Student's t-distribution with ν degrees of freedom are respectively:

$$f(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{x\nu}\,\Gamma(\frac{\nu}{2})}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \tag{1.8}$$

for $x \in (-\infty, \infty)$, and

$$F(x) = \frac{1}{2} + \frac{x}{2\sqrt{x\nu}\,\Gamma(\frac{\nu}{2})} G_{2,2}^{1,2}\left[\frac{x^2}{\nu} \Big|_{0,-\frac{1}{2}}^{\frac{1-\nu}{2},\frac{1}{2}} \right], \tag{1.9}$$

for $x \in (-\infty, \infty)$.

An alternative form for F(t), $t > 0$ is

$$F(t) = 1 - \frac{1}{2} I_{x(t)}\left(\frac{\nu}{2}, \frac{1}{2}\right), \tag{1.10}$$

where $x(t) = \frac{\nu}{t^2+\nu}$ and I is the incomplete Beta function.

The rest of the paper is divided as follows: Graphs for density and distribution functions for various values of the parameters are given in Section 2. Section 3 deals with the approximations of the S-R distribution to mathematical functions and statistical distributions. An expression for y is obtained for given $F(y)$ in Section 4. In Section 5, the density function is expressed as an infinite linear combination of density functions. Section 6 deals with the n-th moments about the origin both in terms of Gamma and H-functions, respectively. The reliability function $P(X < Y)$, for X and Y independent S-R distributions, is obtained in Section 7. Order statistics are indicated in Section 8. Marshall-Olkin-Swamee-Rathie, generalized Gamma and Beta generated distributions are given respectively in Sections 9 to 11. In Section 12, hazard rate function is mentioned. The paper ends with a conclusion section mentioning the possible future works which can be undertaken in the area.

2. Distribution functions

In this section, new density and distribution functions are introduced.

2.1. The Swamee-Rathie distribution

Let $Y \in (-\infty, \infty)$ be a random variable with density function defined by:

$$f(y) = \frac{[a_0 + 2a_1|y| + 3a_2y^2]e^{-y(a_0+a_1|y|+a_2y^2)}}{[1 + e^{-y(a_0+a_1|y|+a_2y^2)}]^2}, \tag{2.1}$$

for $a_0, a_1, a_2 \geq 0$ (not all the three zeros simultaneously).

The corresponding distribution function is given by:

$$F(y) = [1 + e^{-y(a_0+a_1|y|+a_2y^2)}]^{-1}. \tag{2.2}$$

The above symmetric Swamee-Rathie density and distribution functions will be notationally denoted respectively by:

$$Y \sim SRf(y; a_0, a_1, a_2) \tag{2.3}$$

and

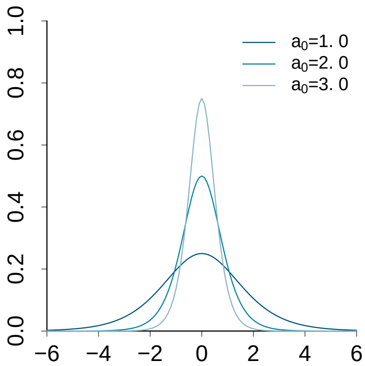
$$Y \sim SRF(y; a_0, a_1, a_2) \tag{2.4}$$

The graphs for $f(y)$ for various values of the parameters a_0, a_1 and a_2 are given in Figure 1. The corresponding cumulative distributions are shown in Figure 2.

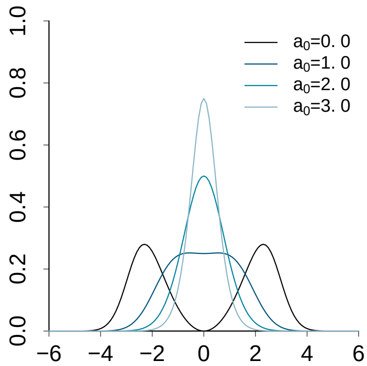
The S-R density function with a position parameter $\mu \in \mathbb{R}$ and a scale parameter $\sigma > 0$ is

$$f(x) = \frac{[a_0 + 2a_1|\frac{x-\mu}{\sigma}| + 3a_2(\frac{x-\mu}{\sigma})^2]}{\sigma\left[1 + \exp\left(-\left(\frac{x-\mu}{\sigma}\right)\left(a_0 + a_1|\frac{x-\mu}{\sigma}| + a_2\left(\frac{x-\mu}{\sigma}\right)^2\right)\right)\right]^2} \times \exp\left[-\left(\frac{x-\mu}{\sigma}\right)\left(a_0 + a_1\left|\frac{x-\mu}{\sigma}\right| + a_2\left(\frac{x-\mu}{\sigma}\right)^2\right)\right] \tag{2.5}$$

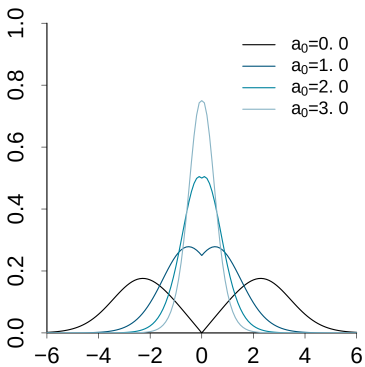
where $x \in (-\infty, \infty)$.



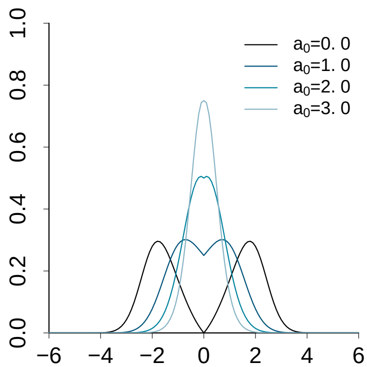
(a) $a_1 = 0,\ a_2 = 0.$



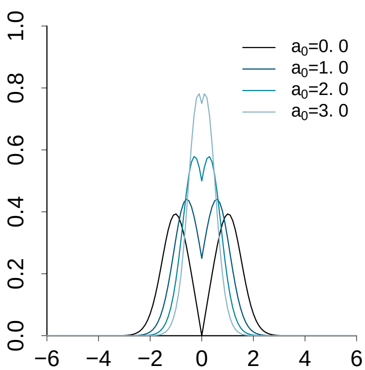
(b) $a_1 = 0,\ a_2 = 0.1.$



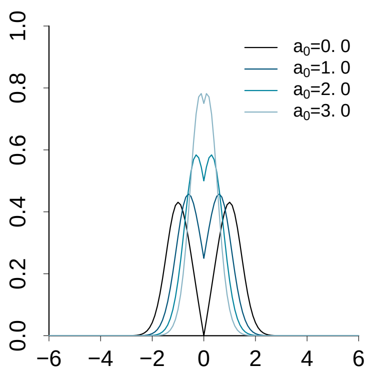
(c) $a_1 = 0.2,\ a_2 = 0.$



(d) $a_1 = 0.2,\ a_2 = 0.1.$

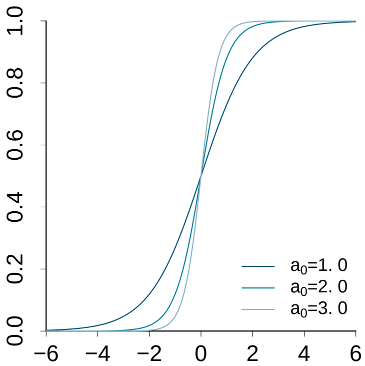


(e) $a_1 = 1,\ a_2 = 0.$

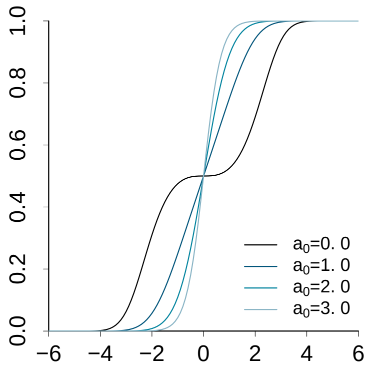


(f) $a_1 = 1,\ a_2 = 0.1.$

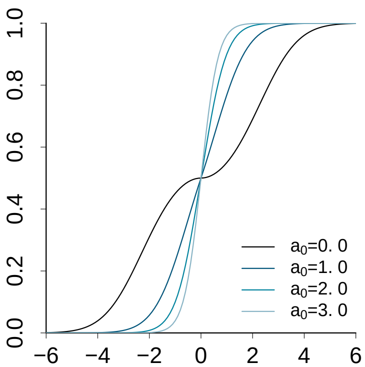
Figure 1: Density function, $f(y)$, for various values of the parameters a_0, a_1 and a_2.



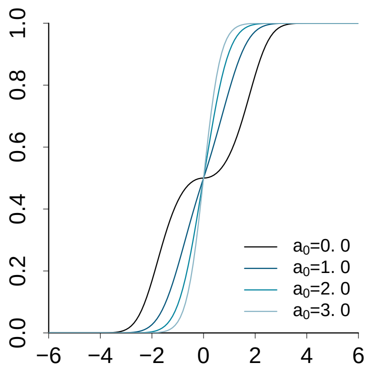
(a) $a_1 = 0,\ a_2 = 0.$



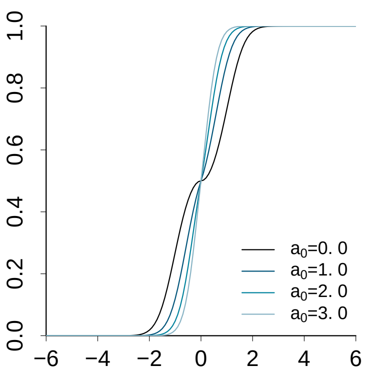
(b) $a_1 = 0,\ a_2 = 0.1.$



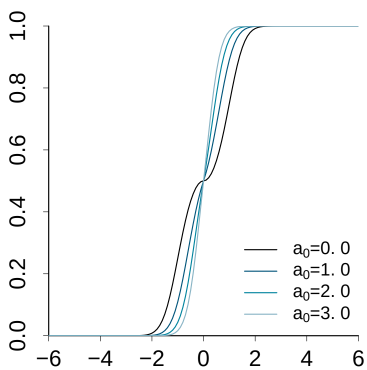
(c) $a_1 = 0.2,\ a_2 = 0.$



(d) $a_1 = 0.2,\ a_2 = 0.1.$



(e) $a_1 = 1,\ a_2 = 0.$



(f) $a_1 = 1,\ a_2 = 0.1.$

Figure 2: Cumulative function, $F(y)$, for various values of the parameters a_0, a_1 and a_2.

The corresponding S-R distribution function is given by

$$F(x) = \left[1 + e^{-\left(\frac{x-\mu}{\sigma}\right)\left(a_0 + a_1\left|\frac{x-\mu}{\sigma}\right| + a_2\left(\frac{x-\mu}{\sigma}\right)^2\right)}\right]^{-1} \tag{2.6}$$

2.2. Special cases

In this subsection, three special cases of the Swamee-Rathie distribution (2.1) are mentioned.

a) For $a_1 = 0$ and $a_2 = 0$, (2.1) reduces to the logistic distribution:

$$f(y) = \frac{a_0 e^{-a_0 y}}{[1 + e^{-a_0 y}]^2}, \tag{2.7}$$

for $a_0 > 0$ and $y \in (-\infty, \infty)$.

b) For $a_2 = 0$, (2.1) reduces to a particular case of R-S. distribution (Rathie and Swamee (2006) [4]) with p=1:

$$f(y) = \frac{[a_0 + 2a_1|y|]e^{-y(a_0 + a_1|y|)}}{[1 + e^{-y(a_0 + a_1|y|)}]^2}, \tag{2.8}$$

for $a_0, a_1 \geq 0$ (not both zeros simultaneously), and $y \in (-\infty, \infty)$.

c) For $a_1 = 0$, (2.1) reduces to a particular case of S-R. distribution (Rathie and Swamee (2006)) with p=2:

$$f(y) = \frac{[a_0 + 3a_2 y^2]e^{-y(a_0 + a_2 y^2)}}{[1 + e^{-y(a_0 + a_2 y^2)}]^2}, \tag{2.9}$$

for $a_0, a_2 \geq 0$ (not both zeros simultaneously), and $y \in (-\infty, \infty)$.

3. Approximations to Mathematical Functions and Statistical Distributions

A few new approximations of the Swamee-Rathie distribution (2.2) are obtained here.

a) The following approximation of (2.2) to Student t-distribution is found out:

$$F(x) = \left[1 + e^{-x(b_0 + b_1|x| + b_2 x^2)}\right]^{-1}, \tag{3.1}$$

where

$$b_0 = \frac{1.593}{\left(1 + \frac{0.34}{\nu}\right)}, \tag{3.2}$$

$$b_1 = 0.075\left[1+\left(\frac{5.4}{\nu}\right)^{1.35}\right]^{-1.5}, \tag{3.3}$$

$$b_2 = 1.931\left[1+\left(\frac{1.69}{\nu}\right)^{1.4}\right]^{-7}. \tag{3.4}$$

The average error involved in the use of (3.1) is 0.000172, whereas the maximum error is 0.0011 which occurs at $x = \pm 2.9$ for $\nu = 8$.

Comparing (3.1) and (1.8), we arrive at the following approximation for G-function:

$$G^{1,2}_{2,2}\left[\frac{x^2}{\nu}\Big|^{\frac{1-\nu}{2},\frac{1}{2}}_{0,-\frac{1}{2}}\right] \approx 2\sqrt{\pi\nu}\,\Gamma\left(\frac{\nu}{2}\right)\left(\left[1+e^{-x(b_0+b_1|x|+b_2x^2)}\right]^{-1}-\frac{1}{2}\right) \tag{3.5}$$

For $\nu \to \infty$, (3.1) gets converted to the following form of normal distribution:

$$F(x) = [1+e^{-x(1.593+0.075|x|+1.931x^2)}]^{-1} \tag{3.6}$$

b) The three approximations for mathematical functions are detailed in what follows:

i) For $\nu = 1$, (1.9) simplifies to:

$$F(x) = \frac{1}{2}+\frac{1}{\pi}\tan^{-1}x. \tag{3.7}$$

On the other hand, putting $\nu = 1$ in (3.1), one gets:

$$F(x) = [1+e^{-x(1.189+0.00213|x|+0.000727x^2)}]^{-1}. \tag{3.8}$$

Equating (3.7) and (3.8) and simplifying:

$$\tan^{-1}x \approx \pi[1+e^{-x(1.189+0.00213|x|+0.000727x^2)}]^{-1}-\frac{\pi}{2}. \tag{3.9}$$

ii) For $\nu = 2$, (1.9) simplifies to:

$$F(x) = \frac{1}{2}+\frac{x}{2\sqrt{2}\sqrt{1+\frac{x^2}{2}}}. \tag{3.10}$$

Putting $\nu = 2$ in (3.1) one gets:

$$F(x) = [1+e^{-x(1.362+0.00708|x|+0.0328x^2)}]^{-1}. \tag{3.11}$$

Equating (3.10) and (3.11) and simplifying:

$$\frac{\sqrt{2+x^2}-x}{\sqrt{2+x^2}+x} = e^{-x(1.362+0.00708|x|+0.0328x^2)}. \tag{3.12}$$

iii) For $\nu \to \infty$, (1.9) becomes:

$$F(x) = \frac{1}{2} + \frac{1}{2}\text{erf}\left(\frac{x}{\sqrt{2}}\right). \tag{3.13}$$

Equations (3.6) and (3.13) imply:

$$\text{erf}\left(\frac{x}{\sqrt{2}}\right) = 2[1 + e^{-x(1.593+0.075|x|+1.931x^2)}]^{-1} - 1. \tag{3.14}$$

4. Inversion Result

In this Section, an expression for y is obtained for given values of $F(y)$.

For $y \geq 0$, (2.2) yields the following third degree equation in y:

$$a_2 y^3 + a_1 y^2 + a_0 y + \ln\left(\frac{1-F(y)}{F(y)}\right) = 0. \tag{4.1}$$

The real solution of the third degree equation:

$$b_3 z^3 + b_2 z^2 + b_1 z + b_0 = 0 \tag{4.2}$$

is given by

$$z = w + a, \tag{4.3}$$

with

$$a = -\frac{b_2}{3b_3}, \tag{4.4}$$

and

$$w = \left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{3}} + \left(\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{3}}, \tag{4.5}$$

where

$$p = \frac{3a^2 b_3 + 2ab_2 + b_1}{b_3}, \tag{4.6}$$

$$q = \frac{a^3 b_3 + a^2 b_2 + ab_1 + b_0}{b_3}. \tag{4.7}$$

Thus, for $y \geq 0$,

$$y = w + a \tag{4.8}$$

is the solution of (4.1) for $b_3 = a_2$, $b_2 = a_1$, $b_1 = a_0$ and $b_0 = \ln\left[\frac{1-F(y)}{F(y)}\right]$.

For $y \geq 0$,

$$F(-y) = [1 + e^{y(a_0 - a_1 y + a_2 y^2)}]^{-1} \tag{4.9}$$

which yields the following three degree equation:

$$a_2 y^3 - a_1 y^2 + a_0 y - \ln\left[\frac{1 - F(y)}{F(-y)}\right] = 0 \tag{4.10}$$

Thus, (4.8) is the solution of (4.10) for $b_3 = a_2$, $b_2 = -a_1$, $b_1 = a_0$ and $b_0 = \ln\left[\frac{1-F(-y)}{F(-y)}\right]$. Hence, $-y$ is the value corresponding to given value of $F(-y)$.

5. Infinite linear combination

Alternatively, the symmetric S-R density function (2.1) may be written as

$$f(y) = \frac{[a_0 + 2a_1|y| + 3a_2 y^2]e^{-|y|(a_0 + a_1|y| + a_2 y^2)}}{[1 + e^{-|y|(a_0 + a_1|y| + a_2 y^2)}]^2}, \tag{5.1}$$

for $y \in (-\infty, \infty)$. Using (1.3), one obtains,

$$\begin{aligned} f(y) &= \sum_{r=0}^{\infty} (-1)^r (1 + r)[a_0 + 2a_1|y| + 3a_2 y^2]e^{-(1+r)|y|(a_0 + a_1|y| + a_2 y^2)} \\ &= 2\sum_{r=0}^{\infty} (-1)^r g_r(y) \end{aligned} \tag{5.2}$$

where the density function $g_r(y)$ is given by

$$g_r(y) = \frac{r+1}{2}[a_0 + 2a_1|y| + 3a_2 y^2]e^{-(1+r)|y|(a_0 + a_1|y| + a_2 y^2)} \tag{5.3}$$

with $a_0, a_1, a_2 \geq 0$ (not all zeros simultaneously) and $y \in (-\infty, \infty)$. Hence, from (5.2) one concludes that $f(y)$ is an infinite linear combination of $g_r(y)$ due to the fact that

$$2\sum_{r=0}^{\infty} (-1)^r = 1. \tag{5.4}$$

6. Moments

In this Section, n-th moments about origin are obtained. Using (5.2) and expanding two exponential functions, one gets:

$$f(y)=\sum_{r=0}^{\infty}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\Big[(-1)^r(1+r)^{1+u+v}(-a_0)^u(-a_1)^v \times[a_0+2a_1|y|+3a_2y^2]|y|^{u+2v}e^{-(1+r)a_2y^2|y|}\Big] \tag{6.1}$$

The n-th moments are given by:

$$E(Y^n)=\sum_{r=0}^{\infty}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}(-1)^r(1+r)^{1+u+v}(-a_0)^u(-a_1)^vI_n, \tag{6.2}$$

where

$$I_n=\int_{-\infty}^{\infty}[a_0+2a_1|y|+3a_2y^2]|y|^{u+2v}y^ne^{-(1+r)a_2y^2|y|}dy \tag{6.3}$$

Hence, $I_n=0$ for n odd integers. For n even integers,

$$I_n=2\int_0^{\infty}[a_0+2a_1y+3a_2y^2]y^{u+2v+n}e^{-(1+r)a_2y^3}dy \tag{6.4}$$

Using (1.5) in (6.4), one gets:

$$I_n=\frac{2}{3[(1+r)a_2]^{(u+2v+n+3)/3}}\Bigg[a_0[(1+r)a_2]^{\frac{2}{3}}\Gamma\left(\frac{u+2v+n+1}{3}\right) +2a_1[(1+r)a_2]^{\frac{1}{3}}\Gamma\left(\frac{u+2v+n+2}{3}\right)+3a_2\Gamma\left(\frac{u+2v+n+3}{3}\right)\Bigg], \tag{6.5}$$

Substituting I_n from (6.5) into (6.2) yields the moments for n even integer.

Alternatively, the n-th moments are derived in terms of H-function. Using (5.2) and expanding one exponential function, one gets:

$$f(y)=\sum_{r=0}^{\infty}\sum_{v=0}^{\infty}(-1)^r(-a_1)^v(1+r)^{1+v}[a_0+2a_1|y|+3a_2y^2]|y|^{2v}e^{-(1+r)|y|(a_0+a_2y^2)} \tag{6.6}$$

Hence, the n-th moments are

$$E(Y^n)=\sum_{r=0}^{\infty}\sum_{v=0}^{\infty}(-1)^r(1+r)^{1+v}(-a_1)^vK_n, \tag{6.7}$$

where $K_n = 0$ for n odd integer, and

$$K_n = 2\int_0^\infty [a_0 + 2a_1|y| + 3a_2y^2]|y|^{2v}y^n \exp[-(1+r)|y|(a_0 + a_2y^2)]dy \tag{6.8}$$

for n even integer.

Evaluating the integral in (6.8) with the help of (1.6), one gets:

$$K_n = 2\{(1+r)a_0\}^{-2v-n-3}\Bigg[\{(1+r)a_0\}^2 H_{1,1}^{1,1}\left[\frac{a_2}{a_0^3(1+r)^2}\Bigg|_{(0,1)}^{(-2v-n,3)}\right] + (1+r)a_0 H_{1,1}^{1,1}\left[\frac{a_2}{a_0^3(1+r)^2}\Bigg|_{(0,1)}^{(-2v-n-1,3)}\right] + H_{1,1}^{1,1}\left[\frac{a_2}{a_0^3(1+r)^2}\Bigg|_{(0,1)}^{(-2v-n-2,3)}\right]\Bigg]. \tag{6.9}$$

7. Reliability $P(X < Y)$

Let the distribution function of $X \sim SRF(b_1, b_2, b_3)$ be $F_X(t)$ and the density function of $Y \sim SRf(a_1, a_2, a_3)$ be $f_Y(t)$. Then, for X and Y independent, the reliability $P(X < Y)$ is given by:

$$P(X < Y) = \int_{-\infty}^\infty F_X(t)f_Y(t)dt = I_1 + I_2, \tag{7.1}$$

where

$$I_1 = \int_0^\infty F_X(-t)f_Y(-t)dt \tag{7.2}$$

and

$$I_2 = \int_0^\infty F_X(t)f_Y(t)dt \tag{7.3}$$

Following the procedure used in Section 6 with the help of (1.3) and (1.4), we obtain:

$$I_1 = \sum_{r=0}^\infty\sum_{s=0}^\infty\sum_{u=0}^\infty\sum_{v=0}^\infty \Big[(-1)^{r+s+u+v}(1+s)[(s+1)a_0 + (r+1)b_0]^u \times [(s+1)a_1 + (r+1)b]^v I_{1,r+1}(s+1, r+1, u, v)\Big] \tag{7.4}$$

with

$$I_{1,r+1}(s+1, r+1, u, v) = \int_0^\infty (a_0 + 2a_1t + 3a_2t^2)t^{u+2v}e^{-[(s+1)a_2+(r+1)b_2]t^3}dt = \frac{a_0\Gamma(\frac{u+2v+1}{3})[(s+1)a_2 + (r+1)b_2]^{\frac{2}{3}} + 2a_1\Gamma(\frac{u+2v+2}{3})}{3[(s+1)a_2 + (r+1)b_2]^{(u+2v+3)/3}} \times [(s+1)a_2 + (r+1)b_2]^{\frac{1}{3}} + 3a_2\Gamma\left(\frac{u+2v+3}{3}\right) \tag{7.5}$$

Similarly, I_2 is obtained from (7.4) by replacing $I_{1,r+1}(s+1,r+1,u,v)$ by $I_{1,r}(s+1,r,u,v)$.

In what follows, the reliability $P(X<Y)$ given in (7.1) will be expressed in terms of the H-function.

I_1 in (7.2) is rewritten as

$$I_1=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{w=0}^{\infty}(-1)^{r+s+w}(s+1)[(s+1)a_2+(r+1)b_2]^3 J_{1,r+1}(s+1,r+1,w), \quad (7.6)$$

where

$$J_{1,r+1}(s+1,r+1,w)=\int_0^{\infty}(a_0+2a_1t+3a_2t^2)t^{3w}e^{-\sum_{m=1}^{2}[(s+1)a_{m-1}+(r+1)b_{m-1}]t^m}dt \quad (7.7)$$

Applying (1.6) to (7.7) yields

$$\begin{aligned} J_{1,r+1}(s+1,r+1,w) &= [(s+1)a_0+(r+1)b_0]^{-3w-3} \\ &\times \Bigg[((s+1)a_0+(r+1)b_0)^2 \\ &\times H_{1,1}^{1,1}\left[\frac{(s+1)a_1+(r+1)b_1}{[(s+1)a_0+(r+1)b_0]^2}\Bigg|_{(0,1)}^{(-3w,2)}\right] \\ &+((s+1)a_0+(r+1)b_0) \\ &\times H_{1,1}^{1,1}\left[\frac{(s+1)a_1+(r+1)b_1}{[(s+1)a_0+(r+1)b_0]^2}\Bigg|_{(0,1)}^{(-3w-1,2)}\right] \\ &+ \ H_{1,1}^{1,1}\left[\frac{(s+1)a_1+(r+1)b_1}{[(s+1)a_0+(r+1)b_0]^2}\Bigg|_{(0,1)}^{(-3w-2,2)}\right]\Bigg] \end{aligned} \quad (7.8)$$

I_2 in (7.3) is obtained from (7.6) by replacing $J_{1,r+1}(s+1,r+1,w)$ by $J_{1,r}(s+1,r,w)$

8. Order statistics

For a random sample $X_1,...,X_n$ with cumulative distribution $F_X(x)$, the r-th order statistics has the following distribution function:

$$F_{X_{(r)}}(x)=\sum_{j=r}^{n}\binom{n}{j}[F_X(x)]^j[1-F_X(x)]^{n-j}, \quad (8.1)$$

where $F_X(x)$ and $f_X(x)$ are given respectively in (2.2) and (2.1).

The corresponding density function is

$$f_{X_{(r)}}(x)=r\binom{n}{r}f_X(x)[F_X(x)]^{r-1}[1-F_X(x)]^{n-r}. \quad (8.2)$$

In particular, the n-th order statistics has distribution and density functions given by

$$F_{X_{(n)}}(x) = [F_X(x)]^n, \tag{8.3}$$

and

$$f_{X_{(n)}}(x) = n[F_X(x)]^{n-1} f_X(x). \tag{8.4}$$

9. Marshall-Olkin-Swamee-Rathie distribution

Using the Marshall-Olkin (1997) [2] expression

$$G(x) = \frac{F(x)}{\beta + (1-\beta)F(x)}, \tag{9.1}$$

for $\beta > 0$ and $x \in \mathbb{R}$, one gets from (2.2), the following tilted generalized M-D-S-R distribution.

$$G(x) = \frac{1}{1 + \beta e^{-x(a_0 + a_1|x| + a_2 x^2)}}. \tag{9.2}$$

Here, β may be taken as a tilt parameter. The corresponding density function is:

$$g(x) = \frac{\beta[a_0 + 2a_1|x| + 3a_2x^2] e^{-x(a_0+a_1|x|+a_2x^2)}}{[1 + \beta e^{-x(a_0+a_1|x|+a_2x^2)}]^2}. \tag{9.3}$$

10. Generalized Gamma generated distributions

Two generalized gamma generated S-R distributions are mentioned in what follows by using the results from Rathie and Silva (2017) [5]:

$$\begin{aligned} H_1(x) &= \frac{cb^{\frac{a}{c}}}{\Gamma(\frac{a}{c})} \int_0^{-\ln(1-F(x))} w^{a-1} e^{-bw^c} dw \\ &= \frac{b^{\frac{a}{c}}}{\Gamma(1+\frac{a}{c})} \{-\ln(1-F(x)\}^{a-1} \\ &\quad \times {}_1F_1\left(\frac{a}{c}; 1+\frac{a}{c}; -b\{-\ln(1-F(x)\}^c\right), \end{aligned} \tag{10.1}$$

and

$$\begin{aligned} H_2(x) &= 1 - \frac{cb^{\frac{a}{c}}}{\Gamma(\frac{a}{c})} \int_0^{-ln(F(x))} w^{a-1} e^{-bw^c} dw \\ &= 1 - \frac{b^{\frac{a}{c}}}{\Gamma(1+\frac{a}{c})} \{-\ln(F(x)\}^{a} \\ &\quad \times {}_1F_1\left(\frac{a}{c}; 1+\frac{a}{c}; -b\{-\ln(1-F(x)\}^c\right). \end{aligned} \tag{10.2}$$

In both (10.1) and (10.2), $F(x)$ is given by (2.2).

11. Beta generated distributions

Using the result in Andrade and Rathie (2016) [1], the Beta generated S-R distribution is given by

$$G(x) = \frac{1}{B(\alpha,\beta)} \int_0^{F(x)} w^{\alpha-1}(1-w)^{\beta-1} dw$$
$$= \frac{F^{\alpha}(x)}{\alpha B(\alpha,\beta)} {}_2F_1(\alpha, 1-\beta; 1+\alpha; F(x)), \tag{11.1}$$

where $F(x)$ is given in (2.2).

By using the series expression for ${}_2F_1$, one gets

$$G(x) = \frac{1}{\alpha B(\alpha,\beta)} \sum_{r=0}^{\infty} \frac{(\alpha)_r (1-\beta)_r}{(1+\alpha)_r r!} F^{\alpha+r}(x), \tag{11.2}$$

indicating that G(x) is an infinite linear combination of the distribution functions $F^{\alpha+r}(x)$.

12.1 Hazard rate function

The hazard rate function for the S-R distribution is

$$h(x) = \frac{f(x)}{1-F(x)}, \tag{12.1}$$

where $f(x)$ and $F(x)$ are respectively given in (2.1) and (2.2). Substitution of the expressions in (12.1) and simplifying yields:

$$h(x) = \frac{a_0 + 2a_1|x| + 3a_2x^2}{1 + e^{-x(a_0+a_1|x|+a_2x^2)}}. \tag{12.2}$$

13. The Swamee-Rathie skew distribution

Various Swamee-Rathie skew distributions may be obtained from

$$h(x) = 2f(x)G(w(x)), \tag{13.1}$$

where $f(x)$ is a symmetric density function, $G(x)$ is a cumulative distribution function of any symmetric density function, and $w(x)$ is an odd function.

For example, taking $w(x) = cx$, $c > 0$, Swamee-Rathie density function $f(x)$ defined by (2.1) and Swamee-Rathie distribution function $G(x)$ given in (2.2), we have

$$h_c(x) = 2\frac{[a_0 + 2a_1|x| + 3a_2x^2]e^{-x(a_0+a_1|x|+a_2x^2)}}{[1+e^{-x(a_0+a_1|x|+a_2x^2)}]^2} \frac{1}{[1+e^{-cx(a_0+a_1c|x|+a_2c^2x^2)}]} \tag{13.2}$$

As a special case for $c = 1$, (13.2) reduces to

$$h_1(x) = 2\frac{[a_0 + 2a_1|x| + 3a_2x^2]e^{-x(a_0+a_1|x|+a_2x^2)}}{[1 + e^{-x(a_0+a_1|x|+a_2x^2)}]^3}. \tag{13.3}$$

The corresponding cumulative distribution function is

$$H_1(x) = \frac{1}{[1 + e^{-x(a_0+a_1|x|+a_2x^2)}]^2}. \tag{13.4}$$

The Skew S-R distribution with a position parameter $\mu \in \mathbb{R}$ and a scale parameter $\sigma > 0$ corresponding to (13.2) is given by

$$f(x) = 2\frac{[a_0 + 2a_1|\frac{x-\mu}{\sigma}| + 3a_2(\frac{x-\mu}{\sigma})^2]}{\sigma[1 + e^{-(\frac{x-\mu}{\sigma})(a_0+a_1|\frac{x-\mu}{\sigma}|+a_2(\frac{x-\mu}{\sigma})^2)}]} \frac{e^{-(\frac{x-\mu}{\sigma})(a_0+a_1|\frac{x-\mu}{\sigma}|+a_2(\frac{x-\mu}{\sigma})^2)}}{[1 + e^{-c(\frac{x-\mu}{\sigma})(a_0+a_1|c\frac{(x-\mu)}{\sigma}|+a_2c^2(\frac{x-\mu}{\sigma})^2)}]}. \tag{13.5}$$

where $x \in (-\infty, \infty)$, $a_0, a_1, a_2 \geq 0$ (not all three zeros simultaneously), and $c > 0$.

14. Conclusions

The authors plan to apply the Swamee-Rathie distribution and its skew version to analyze a few real data sets in the near future. In addition, plan to study extensively Marshall-Olkin-Swamee-Rathie, Generalized Gamma and Beta generated distributions.

References

[1] Andrade B. B. and Rathie P. N., Fitting bimodal data with selected asymmetric multimodal distributions, springer, New York, J. Stat. Comp. and Simulation, 86(16) (2016), 3205-3224.

[2] Marshall A. W. and Olkin I., A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families, Biometrika, 84(3) (1997), 641-652.

[3] Mathai A. M., Saxena R. K., and Haubold H. J., The H-function: Theory and Applications, Springer-Verlag New York, (2009).

[4] Rathie P. N. and Swamee P. K., On a new invertible generalized logistic distribution to normal distribution, Technical Research Report in Statistics 07/2006, Dept. of Statistics, Univ. of Brasilia, Brasilia, Brazil, (2006).

[5] Rathie P. N. and Silva P. H. D., On the generalized gamma-generated distributions and applications, J. Ramanujan Society of Mathematics and Mathematical Sciences, 6 (2017), 23-40.