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A SERIES EXPANSION FOR THE $b(s)$ BROUNCKER-RAMANUJAN FUNCTION

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Abstract: Our basic aim is to provide a power series representation for $b(s)$, $0 < s < 3$, the well-known function satisfying $b(s-1)b(s+1) = s^2$. We will do this by using integer compositions of n. In the last section, some properties involving the coefficients of s^n in the power series expansion of $b(s)$ are given, as well an expression for $\frac{4}{\pi}$.

Keywords and Phrases: Brouncker-Ramanujan function, Integer Compositions, Convergent series, Infinite Products, Functional Equations.

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1. Introduction

In 1655, the mathematician John Wallis concerning in the quadrature of the unit circle wrote a letter to William Brouncker in the attempt to solve a special problem. He want to find an arithmetical expression to

$$\int_0^1 \sqrt{1-x^2}dx,$$

wrote in contemporaneous mathematical language.
The answer of Brouncker is the $b(s)$ function satisfying $b(s-1)b(s+1) = s^2$. Writing using continued fraction, we have

$$b(s) = s + \cfrac{1^2}{2s + \cfrac{3^2}{2s + \cfrac{5^2}{2s + \cdots}}}$$

An important mention in the answer of Brouncker is $b(1) = \frac{4}{\pi}$.

There is a rich history about the function $b(s)$ which is detailed in Khrushchev [7, 8], and the proper problems in Kramar [9], Dutka [4] and Brouncker [3].

Later, Ramanujan my able to find an expression for $b(s)$ into infinite product using the gamma function, for all $s > 0$.

$$b(s) = 4\left(\frac{\Gamma\left(\frac{s+3}{4}\right)}{\Gamma\left(\frac{s+1}{4}\right)}\right)^2 = (s+1)\prod_{n=1}^{\infty}\frac{(s+4n-3)(s+4n+1)}{(s+4n-1)^2} \tag{1}$$

wherein

$$\Gamma(z) = \frac{1}{z}e^{-\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}e^{\frac{z}{n}},$$

and γ is the Euler-Mascheroni constant. Some kind of generalization of $b(s)$ is given by Friedmann and Webb in [5].

In this paper, our intention is to find a power series expansion for $b(s)$, for $0 < s < 3$. Obviously, we will achieve this aim, finding the coefficient of s^n in the equation (1). An alternative for this purpose is using integer compositions of n. To be more precise, the formal definition of integer composition is given next.

Definition 1. *An integer composition of a positive integer n is an ordered collection of positive integers whose sum is n. The set of compositions of n is denoted by $C(n)$.*

Example 1. There are 8 integer compositions of 4:

$$C(4) = \{4, 3+1, 1+3, 2+2, 2+1+1, 1+2+1, 1+1+2, 1+1+1+1\}$$

For more detailed information about integer compositions, see Heubach and Mansour [6] and Sills [10]. The simple technique employed here is similar to the used by Alegri [1] and Alegri, Santos and Silva [2].

2. b(s) Into Power Series

Lemma 1. *The series*

$$\sum_{w_1+\ldots+w_m\in C(n)}\ \sum_{1\le l_1<\ldots<l_m}\frac{(-4)^m(1+w_1)(1+w_2)\cdots(1+w_m)}{(4l_1-1)^{w_1+2}(4l_2-1)^{w_2+1}\cdots(4l_m-1)^{w_m+2}}\prod_{\substack{l\neq\\ l_1,\ldots,l_m}}\left(1-\frac{4}{(4l-1)^2}\right)$$

is convergent.

Proof. Firstly, since $(4l_1-1)^{w_1+2} > l_1^{w_1+2}$, we have $\frac{1}{(4l_1-1)^{w_1+2}} < \frac{1}{l_1^{w_1+2}}$, and, as the series $\sum_{l_1=1}^{\infty} \frac{1}{l_1^{w_1+2}}$ converges, so the same happens to the series $\sum_{l_1=1}^{\infty} \frac{1}{(4l_1-1)^{w_1+2}}$, by the comparison criterion. Since the following finite product of series

$$\left(\sum_{l_1=1}^{\infty} \frac{1}{(4l_1-1)^{w_1+2}}\right)\left(\sum_{l_2=1}^{\infty} \frac{1}{(4l_2-1)^{w_2+2}}\right)\cdots\left(\sum_{l_m=1}^{\infty} \frac{1}{(4l_m-1)^{w_m+2}}\right)$$

converges, then a partial sum as

$$\sum_{1\le l_1<\ldots<l_m} \frac{1}{(4l_1-1)^{w_1+2}(4l_2-1)^{w_2+1}\cdots(4l_m-1)^{w_m+2}}$$

also converges.

For $l \neq l_1, l_2, \ldots, l_m$, we have

$$0 \le \prod_{\substack{l\neq \\ l_1,\ldots,l_m}} \left(1-\frac{4}{(4l-1)^2}\right) < 1$$

As the series

$$\sum_{1\le l_1<\ldots<l_m} \frac{1}{(4l_1-1)^{w_1+2}(4l_2-1)^{w_2+1}\cdots(4l_m-1)^{w_m+2}} \prod_{\substack{l\neq \\ l_1,\ldots,l_m}} \left(1-\frac{4}{(4l-1)^2}\right)$$

is convergent, and $(-4)^m(1+w_1)(1+w_2)\cdots(1+w_m)$ is a constant, we obtain the convergence of

$$\sum_{1\le l_1<\ldots<l_m} \frac{(-4)^m(1+w_1)(1+w_2)\cdots(1+w_m)}{(4l_1-1)^{w_1+2}(4l_2-1)^{w_2+1}\cdots(4l_m-1)^{w_m+2}} \prod_{\substack{l\neq \\ l_1,\ldots,l_m}} \left(1-\frac{4}{(4l-1)^2}\right).$$

As we known the number of integer composition of n is 2^{n-1}, so taking the sum over all integer compositions $w_1+w_2+\ldots+w_m \in C(n)$, the next series is convergent.

$$\sum_{w_1+\ldots+w_m\in C(n)} \sum_{1\le l_1<\ldots<l_m} \frac{(-4)^m(1+w_1)(1+w_2)\cdots(1+w_m)}{(4l_1-1)^{w_1+2}(4l_2-1)^{w_2+1}\cdots(4l_m-1)^{w_m+2}} \prod_{\substack{l\neq \\ l_1,\ldots,l_m}} \left(1-\frac{4}{(4l-1)^2}\right)$$

Theorem 1. *For $0 < s < 3$, we have*

$$b(s) = \sum_{n=0}^{\infty} f_n s^n,$$

for $f_n = x_n - x_{n-1}$, *and*

$$x_n = \sum_{w_1+w_2+\ldots+w_m \in C(n)} \sum_{1\leq l_1<\ldots<l_m} \frac{(-4)^m(1+w_1)(1+w_2)\cdots(1+w_m)}{(4l_1-1)^{w_1+2}(4l_2-1)^{w_2+1}\cdots(4l_m-1)^{w_m+2}} \prod_{\substack{l\neq \\ l_1,\ldots,l_m}} \left(1-\frac{4}{(4l-1)^2}\right)$$

Proof. First let consider the function

$$g(s) = \prod_{n=1}^{\infty} \frac{(s+4n-3)(s+4n+1)}{(s+4n-1)^2},$$

for $0 < s < 3$.

Is easy to see that

$$g(s) = \frac{(s+4n-3)(s+4n+1)}{(s+4n-1)^2} = 1 - \frac{4}{(s+4n-1)^2} = 1 - \frac{4}{(4n-1)^2\left(1+\frac{s}{4n-1}\right)^2}.$$

Then

$$g(-s) = \prod_{n=1}^{\infty}\left(1 - \frac{4}{(4n-1)^2}.\frac{1}{\left(1-\frac{s}{4n-1}\right)}\right)$$

Since $0 < s < 3$, we have $|\frac{s}{4n-1}| < 1$, and then

$$\frac{1}{\left(1-\frac{s}{4n-1}\right)} = \left(1 + \frac{s}{4n-1} + \frac{s^2}{(4n-1)^2} + \frac{s^3}{(4n-1)^3} + \ldots\right)^2,$$

and the previous equation reveals

$$\frac{1}{\left(1-\frac{s}{4n-1}\right)} = 1 + \frac{2s}{4n-1} + \frac{3s^2}{(4n-1)^2} + \frac{4s^3}{(4n-1)^3} + \ldots,$$

so $g(-s)$ can be write as

$$g(-s) = \prod_{n=1}^{\infty}\left(1 - \frac{4}{4n-1}\left[1 + \frac{2s}{4n-1} + \frac{3s^2}{(4n-1)^2} + \frac{4s^3}{(4n-1)^3} + \ldots\right]\right). \quad (2)$$

The next step is to find the coefficient of s^n in the previous infinite product. A simple way to find a portion of the sum that make up the coefficient of s^n is taking an integer composition $w_1 + w_2 + \ldots + w_m \in C(n)$. Here each part w_i of the composition is considered as a contribution of s^{w_i}, since $s^{w_1+\ldots+w_m} = s^n$. But this contribution came from infinite ways, so the manner to proceed with this task is considering the sums as

$$\sum_{1\le l_1<l_2<\ldots<l_m}.$$

Looking in the equation (2), for the integer composition $w_1+w_2+\ldots+w_m \in C(n)$, the contribution in the coefficient s^n is

$$\left(\sum_{1\le l_1<\ldots<l_m} \frac{(-4)^m(1+w_1)(1+w_2)\cdots(1+w_m)}{(4l_1-1)^{w_1+2}(4l_2-1)^{w_2+1}\cdots(4l_m-1)^{w_m+2}}\right)\times\Theta$$

where Θ is the product of the constant terms

$$1-\frac{4}{(4l-1)^2},$$

for $l\neq l_1,l_2,\ldots,l_m$.

Thus the coefficient of s^n in the power series expansion of $g(-s)$ is equal to

$$\sum_{w_1+\ldots+w_m\in C(n)}\sum_{1\le l_1<\ldots<l_m} \frac{(-4)^m(1+w_1)(1+w_2)\cdots(1+w_m)}{(4l_1-1)^{w_1+2}(4l_2-1)^{w_2+1}\cdots(4l_m-1)^{w_m+2}}\prod_{\substack{l\neq\\ l_1,\ldots,l_m}}\left(1-\frac{4}{(4l-1)^2}\right).$$

Since $b(s)=(1+s)g(s)$, we have

$$b(s)=\sum_{n=0}^{\infty} f_n s^n,$$

where $f_n = x_n - x_{n-1}$, and

$$x_n = \sum_{w_1+w_2+\ldots+w_m\in C(n)}\sum_{1\le l_1<\ldots<l_m} \frac{(-4)^m(1+w_1)(1+w_2)\cdots(1+w_m)}{(4l_1-1)^{w_1+2}(4l_2-1)^{w_2+1}\cdots(4l_m-1)^{w_m+2}}\prod_{\substack{l\neq\\ l_1,\ldots,l_m}}\left(1-\frac{4}{(4l-1)^2}\right)$$

for $0<s<3$, completing the proof.

3. Some Consequences

Returning in $b(1) = \frac{4}{\pi}$, we have

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} f_n = f_0 + \lim_{k\to\infty} \sum_{n=1}^{k} f_n = f_0 - x_0 + \lim_{k\to\infty} x_k = \lim_{k\to\infty} x_k,$$

or, more explicit,

$$\frac{4}{\pi} = \lim_{k\to\infty} \sum_{w_1+w_2+\ldots+w_m \in C(k)} \sum_{1\le l_1<\ldots<l_m} \frac{(-4)^m(1+w_1)\cdots(1+w_m)}{(4l_1-1)^{w_1+2}\cdots(4l_m-1)^{w_m+2}} \prod_{\substack{l\neq \\ l_1,\ldots,l_m}} \left(1 - \frac{4}{(4l-1)^2}\right).$$

Here

$$0 < f_0 = \prod_{n=1}^{\infty} \left(1 - \frac{4}{(4n-1)^2}\right).$$

and we can write

$$f_0 = \prod_{n=1}^{\infty} \left(1 - \frac{4}{(4n-1)^2}\right) = cos(\pi^2) \prod_{n=0}^{\infty} \left(\frac{4n+1}{4n-3}\right).$$

Since $b(s)$ obeys the function equation:

$$\left[\sum_{n=0}^{\infty} f_n (s-1)^n\right] \left[\sum_{n=0}^{\infty} f_n (s+1)^n\right] = s^2,$$

searching for the coefficient of s^0, s^1 and s^2 in the previous equation, we are able to state the next corollary.

Corollary 1.
a) $f_0 + 2f_2 + 2f_4 + \ldots = 0$;

b) $(f_0 - f_1 + f_2 - f_3 + \ldots)(f_1 + 2f_2 + 3f_3 + \ldots) + (f_1 - 2f_2 + 3f_3 + \ldots)(f_0 + f_1 + f_2 + \ldots) = 0$;
c) $(f_0 - f_1 + f_2 - f_3 + \ldots)(f_2 + \binom{3}{2} f_3 + \binom{4}{2} f_4 + \ldots) + (f_1 - 2f_2 + 3f_3 + \ldots)(f_1 + 2f_2 + 3f_3 + \ldots) + (f_2 - \binom{3}{2} f_3 + \binom{4}{2} f_4 + \ldots)(f_1 + f_2 + f_3 + \ldots) = 1$;

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