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INTEGRAL FORMULAS FOR WRIGHT FUNCTION

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Abstract: The aim of this paper is to present integrals form of Wright function associated with algebraic function, special functions. Results are key to the analysis of the Wright function in various type of integrals.

Keywords and Phrases: Wright function, Jacobi polynomial, Bessels Maitland function, Legendre function and Hypergeometric function.

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1. Introduction

Wright Function.

Wright function is a powerful technique for the solution of problems in mathematics, mathematical physics and engineering. From the early days of the Wright function the subject has been an area of great theoretical research and practical applications and it continues to be in so our day. Many studies related to the Wright function are found in numerous research papers [1, 2, 3, 4, 5, 9, 10, 11, 12, 13, 18]. The generalized hypergeometric Wright function introduced by Wright [17, 18] in terms of generalized hypergeometric function form and defined by

$$ {}_p\Psi_q(z) = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + \beta_i k)}{\prod_{j=1}^{q} \Gamma(\alpha_j + \beta_j k)} \frac{z^k}{k!} \tag{1.1} $$

where

$$\alpha_i, \beta_j \in R = (-\infty, \infty), (\alpha_i, \beta_j \neq 0, \ i = 1, 2, ...p \ \ \& \ j = 1, 2, ...q)$$

where $z \in C$ is the set of complex numbers and $\Gamma(z)$ is Euler Gamma function [14, sec 1.1] condition for the existence of equation (1.1) together with its representation in terms of Mellin Barnes integral and of H-function where established in [14, 17]. In particular Wright hypergeometric function in an entire function if therefor the condition

$$\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > -1. \tag{1.2}$$

This paper deal with various integral formulas involving the Wright function. Each formula given here can be established by employing the definition for the Wright function evaluating the inner integral with the help of some already known integral and Changing the order of integration. We mention below a some interesting results.

Main Results

2. Integral with Algebraic Function.

$$I_1 = \int_0^1 x^{-\rho}(1-x)^{\rho-\sigma-1} {}_p\Psi_q(zx)dx$$

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + \beta_i k)}{\prod_{j=1}^{q} \Gamma(\alpha_j + \beta_j k)} \frac{z^k}{k!} \int_0^1 x^{1-\rho+k-1}(1-x)^{\rho-\sigma-1} dx$$

$$= \Gamma(\rho - \sigma)_{p+1}\Psi_{q+1} \left[z \left| \begin{matrix} (\alpha_i, \beta_i)_{1,p}, & (1-\rho, 1) \\ (\alpha_j, \beta_j)_{1,q}, & (1-\sigma, 1) \end{matrix} \right. \right]. \tag{2.1}$$

$$I_2 = \int_0^1 x^{\rho-1}(1-x)^{\sigma-1} {}_p\Psi_q(zx)dx$$

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + \beta_i k)}{\prod_{j=1}^{q} \Gamma(\alpha_j + \beta_j k)} \frac{z^k}{k!} \int_0^1 x^{\rho+k-1}(1-x)^{\sigma-1} dx$$

$$= \Gamma(\sigma)_{p+1}\Psi_{q+1}\left[\alpha\left|\begin{matrix}(\alpha_i,\beta_i)_{1,p}, & (\rho,1)\\(\alpha_j,\beta_j)_{1,q}, & (\rho+\sigma,1)\end{matrix}\right.\right]. \tag{2.2}$$

$$I_3 = \int_1^\infty x^{-\rho}(x-1)^{\sigma-1}{}_p\Psi_q(zx)dx$$

$$= \sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(\alpha_i+\beta_i k)}{\prod_{j=1}^{q}\Gamma(\alpha_j+\beta_j k)}\frac{z^k}{k!}\int_1^\infty x^{-\rho}(x-1)^{\sigma-1}x^k dx.$$

Putting, $x = t+1$ & $dx = dt$ we get

$$= \sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(\alpha_i+\beta_i k)}{\prod_{j=1}^{q}\Gamma(\alpha_j+\beta_j k)}\frac{z^k}{k!}\int_0^\infty t^{\sigma-1}(t+1)^{-(\rho-k-\sigma+\sigma)}dt.$$

By using

$$\int_0^\infty t^{\alpha-1}(1+t)^{-(\beta+\alpha)}dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

$$= \Gamma(\sigma)_{p+1}\Psi_{q+1}\left[z\left|\begin{matrix}(\alpha_i,\beta_i)_{1,p}, & (\rho-\sigma,1)\\(\alpha_j,\beta_j)_{1,q}, & (\rho,-1)\end{matrix}\right.\right]. \tag{2.3}$$

$$I_4 = \int_0^\infty x^{\rho-1}(x+\mu)^{-\sigma}{}_p\Psi_q(zx)dx$$

$$= \sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(\alpha_i+\beta_i k)}{\prod_{j=1}^{q}\Gamma(\alpha_j+\beta_j k)}\frac{z^k}{k!}\mu^{-\sigma}\int_0^\infty x^{\rho+k-1}\left(\frac{x}{\mu}+1\right)^{-\sigma}dx.$$

Putting $x = t\mu$ & $dx = \mu dt$, we get

$$= \mu^{\rho-\sigma}\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(\alpha_i+\beta_i k)}{\prod_{j=1}^{q}\Gamma(\alpha_j+\beta_j k)}\frac{z^k\mu^k}{k!}\int_0^\infty t^{\rho+k-1}(t+1)^{-(\sigma-\rho-k+\rho+k)}dt$$

$$= \frac{\mu^{\rho-\sigma}}{\Gamma(\sigma)}{}_{p+2}\Psi_q\left[\mu z \left| \begin{matrix} (\alpha_i,\beta_i)_{1,p}, & (\rho,1), & (\sigma-\rho,-1) \\ (\alpha_j,\beta_j)_{1,q}, & - & \end{matrix} \right.\right]. \quad (2.4)$$

$$I_5 = \int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}{}_p\Psi_q(z(1-x)^{\mu})dx$$

$$= \sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(\alpha_i+\beta_i k)}{\prod_{j=1}^{q}\Gamma(\alpha_j+\beta_j k)}\frac{z^k}{k!}\int_{-1}^{1}(1-x)^{1+\rho+\mu k-1}(1+x)^{1+\sigma-1}dx. \quad (2.5)$$

Using the formula ([14], p. 261)

$$\int_{-1}^{+1}(1-x)^{n+\alpha}(1+x)^{n+\beta}dx = 2^{2n+\alpha+\beta+1} \quad B(1+\alpha+n, 1+\beta+n). \quad (2.6)$$

Hence (2.5) becomes

$$= \sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(\alpha_i+\beta_i k)}{\prod_{j=1}^{q}\Gamma(\alpha_j+\beta_j k)}\frac{z^k}{k!}2^{2+\rho+\mu k-1+\sigma-1+1}\frac{\Gamma(1+\rho+\mu k)\Gamma(1+\sigma)}{\Gamma(2+\rho+\mu k+\sigma)}$$

$$= 2^{\rho+\sigma+1}\Gamma(1+\sigma)_{p+1}\Psi_{q+1}\left[2^{\mu}z\left|\begin{matrix}(\alpha_i,\beta_i)_{1,p}, & (1+\rho,\mu) \\ (\alpha_j,\beta_j)_{1,q}, & (2+\rho+\sigma,\mu)\end{matrix}\right.\right]. \quad (2.7)$$

3. Integral with Jacobi Polynomials

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ ([14]. P. 254) may be defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\,{}_2F_1\left[\begin{matrix}-n, & 1+\alpha+\beta+n; \\ & 1+\alpha;\end{matrix}\quad \frac{1-x}{2}\right]. \quad (3.1)$$

When $\alpha = \beta = 0$. The polynomial in (3.1) becomes the Legendre polynomial ([8] P. 157). From (3.1) it follows that $P_n^{(\alpha,\beta)}(x)$ is a polynomial of degree precisely n and that

$$P_n^{(\alpha,\beta)}(1) = \frac{(1+\alpha)_n}{n!}.$$

$$I_6 = \int_{-1}^{1}x^{\lambda}(1-x)^{\mu}(1+x)^{\delta}P_n^{(\mu,\nu)}(x)_p\Psi_q(z(1+x)^h)dx$$

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + \beta_i k)}{\prod_{j=1}^{q} \Gamma(\alpha_j + \beta_j k)} \frac{z^k}{k!} \int_{-1}^{1} x^{\lambda}(1-x)^{\mu}(1+x)^{\delta}(1+x)^{hk} P_n^{(\mu,\nu)}(x)dx.$$

By using the formula ([14], p. 52)

$$\int_{-1}^{+1} x^{\lambda}(1-x)^{\mu}(1+x)^{\delta} P_n^{(\mu,\nu)}(x)dx$$

$$= (-1)^n \frac{2^{\mu+\delta+1}\Gamma(\delta+1)\Gamma(n+\mu+1)\Gamma(\delta+\nu+1)}{n!\Gamma(\delta+\nu+n+1)\Gamma(\delta+\mu+n+2)}$$

$$\times {}_3F_2\left[\begin{matrix} -\lambda, & \delta+\nu+1, & \delta+1 \\ \delta+\nu+n+1, & \delta+\mu+n+2 & \end{matrix}; 1\right].$$

Providing : $\mu > -1$ and $\nu > -1$. We arrive at

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + \beta_i k)}{\prod_{j=1}^{q} \Gamma(\alpha_j + \beta_j k)} \frac{z^k}{k!}$$

$$\times \frac{(-1)^n 2^{\mu+\delta+hk+1}\Gamma(\delta+hk+1)\Gamma(n+\mu+1)\Gamma(\delta+hk+\nu+1)}{n!\Gamma(\delta+hk+\nu+n+1)\Gamma(\delta+hk+\mu+n+2)}$$

$$\times {}_3F_2\left[\begin{matrix} -\lambda, & \delta+hk+\nu+1, & \delta+hk+1 \\ \delta+hk+\nu+n+1, & \delta+hk+\mu+n+2 & \end{matrix}; 1\right]$$

$$= \frac{(-1)^n 2^{\mu+\delta+1}\Gamma(n+\mu+1)}{n!} \sum_{s=0}^{\infty} \frac{(-\lambda)_s (1)_s}{s!} \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + \beta_{ik})}{\prod_{j=1}^{q} \Gamma(\alpha_j + \beta_{jk})}$$

$$\times \frac{\Gamma(\delta+\nu+s+1+hk)\Gamma(\delta+s+1+hk)}{\Gamma(\delta+\nu+n+s+1+hk)\Gamma(\delta+\mu+s+n+2+hk)} \frac{(2^h z)^k}{k!}$$

$$= \frac{(-1)^n 2^{\mu+\delta+1}\Gamma(n+\mu+1)}{n!} {}_1F_0\left[-\lambda; \ -1\right]$$

$$\times {}_{p+2}\Psi_{q+2}\left[2^h z \left| \begin{matrix} (\alpha_i,\beta_i)_{1,p}, & (\delta+s+\nu+1,h), & (\delta+s+1,h) \\ (\alpha_j,\beta_j)_{1,q}, & (\delta+s+\nu+n+,h), & (\delta+s+\mu+n+2,h) \end{matrix} \right.\right]. \quad (3.2)$$

Provided $\mathrm{Re}(\nu) > -1, |\arg z| < \frac{1}{2}\pi\Omega,$ h and δ are positive numbers.

$$I_7 = \int_{-1}^{+1} (1-x)^{\delta}(1+x)^{\nu}\ P_n^{(\mu,\nu)}(x) P_m^{(\rho,\sigma)}(x) \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i+\beta_i k)}{\prod_{j=1}^{q} \Gamma(\alpha_j+\beta_j k)} \frac{(z(1-x)^h)^k}{k!} dx$$

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i+\beta_i k)}{\prod_{j=1}^{q} \Gamma(\alpha_j+\beta_j k)} \frac{z^k}{k!} \int_{-1}^{1} (1-x)^{\delta+hk}(1+x)^{\nu} P_n^{(\mu,\nu)}(x) P_m^{(\rho,\sigma)}(x) dx.$$

Now using (3.1) in above expression, we have

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i+\beta_i k)}{\prod_{j=1}^{q} \Gamma(\alpha_j+\beta_j k)} \frac{z^k}{k!} \frac{(1+\rho)_m}{m!} \sum_{s=0}^{\infty} \frac{(-m)_s(1+\rho+\sigma+m)_s}{(1+\rho)_s 2^s s!}$$

$$\times \int_{-1}^{1} (1-x)^{\delta+hk+s}(1+x)^{\nu} P_n^{(\mu,\nu)}(x) dx.. \quad (3.3)$$

Again using (3.1) in (3.3), we get

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i+\beta_i k)}{\prod_{j=1}^{q} \Gamma(\alpha_j+\beta_j k)} \frac{z^k}{k!} \frac{\Gamma(1+\rho+m)\Gamma(1+\mu+n)}{m!n!\Gamma(1+\rho)\Gamma(1+\mu)}$$

$$\times \sum_{s=0}^{\infty} \frac{(-m)_s(-n)_s(1+\rho+\sigma+m)_s(1+\mu+\nu+n)_s}{2^{2s}(s!)^2\Gamma(1+\rho+s)\Gamma(1+\mu+s)}$$

$$\times \int_{-1}^{1} (1-x)^{1+\delta+hk+2s-1}(1+x)^{1+\nu-1} dx.$$

Using formula (2.6), we arrive at

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + \beta_i k)}{\prod_{j=1}^{q} \Gamma(\alpha_j + \beta_j k)} \frac{z^k}{k!} \frac{\Gamma(1+\rho+m)\Gamma(1+\mu+n)}{m!n!}$$

$$\times \sum_{s=0}^{\infty} \frac{(-m)_s(-n)_s(1+\rho+\sigma+m)_s(1+\mu+\nu+n)_s}{2^{2s}(s!)^2\Gamma(1+\rho+s)\Gamma(1+\mu+s)}$$

$$\times\ 2^{2+\delta+hk+2s-1+\nu-1+1} \frac{\Gamma(1+\delta+hk+2s)\Gamma(1+\nu)}{\Gamma(2+\delta+\nu+2s+hk)}$$

$$= \frac{\Gamma(1+\rho+m)\Gamma(1+\mu+n)}{m!n!} \Gamma(1+\nu) 2^{\delta+\nu+1}$$

$$\times \sum_{s=0}^{\infty} \frac{(-m)_s(-n)_s(1+\rho+\sigma+m)_s(1+\mu+\nu+n)_s}{(s!)^2\Gamma(1+\rho+s)\Gamma(1+\mu+s)}$$

$$\times {}_{p+1}\Psi_{q+1} \left[2^h z \left| \begin{matrix} (\alpha_i, \beta_i)_{1,p}, & (1+\delta+2s, h) \\ (\alpha_j, \beta_j)_{1,q}, & (2+\delta+\nu+2s, h) \end{matrix} \right. \right]. \qquad (3.4)$$

Provided $\mathrm{Re}(\nu) > -1, |\arg z| < \frac{1}{2}\pi\Omega,$ h and δ are positive numbers.

$$I_8 = \int_{-1}^{1} (1-x)^{\rho}(1+x)^{\sigma} P_n^{(\mu,\nu)}(x) {}_p\Psi_q(z(1-x)^h(1+x)^t) dx$$

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + \beta_i k)}{\prod_{j=1}^{q} \Gamma(\alpha_j + \beta_j k)} \frac{z^k}{k!} \int_{-1}^{1} (1-x)^{\rho+hk}(1+x)^{\sigma+tk} \frac{(1+\mu)_n}{n!}$$

$$\times {}_2F_1 \left[\begin{matrix} -n, & 1+\mu+\nu+n \\ & 1+\mu \end{matrix} ; \frac{1-x}{2} \right] dx.$$

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + \beta_i k)}{\prod_{j=1}^{q} \Gamma(\alpha_j + \beta_j k)} \frac{z^k}{k!} \frac{(1+\mu)_n}{n!} \sum_{s=0}^{\infty} \frac{(-n)_s (1+\mu+\nu+n)_s}{2^s s! (1+\mu)_s}$$

$$\times \int_{-1}^{1} (1-x)^{1+\rho+hk+s-1} (1+x)^{1+\sigma+tk-1} dx. \tag{3.5}$$

Using (2.6) in (3.5), we get

$$= \frac{(1+\mu)_n}{n!} 2^{\rho+\sigma+1} \sum_{s=0}^{\infty} \frac{(-n)_s (1+\mu+\nu+n)_s}{s! (1+\mu)_s}$$

$$\times \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + \beta_i k)}{\prod_{j=1}^{q} \Gamma(\alpha_j + \beta_j k)} \frac{\Gamma(1+\rho+s+hk))\Gamma(1+\sigma+tk)}{\Gamma((2+\rho+\sigma+s+hk+tk)} \frac{(z 2^{h+t})^k}{k!}.$$

$$= \frac{(1+\mu)_n}{n!} 2^{\rho+\sigma+1} {}_2F_1 \left[\begin{matrix} -n, 1+\mu+\nu+n \\ 1+\mu \end{matrix} ; 1 \right]$$
$$\times {}_{p+2}\Psi_{q+1} \left[2^{h+t} z \left| \begin{matrix} (\alpha_i, \beta_i)_{1,p}, & (1+\rho+s, h), (1+\sigma, t) \\ (\alpha_j, \beta_j)_{1,q}, & (2+\rho+\sigma+s, h+t) \end{matrix} \right. \right]. \tag{3.6}$$

Provided : $|\arg z| < \frac{1}{2}\pi\Omega$ and $\mathrm{Re}(\mu) > -1$ and $Re(\nu) > -1$.

$$I_9 = \int_{-1}^{1} (1-x)^{\rho} (1+x)^{\sigma} P_n^{(\mu,\nu)}(x) {}_p\Psi_q(z(1+x)^{-h}) dx$$

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + \beta_i k)}{\prod_{j=1}^{q} \Gamma(\alpha_j + \beta_j k)} \frac{z^k}{k!} \frac{(1+\mu)_n}{n!} \sum_{s=0}^{\infty} \frac{(-n)_s (1+\mu+\nu+n)_s}{2^s s! (1+\mu)_s}$$

$$\times \int_{-1}^{1} (1-x)^{1+\rho+s-1} (1+x)^{1+\sigma-hk-1} dx. \tag{3.7}$$

Using (2.6) in (3.7), we obtain

$$= \frac{(1+\mu)_n}{n!} \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + \beta_i k)}{\prod_{j=1}^{q} \Gamma(\alpha_j + \beta_j k)} \frac{z^k}{k!} \sum_{s=0}^{\infty} \frac{(-n)_s (1+\mu+\nu+n)_s}{2^s s! (1+\mu)_s}$$

$$\times 2^{2+\rho+s-1+\sigma-hk-1+1} \frac{\Gamma(1+\rho+s)\Gamma(1+\sigma-hk)}{\Gamma(2+\rho+\sigma+s-hk)}$$

$$= \frac{(1+\mu)_n}{n!} 2^{\rho+\sigma+1} \frac{1}{\Gamma(1+\rho)} {}_3F_1 \left[\begin{matrix} -n, & 1+\mu+\nu+n, & 1+\rho \\ & 1+\mu & \end{matrix} ; 1 \right]$$

$$\times {}_{p+1}\Psi_{q+1} \left[2^{-h} z \left| \begin{matrix} (\alpha_i, \beta_i)_{1,p} & (1+\delta, -h) \\ (\alpha_j, \beta_j)_{1,q} & (2+\rho+\sigma+s, -h) \end{matrix} \right. \right]. \tag{3.8}$$

Here $|\arg z| < \frac{1}{2}\pi\Omega$, $Re(\mu) > -1$, $Re(\nu) > -1$.

$$I_{10} = \int_{-1}^{1} (1-x)^{\rho} (1+x)^{\sigma} P_n^{(\mu,\nu)}(x) {}_p\Psi_q(z(1-x)^h (1+x)^{-t}) dx$$

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + \beta_i k)}{\prod_{j=1}^{q} \Gamma(\alpha_j + \beta_j k)} \frac{z^k}{k!} \int_{-1}^{1} (1-x)^{\rho+hk} (1+x)^{\sigma-tk} \frac{(1+\mu)_n}{n!}$$

$$\times {}_2F_1 \left[\begin{matrix} -n, & 1+\mu+\nu+n \\ & 1+\mu \end{matrix} ; \frac{1-x}{2} \right] dx$$

$$= \frac{(1+\mu)_n}{n!} \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + \beta_i k)}{\prod_{j=1}^{q} \Gamma(\alpha_j + \beta_j k)} \frac{z^k}{k!} \sum_{s=0}^{\infty} \frac{(-n)_s (1+\mu+\nu+n)_s}{2^s s! (1+\mu)_s}$$

$$\times \int_{-1}^{1} (1-x)^{1+\rho+hk+s-1} (1+x)^{1+\sigma-tk-1} dx. \tag{3.9}$$

Using (2.6) in (3.9), we get

$$= \frac{(1+\mu)_n}{n!} \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + \beta_i k)}{\prod_{j=1}^{q} \Gamma(\alpha_j + \beta_j k)} \frac{z^k}{k!} \sum_{s=0}^{\infty} \frac{(-n)_s (1+\mu+\nu+n)_s}{(1+\mu)_s s! 2^s}$$

$$\times 2^{2+\rho+hk+s-1+\sigma-tk-1+1} \frac{\Gamma(1+\rho+s+hk)\Gamma(1+\sigma-tk)}{\Gamma(2+\rho+\sigma+s-hk-tk)}$$

$$= \frac{(1+\mu)_n}{n!} 2^{\rho+\sigma+1} \sum_{s=0}^{\infty} \frac{(-n)_s (1+\mu+\nu+n)_s}{(1+\mu)_s s!} \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + \beta_i k)}{\prod_{j=1}^{q} \Gamma(\alpha_j + \beta_j k)}$$

$$\times \frac{\Gamma(1+\rho+s+hk)\Gamma(1+\sigma-tk)}{\Gamma(2+\rho+\sigma+s-hk-tk)} \frac{z^k (2^{h-t})^k}{k!}$$

$$= 2^{\rho+\sigma+1} \frac{(1+\mu)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, \quad 1+\mu+\nu+n \\ 1+\mu \end{matrix} ; 1 \right]$$

$$\times {}_{p+2}\Psi_{q+1} \left[2^{h-t} z \left| \begin{matrix} (\alpha_i, \beta_i)_{1,p}, & (1+\rho+s, h), & (1+\sigma, -t) \\ (\alpha_j, \beta_j)_{1,q}, & (2+\rho+\sigma+s, -(h+t)), & \end{matrix} \right. \right]. \quad (3.10)$$

Provided $|\arg z| < \frac{1}{2}\pi\Omega$.

4. Integral with Bessel Maitland function

The special case of the Wright function (1.1) ([3], Vol. 3, Section 18.1)

$$\phi(B, b; z) \equiv {}_0\Psi_1 \left[-_{(b,B)} | z \right]$$
$$= \sum_{k=0}^{\infty} \frac{1}{\Gamma(Bk+b)} \frac{z^k}{k!} \quad (4.1)$$

with complex $z, b \in C$ and real $B \in R$. When $B = \delta$, $b = \nu + 1$ and z is replaced by $-z$, the function $\phi(\delta, \nu+1; -z)$ is defined by $J_v^{\delta}(z)$

$$J_{\nu}^{\delta}(z) \equiv \phi(\delta, \nu+1; -z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\delta k + \nu + 1)} \frac{(-z)^k}{k!} \quad (4.2)$$

and such a function is known as the Bessel Maitland function or the Wright generalized Bessel function. see ([10], p. 352).

$$I_{11} = \int_0^\infty x^l J_\nu^\tau(x)_p\Psi_q(zx^\gamma)dx.$$

$$= \sum_{k=0}^\infty \frac{\prod_{i=1}^p \Gamma(\alpha_i+\beta_i k)}{\prod_{j=1}^q \Gamma(\alpha_j+\beta_j k)} \frac{z^k}{k!} \int_0^\infty x^{l+\gamma k} J_\nu^\tau(x)dx.$$

Using formula ([14], p. 55)

$$\int_0^\infty x^l J_\nu^\tau(x)dx = \frac{\Gamma(l+1)}{\Gamma(1+\nu-\tau-\tau l)}, \text{Re } (l) > -1, \ 0 < \tau < 1.$$

in above expression, then we obtain

$$= \sum_{k=0}^\infty \frac{\prod_{i=1}^p \Gamma(\alpha_i+\beta_i k)}{\prod_{j=1}^q \Gamma(\alpha_j+\beta_j k)} \frac{\Gamma(l+\gamma k+1)}{\Gamma(1+\nu-\tau-\tau(l+\gamma k)} \frac{z^k}{k!}$$

$$= {}_{p+1}\Psi_{q+1}\left[z \middle| \begin{matrix} (\alpha_i,\beta_i)_{1,p}, & (l+1,\gamma) \\ (\alpha_j,\beta_j)_{1,q}, & (1+\nu-\tau-\tau l, \gamma\tau) \end{matrix} \right]. \tag{4.3}$$

Where

(i) $|\arg z| < \frac{1}{2}\pi\Omega$

(ii) $\gamma - \tau\gamma > 0, \gamma > 0$

(iii) $0 < \tau < 1$ and $R(l+1) > 0$

5. Integral with Legender Function

Legender Function

The Legendre functions are solution of Legendre's differential equation ([2], sec 3.1, vol. 1)

$$(1-z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + [\nu(\nu+1) - \mu^2(1-z^2)^{-1}]w = 0 \tag{5.1}$$

where z, ν, μ unrestricted. Under the substitution $w = (z^2-1)^{\frac{1}{2\mu}\nu}$, (5.1) becomes

$$(1-z^2)\frac{d^2\nu}{dz^2} - 2(\mu+1)z\frac{d\nu}{dz} + (\mu-\nu)(\nu+\mu+1)\nu = 0 \tag{5.2}$$

and with $\xi = \frac{1}{2} - \frac{1}{2}z$ as the independent variables this differential equation becomes

$$\xi(1-\xi)\frac{d^2\nu}{dz^2} - (\mu+1)(1-2\xi)\frac{d\nu}{d\xi} + (\nu-\mu)(\nu+\mu+1)\nu = 0. \tag{5.3}$$

This is the Gauss hypergeometric type equation with $a = \mu - \nu, b = \nu + \mu + 1$ and $c = \mu + 1$. Hence, it follows that the function

$$w = P_\nu^\mu(z)\frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\frac{1}{2\mu}} F\left[-\nu, \nu+1 : 1-\mu : \frac{1}{2} - \frac{1}{2}z\right] \cdot |1-z| < 2$$

is a solution of (5.1). The function $P_\nu^\mu(z)$ is known as the Legendre function of first kind ([16], vol. 1). It is one valued and regular in z-plane supposed cut along the real axis form 1 to $-\infty$.

$$I_{12} = \int_0^1 x^{\sigma-1}(1-x^2)^{\delta/2} P_\nu^\delta(x)_p\Psi_q(zx^\gamma)dx$$

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(\alpha_i+\beta_i k)}{\prod_{j=1}^{q}\Gamma(\alpha_j+\beta_j k)} \frac{z^k}{k!} \int_0^1 x^{\sigma-1+\gamma k}(1-x^2)^{\delta/2} P_\nu^\delta(x)dx. \tag{5.4}$$

Now using the formula ([2], sec. 3.12, vol. 1)

$$\int_0^1 x^{\sigma-1}(1-x)^{\delta/2} P_\nu^\delta(x)dx$$

$$= \frac{(-1)^\delta \pi^{1/2} 2^{-\sigma-\delta}\Gamma(\sigma)\Gamma(1+\delta+\nu)}{\Gamma(\frac{1}{2}+\frac{\sigma}{2}+\frac{\delta}{2}-\frac{\nu}{2})\Gamma(1+\frac{\sigma}{2}+\frac{\delta}{2}+\frac{\nu}{2})\Gamma(1-\delta+\nu)}$$

Provided $\text{Re}(\sigma) > 0,\ \delta = 1, 2, 3, \ldots$.
Then, (5.4) becomes

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(\alpha_i+\beta_i k)}{\prod_{j=1}^{q}\Gamma(\alpha_j+\beta_j k)} \frac{z^k}{k!} \frac{(-1)^\delta \pi^{1/2} 2^{-\sigma-\gamma k-\delta}\Gamma(\sigma+\gamma k)\Gamma(1+\delta+\nu)}{\Gamma(\frac{1}{2}+\frac{\sigma+\gamma k}{2}+\frac{\delta}{2}-\frac{\nu}{2})\Gamma(1+\frac{\sigma+\gamma k}{2}+\frac{\delta}{2}+\frac{\nu}{2})\Gamma(1-\delta+\nu)}$$

$$= \frac{(-1)^{\delta}\pi^{1/2}2^{-\sigma-\delta}\Gamma(1+\delta+\nu)}{\Gamma(1-\delta+\nu)} \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(\alpha_i+\beta_i k)}{\prod_{j=1}^{q}\Gamma(\alpha_j+\beta_j k)}$$

$$\times \frac{\Gamma(\sigma+\gamma k)}{\Gamma(\frac{1}{2}+\frac{\sigma+\gamma k}{2}+\frac{\delta}{2}-\frac{\nu}{2})\Gamma(1+\frac{\sigma+\gamma k}{2}+\frac{\delta}{2}+\frac{\nu}{2})} \frac{z^k 2^{-\gamma k}}{k!}$$

$$= \frac{(-1)^{\delta}(\pi)^{1/2}2^{-\sigma-\delta}\Gamma(1+\delta+\nu)}{\Gamma(1-\delta+\nu)}$$

$$\times {}_{p+1}\Psi_{q+1}\left[2^{-\gamma}z \left| \begin{matrix} (\alpha_i,\beta_i)_{1,p}, & (\sigma,\gamma) \\ (\alpha_j,\beta_j)_{1,q}, & \left(\frac{1}{2}+\frac{\sigma}{2}+\frac{\delta}{2}-\frac{\nu}{2},\frac{\gamma}{2}\right),\left(1+\frac{\sigma}{2}+\frac{\delta}{2}+\frac{\nu}{2},\frac{\gamma}{2}\right) \end{matrix}\right.\right]. \tag{5.5}$$

Provided : $|\arg z| < \frac{1}{2}\pi\Omega,\ \sigma > 0$ and δ is non negative integer.

$$I_{13} = \int_0^1 x^{\sigma-1}(1-x^2)^{\frac{-\delta}{2}} P_{\nu}^{\delta}(x)_p\Psi_q(zx^{\gamma})dx$$

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(\alpha_i+\beta_i k)}{\prod_{j=1}^{q}\Gamma(\alpha_j+\beta_j k)} \frac{z^k}{k!} \int_0^1 x^{\sigma-1+\gamma k}(1-x^2)^{-\delta/2}P_{\nu}^{\delta}(x)dx. \tag{5.6}$$

Now using formula ([2], see 3.12, vol. 1)

$$\int_0^1 x^{\sigma-1}(1-x^2)^{-\delta/2}P_{\nu}^{\delta}(x)dx = \frac{\pi^{1/2}2^{-\sigma-\delta}\Gamma(\sigma)}{\Gamma(\frac{1}{2}+\frac{\sigma}{2}-\frac{\delta}{2}-\frac{\nu}{2})\Gamma(1+\frac{\sigma}{2}-\frac{\delta}{2}-\frac{\nu}{2})}.$$

Equation (5.6) becomes

$$= 2^{-\sigma+\delta}(\pi)^{1/2} \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(\alpha_i+\beta_i k)}{\prod_{j=1}^{q}\Gamma(\alpha_j+\beta_j k)} \frac{\Gamma(\sigma+\gamma k)}{\Gamma(\frac{1}{2}+\frac{\sigma+\gamma k}{2}-\frac{\delta}{2}-\frac{\nu}{2})\Gamma(1+\frac{\sigma+\gamma k}{2}-\frac{\delta}{2}-\frac{\nu}{2})} \frac{(2^{-\gamma}z)^k}{k!}$$

$$= (\pi)^{1/2} 2^{\delta-\sigma} {}_{p+1}\Psi_{q+2} \left[\frac{z}{2^{\gamma}} \middle| \begin{matrix} (\alpha_i, \beta_i)_{1,p}, & (\sigma, \gamma) \\ (\alpha_j, \beta_j)_{1,q}, & \left(\frac{1}{2} + \frac{\sigma}{2} - \frac{\delta}{2} - \frac{\nu}{2}, \frac{\gamma}{2}\right), \left(1 + \frac{\sigma}{2} - \frac{\delta}{2} - \frac{\nu}{2}, \frac{\gamma}{2}\right) \end{matrix} \right]. \tag{5.7}$$

Provided : $|\arg z| < \frac{1}{2}\pi\Omega,\ Re(\sigma) > 0$ and $Re(\delta) > 0$.

6. Integral involving Hypergeometric function

Hypergeometric Function

In the study of second-order linear differential equations with three regular singular points, there arise the function

$$F(a, b, c; z) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}. \tag{6.1}$$

For c neither zero nor a negative integer in (6.1) the notation

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)...(\alpha+n-1),\ n \geq 1.$$

$$(\alpha)_0 = 1,\ \alpha \neq 0$$

is called the factorial function and the function in (6.1) is called the Hypergeometric function ([14], p.45).

$$I_{14} = \int_1^{\infty} x^{-\rho}(x-1)^{\rho-1} {}_2F_1 \left[\begin{matrix} \nu+\sigma-\rho, \lambda+\sigma-\rho \\ \sigma \end{matrix} ; (1-x) \right] {}_p\Psi_q(zx) dx.$$

$$= \int_1^{\infty} x^{-\rho}(x-1)^{\rho-1} {}_2F_1 \left[\begin{matrix} \nu+\sigma-\rho, \lambda+\sigma-\rho \\ \sigma \end{matrix} ; (1-x) \right]$$

$$\times \left\{ \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + \beta_i k)}{\prod_{j=1}^{q} \Gamma(\alpha_j + \beta_j k)} \frac{(zx)^k}{k!} \right\} dx$$

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + \beta_i k)}{\prod_{j=1}^{q} \Gamma(\alpha_j + \beta_j k)} \frac{z^k}{k!} \int_1^{\infty} x^{-\rho+k}(x-1)^{\sigma-1} {}_2F_1 \left[\begin{matrix} \nu+\sigma-\rho, \lambda+\sigma-\rho \\ \sigma \end{matrix} ; (1-x) \right] dx.$$

Putting $x = t + 1$ and $dx = dt$

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + \beta_i k)}{\prod_{j=1}^{q} \Gamma(\alpha_j + \beta_j k)} \frac{z^k}{k!} \int_0^\infty (t+1)^{k-\rho} t^{\sigma-1} {}_2F_1 \left[\begin{matrix} \nu+\sigma-\rho, \lambda+\sigma-\rho \\ \sigma \end{matrix} ; -t \right] dt$$

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + \beta_i k)}{\prod_{j=1}^{q} \Gamma(\alpha_j + \beta_j k)} \frac{z^k}{k!} \left[\sum_{s=0}^{\infty} \frac{(\nu+\sigma-\rho)_s (\lambda+\sigma-\rho)_s (-1)^s}{(\sigma)_s s!} \right] \int_0^\infty t^{s+\sigma-1} (t+1)^{k-\rho} dt$$

$$= \frac{1}{\Gamma(\sigma)} {}_3F_1 \left[\begin{matrix} v+\sigma-\rho, \; \lambda+\sigma-\rho \\ \sigma \end{matrix} ; -1 \right] {}_{p+1}\Psi_{q+1} \left[z \left| \begin{matrix} (\alpha_i, \beta_i)_{1,p}, & (\rho-\sigma-s, -1) \\ (\alpha_j, \beta_j)_{1,q}, & (\rho, -1) \end{matrix} \right. \right]. \tag{6.2}$$

7. Conclusion

Only an elementary exposition of the subject is given here with the sole aim of introducing the researchers to a topic whose important is fast growing in science and engineering.

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