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A STUDY ON GROWTH PROPERTIES OF GENERALISED ITERATED INTEGRAL FUNCTIONS

Dibyendu Banerjee and Sumanta Ghosh*

Department of Mathematics,
Visva-Bharati, Santiniketan - 731235, West Bengal, INDIA

E-mail : dibyendu192@rediffmail.com

*Ranaghat P.C. High School,
Ranaghat - 741201, Nadia, West Bengal, INDIA

E-mail : sumantarpc@gmail.com

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Abstract: In the present paper we investigate some growth properties of generalised iterated integral functions.

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1. Introduction and Definitions

Let $f(z)$ and $g(z)$ be two integral functions. In [3], $T(r,f)$, $M(r,f)$, $N(r,a,f)$, $\delta(a,f)$, $\delta(a(z).f)$, $log^{+}x$ have their usual meanings in the Nevanlinna theory of meromorphic functions.

After that in [2], Clunie studied some comparative growths of $T(r,fg)$ with $T(r,f)$ and $T(r,g)$ and showed that

$$\lim_{r\to\infty}\frac{T(r,fg)}{T(r,f)}=\infty \text{ and } \lim_{r\to\infty}\frac{T(r,fg)}{T(r,g)}=\infty,$$

where $f(z)$ and $g(z)$ are transcendental integral functions. In [7], Singh proved some comparative growths of $\log T(r,fg)$ and $T(r,f)$. In [4] Lahiri proved some theorems on the comparative growth of $\log T(r, fg)$ with $T(r, f)$ and, as well as, with $T(r, g)$.

Also the order and the lower order of the integral function $f(z)$ are respectively denoted by ρ_f and λ_f and are defined by

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r}$$

and

$$\lambda_f = \liminf_{r\to\infty} \frac{\log\log M(r, f)}{\log r}.$$

Definition 1.1. *The number* $\overline{\lambda}_f$ *is said to be the hyper lower order of* $f(z)$ *if and only if*

$$\overline{\lambda}_f = \liminf_{r\to\infty} \frac{\log\log\log M(r, f)}{\log r}.$$

It is clear that $\overline{\lambda}_f \le \lambda_f$.

Definition 1.2. *A function* $\rho_f(r)$ *is called a proximate order of* $f(z)$ *relative to* $T(r, f)$ *if and only if*

(*i*) $\rho_f(r)$ *is real, continuous and piecewise differentiable for* $r > r_0$

(*ii*) $\lim_{r\to\infty} \rho_f(r) = \rho_f$,

(*iii*) $\lim_{r\to\infty} r \log r \rho'_f(r) = 0$ *and*

(*iv*) $\limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f(r)}} = 1$.

Note 1.3. *For* $\delta > 0$ *the function* $r^{\rho_f+\delta-\rho_f(r)}$ *is ultimately an increasing function of* r.

Since $\frac{d}{dr} r^{\rho_f+\delta-\rho_f(r)} = \left\{\rho_f + \delta - \rho_f(r) - r\log r \rho'_f(r)\right\} r^{\rho_f+\delta-1-\rho_f(r)} > 0$ *for sufficiently large values of* r.

In [8], Zhou proved the following theorem.

Theorem 1.4. *Let* $f(z)$ *and* $g(z)$ *be two integral functions of finite orders such that* $g(0) = 0$ *and* $\rho_g < \lambda_f \le \rho_f$. *Then*

$$\lim_{r\to\infty} \frac{\log T(r, fg)}{T(r, f)} = 0.$$

In [4], Lahiri proved the following theorem.

Theorem 1.5. *Let* $f(z)$ *and* $g(z)$ *be two non-constant integral functions such that*

$\lambda_g < \lambda_f \leq \rho_f < \infty$. *Then*

$$\liminf_{r\to\infty} \frac{\log T(r, fg)}{T(r, f)} = 0.$$

If $\rho_g > \rho_f$, Singh [7] proved the following theorem.

Theorem 1.6. *Let $f(z)$ and $g(z)$ be two integral functions of finite orders with $\rho_g > \rho_f$. Then*

$$\limsup_{r\to\infty} \frac{\log T(r, fg)}{T(r, f)} = \infty.$$

In [4], Lahiri proved the following four theorems.

Theorem 1.7. *Let $f(z)$ and $g(z)$ be two integral functions such that $0 < \lambda_f < \lambda_g < \infty$. Then*

$$\limsup_{r\to\infty} \frac{\log T(r, fg)}{T(r, f)} = \infty.$$

Theorem 1.8. *Let $f(z)$ and $g(z)$ be two non-constant integral functions such that ρ_f and ρ_g are finite. Then*

$$\liminf_{r\to\infty} \frac{\log T(r, fg)}{T(r, g)} \leq 3.\rho_f.2^{\rho_g}.$$

Theorem 1.9. *Let f and g be non-constant integral functions such that ρ_f and λ_g are finite. Also suppose that there exist integral functions $a_i(z)$ $(i = 1, 2, ..., n; n \leq \infty)$ such that (i) $T(r, a_i(z)) = o\{T(r, g)\}$ as $r \to \infty$ for $i = 1, 2, ..., n$ and (ii) $\sum_{i=1}^{n} \delta(a_i(z), g) = 1$. Then*

$$\limsup_{r\to\infty} \frac{\log T(r, fg)}{T(r, g)} < \pi.\rho_{f_k}.$$

Theorem 1.10. *Let f and g be transcendental integral functions such that*

(i) $\rho_g < \infty$ and the hyper lower order of $g(z)$, $\overline{\lambda}_g$ is positive
(ii) $\lambda_f > 0$, and
(iii) $\delta(0, f) < 1$.
Then

$$\limsup_{r\to\infty} \frac{\log T(r, fg)}{T(r, g)} = \infty.$$

A real valued function $\phi(r)$ is said to have the property P [1] if

(i) $\phi(r)$ is non-negative and continuous for $r \geq r_0$, say;

(ii) $\phi(r)$ is strictly increasing and $\phi(r) \to \infty$ as $r \to \infty$;

and

(iii) $\log \phi(r) < \delta \phi(r/2)$, for all $\delta > 0$ and for all sufficiently large values of r.

Therefore a function satisfying the property P also satisfies the following relation :

$\log^{[p]} \phi(r) < \delta \log^{[q]} \phi(r/2)$ for all $\delta > 0$, $p > q$ and for all sufficiently large values of r.

For two non-constant integral functions $f(z)$ and $g(z)$, the inequality

$$\log M(r, f(g)) \leq \log M(M(r, g), f) \text{ is obvious.}$$

In the present paper we consider k non-constant integral functions $f_1, f_2, ..., f_k$ and a constant α with $0 < \alpha \leq 1$ and form the iteration as below:

$$\begin{aligned}
F_1^1(z) &= (1-\alpha) z + \alpha f_1(z) \\
F_2^1(z) &= (1-\alpha) F_1^2(z) + \alpha f_1\left(F_1^2(z)\right) \\
F_3^1(z) &= (1-\alpha) F_2^2(z) + \alpha f_1\left(F_2^2(z)\right) \\
&\vdots \\
F_n^1(z) &= (1-\alpha) F_{n-1}^2(z) + \alpha f_1\left(F_{n-1}^2(z)\right).
\end{aligned}$$

Similarly

$$\begin{aligned}
F_1^2(z) &= (1-\alpha) z + \alpha f_2(z) \\
F_2^2(z) &= (1-\alpha) F_1^3(z) + \alpha f_2\left(F_1^3(z)\right) \\
F_3^2(z) &= (1-\alpha) F_2^3(z) + \alpha f_2\left(F_2^3(z)\right) \\
&\vdots \\
F_n^2(z) &= (1-\alpha) F_{n-1}^3(z) + \alpha f_2\left(F_{n-1}^3(z)\right)
\end{aligned}$$

and

$$\begin{aligned}
F_1^k(z) &= (1-\alpha) z + \alpha f_k(z) \\
F_2^k(z) &= (1-\alpha) F_1^1(z) + \alpha f_k\left(F_1^1(z)\right) \\
F_3^k(z) &= (1-\alpha) F_2^1(z) + \alpha f_k\left(F_2^1(z)\right) \\
&\vdots \\
F_n^k(z) &= (1-\alpha) F_{n-1}^1(z) + \alpha f_k\left(F_{n-1}^1(z)\right).
\end{aligned}$$

Clearly all $F_n^1, F_n^2, ..., F_n^k$ are integral functions. Throughout this paper we assume that maximum modulus functions of $f_1, f_2, ..., f_k$ and all their generalised iterated functions satisfy the property P.

The main purpose of this paper is to extend the results of Lahiri [4] for generalised iterated integral functions.

2. Lemmas

The following lemmas will be needed to prove our results.

Lemma 2.1. [3] *Let $f(z)$ be an integral function. For $0 \le r < R < \infty$, we have*

$$T(r, f) \le \log^+ M(r, f) \le \frac{R+r}{R-r} T(R, f).$$

Putting $R = 2r$, for large values of r, we have

$$T(r, f) \le \log M(r, f) \le 3T(2r, f).$$

Lemma 2.2. [5] *Let f be an integral function of finite lower order. If there exist integral functions a_i $(i = 1, 2, 3..........n;\ n \le \infty)$ satisfying $T(r, a_i) = o\{T(r, f)\}$ and*

$$\sum_{i=1}^{n} \delta(a_i, f) = 1 \quad \text{then} \quad \lim_{r\to\infty} \frac{T(r, f)}{\log M(r, f)} = \frac{1}{\pi}.$$

Lemma 2.3. [2] *Let $f(z)$ and $g(z)$ be two integral functions with $g(0) = 0$. Let β satisfies $0 < \beta < 1$ and $C(\beta) = \frac{(1-\beta)^2}{4\beta}$. Then for $r > 0$*

$$M(r, f \circ g) \ge M(C(\beta) M(\beta r, g), f).$$

Further if $g(z)$ is any integral function, then with $\beta = \frac{1}{2}$, for sufficiently large values of r

$$M(r, f \circ g) \ge M\left(\frac{1}{8} M\left(\frac{r}{2}, g\right) - |g(0)|, f\right).$$

Clearly

$$M(r, f \circ g) \ge M\left(\frac{1}{16} M\left(\frac{r}{2}, g\right), f\right). \tag{2.1}$$

On the other hand the opposite inequality

$$M(r, f \circ g) \le M(M(r, g), f) \tag{2.2}$$

is obvious.

Lemma 2.4. [6] *Let $f(z)$ be transcendental integral function, $g(z)$ a transcendental integral function of finite order, η a constant satisfying $0 < \eta < 1$, and α is a positive number. Then we have*

$$T(r,fg)+O(1)\geq N(r,0,fg)\geq \log\frac{1}{\eta}\left[\frac{N\left\{M\left((\eta r)^{\frac{1}{1+\alpha}},g\right),0,f\right\}}{\log M\left((\eta r)^{\frac{1}{1+\alpha}},g\right)-O(1)}\right]-O(1)$$

as $r\to\infty$ through all values.

3. Main Results

As an extension of Theorem 1.5 we have the following theorem.

Theorem 3.1. *Let $f_1, f_2, ..., f_k$ be non-constant integral functions such that $\lambda_{f_k} < \lambda_{f_1} \leq \rho_{f_1} < \infty$. Then for $n = km, m \in \mathbb{N}$*

$$\liminf_{r\to\infty}\frac{\log^{[n-1]}T(r,F_n^1)}{T(r,f_1)}=0.$$

Proof. Since $\lambda_{f_k} < \lambda_{f_1}$, we can choose $\varepsilon\,(>0)$ such that $\lambda_{f_k}+\varepsilon < \lambda_{f_1}-\varepsilon$. Also for all large values of r, $r^{\lambda_{f_1}-\frac{\varepsilon}{2}} < T(r,f_1)$ and for a sequence of values of r tending to infinity $\log M(r,f_k) < r^{\lambda_{f_k}+\varepsilon}$.

Then we have

$$\begin{aligned}T\left(r,F_n^1\right) &\leq \log M\left(r,F_n^1\right)\\ &= \log M\left\{r,(1-\alpha)F_{n-1}^2+\alpha f_1\left(F_{n-1}^2\right)\right\}\\ &\leq \log M\left(r,F_{n-1}^2\right)+\log M\left(r,f_1\left(F_{n-1}^2\right)\right)+O(1)\\ &\leq \log M\left(r,F_{n-1}^2\right)+\log M\left(M\left(r,F_{n-1}^2\right),f_1\right)+O(1)\\ &\leq \log M\left(r,F_{n-1}^2\right)+\left\{M\left(r,F_{n-1}^2\right)\right\}^{\rho_{f_1}+\varepsilon}+O(1)\\ &\leq M\left(r,F_{n-1}^2\right)+\left\{M\left(r,F_{n-1}^2\right)\right\}^{\rho_{f_1}+\varepsilon}+O(1). \qquad (3.1)\end{aligned}$$

Therefore,

$$\begin{aligned}\log T\left(r,F_n^1\right) &\leq \log M\left(r,F_{n-1}^2\right)+(\rho_{f_1}+\varepsilon)\log M\left(r,F_{n-1}^2\right)+O(1)\\ &\leq (1+\rho_{f_1}+\varepsilon)\log M\left(r,F_{n-1}^2\right)+O(1)\\ &\leq (1+\rho_{f_1}+\varepsilon)\left[M\left(r,F_{n-2}^3\right)+\left\{M\left(r,F_{n-2}^3\right)\right\}^{\rho_{f_2}+\varepsilon}\right]+O(1)\,.\text{using } (3.1)\end{aligned}$$

So,

$$\begin{aligned}\log^{[2]}T\left(r,F_n^1\right) &\leq \log M\left(r,F_{n-2}^3\right)+(\rho_{f_2}+\varepsilon)\log M\left(r,F_{n-2}^3\right)+O(1)\\ &\leq (1+\rho_{f_2}+\varepsilon)\log M\left(r,F_{n-2}^3\right)+O(1).\end{aligned}$$

Therefore,

$$\log^{[km-1]} T\left(r, F_n^1\right) \leq \left(1+\rho_{f_{k-1}}+\varepsilon\right) \log M\left(r, F_{n-(km-1)}^k\right) + O\left(1\right).$$

So,

$$\begin{aligned}
&\log^{[n-1]} T\left(r, F_n^1\right) \\
&\quad \leq \left(1+\rho_{f_{k-1}}+\varepsilon\right) \log M\left(r, F_1^k\right) + O\left(1\right) \\
&\quad = \left(1+\rho_{f_{k-1}}+\varepsilon\right) \log M\left(r, \left(1-\alpha\right) z + \alpha f_k\right) + O\left(1\right) \\
&\quad \leq \left(1+\rho_{f_{k-1}}+\varepsilon\right) \left\{\log M\left(r, z\right) + \log M(r, f_k)\right\} + O\left(1\right) \qquad (3.2) \\
&\quad = \left(1+\rho_{f_{k-1}}+\varepsilon\right) \left\{\log r + \log M(r, f_k)\right\} + O\left(1\right) \qquad (3.3) \\
&\quad \leq \left(1+\rho_{f_{k-1}}+\varepsilon\right) \left\{\log r + r^{\lambda_{f_k}+\varepsilon}\right\} + O\left(1\right) \text{ for a sequence}
\end{aligned}$$

of values of $r = r_s \to \infty$.

So for a sequence of values of $r = r_s$ tending to infinity we obtain

$$\frac{\log^{[n-1]} T\left(r, F_n^1\right)}{T\left(r, f_1\right)} < \frac{\left(1+\rho_{f_{k-1}}+\varepsilon\right) \left\{\log r + r^{\lambda_{f_k}+\varepsilon}\right\} + O\left(1\right)}{r^{\lambda_{f_1}-\frac{\varepsilon}{2}}}$$

and hence

$$\liminf_{r\to\infty} \frac{\log^{[n-1]} T\left(r, F_n^1\right)}{T\left(r, f_1\right)} = 0.$$

This proves the theorem.

Note 3.2. *The conditions of Theorem 3.1 are not strictly sharp, which follows from the following example.*

Example 3.3. Let $f_1 = f_2 = ... = f_k = z$. Then $F_n^1\left(z\right) = z$ for every n. In this case $\lambda_{f_1} = \lambda_{f_2} = ... = \lambda_{f_k} = 0$. But

$$\begin{aligned}
\liminf_{r\to\infty} \frac{\log^{[n-1]} T\left(r, F_n^1\right)}{T\left(r, f_1\right)} &= \liminf_{r\to\infty} \frac{\log^{[n-1]} T\left(r, z\right)}{T\left(r, z\right)} \\
&= \liminf_{r\to\infty} \frac{\log^{[n-1]} \log r}{\log r} \\
&= \liminf_{r\to\infty} \frac{\log^{[n]} r}{\log r} \\
&= 0.
\end{aligned}$$

The following four theorems are extensions of Theorem 1.7, Theorem 1.8, Theorem 1.9 and Theorem 1.10 of Lahiri [4].

Theorem 3.4. *Let $f_1, f_2, ..., f_k$ be non-constant integral functions such that $0 < \lambda_{f_1} < \lambda_{f_k} < \infty$. Then for $n = km, m \in \mathbb{N}$*

$$\limsup_{r\to\infty} \frac{\log^{[n-1]} T\left(r, F_n^1\right)}{T\left(r, f_1\right)} = \infty.$$

Proof. Since $\lambda_{f_1} < \lambda_{f_k}$, we can choose $\varepsilon\,(> 0)$ such that $\lambda_{f_1} + \varepsilon < \lambda_{f_k} - \varepsilon$. Also for all large values of r, $r^{\lambda_{f_k} - \frac{\varepsilon}{2}} < T\left(r, f_k\right)$.

Using Lemma 2.1, we have

$$\begin{aligned}
&T\left(r, F_n^1\right)\\
&\geq \frac{1}{3}\log M\left(\frac{r}{2}, F_n^1\right)\\
&= \frac{1}{3}\log M\left(\frac{r}{2}, (1-\alpha)\, F_{n-1}^2 + \alpha f_1\left(F_{n-1}^2\right)\right)\\
&\geq \frac{1}{3}\log\left\{\alpha M\left(\frac{r}{2}, f_1\left(F_{n-1}^2\right)\right) - (1-\alpha)\, M\left(\frac{r}{2}, F_{n-1}^2\right)\right\}\\
&\geq \frac{1}{3}\log\left\{\alpha M\left(\frac{1}{16}M\left(\frac{r}{2^2}, F_{n-1}^2\right), f_1\right) - (1-\alpha)\, M\left(\frac{r}{2}, F_{n-1}^2\right)\right\}, \text{using (2.1)}\\
&\geq \frac{1}{3}\left\{\log M\left(\frac{1}{16}M\left(\frac{r}{2^2}, F_{n-1}^2\right), f_1\right) - \log M\left(\frac{r}{2}, F_{n-1}^2\right)\right\} + O(1).
\end{aligned}$$

So,

$$\begin{aligned}
&\log T\left(r, F_n^1\right)\\
&\geq \log\log M\left(\frac{1}{16}M\left(\frac{r}{2^2}, F_{n-1}^2\right), f_1\right) - \log\log M\left(\frac{r}{2}, F_{n-1}^2\right) + O(1)\\
&\geq \left(\lambda_{f_1} - \varepsilon\right)\log\left(\frac{1}{16}M\left(\frac{r}{2^2}, F_{n-1}^2\right)\right) - \log\log M\left(\frac{r}{2}, F_{n-1}^2\right) + O(1)\\
&> \left(\lambda_{f_1} - \varepsilon\right)\log M\left(\frac{r}{2^2}, F_{n-1}^2\right) - \frac{1}{2}\left(\lambda_{f_1} - \varepsilon\right)\log M\left(\frac{r}{2^2}, F_{n-1}^2\right) + O(1),\\
&\qquad\text{by property P}\\
&= \frac{1}{2}\left(\lambda_{f_1} - \varepsilon\right)\log M\left(\frac{r}{2^2}, F_{n-1}^2\right) + O(1)\\
&\geq \frac{1}{2}\left(\lambda_{f_1} - \varepsilon\right) T\left(\frac{r}{2^2}, F_{n-1}^2\right) + O(1). \qquad (3.4)
\end{aligned}$$

Therefore,

$$\begin{aligned}&\log^{[2]} T\left(r, F_n^1\right)\\ &\quad > \log T\left(\frac{r}{2^2}, F_{n-1}^2\right)+O(1)\\ &\quad > \frac{1}{2}\left(\lambda_{f_2}-\varepsilon\right) T\left(\frac{r}{2^4}, F_{n-2}^3\right)+O(1), \text{ using (3.4)}\end{aligned}$$

and

$$\begin{aligned}&\log^{[3]} T\left(r, F_n^1\right)\\ &\quad > \log T\left(\frac{r}{2^4}, F_{n-2}^3\right)+O(1)\\ &\quad > \frac{1}{2}\left(\lambda_{f_3}-\varepsilon\right) T\left(\frac{r}{2^6}, F_{n-3}^4\right)+O(1).\end{aligned}$$

So,

$$\begin{aligned}&\log^{[km-1]} T\left(r, F_n^1\right)\\ &\quad > \frac{1}{2}\left(\lambda_{f_{k-1}}-\varepsilon\right) T\left(\frac{r}{2^{2(km-1)}}, F_{n-(km-1)}^k\right)+O(1).\end{aligned}$$

Therefore,

$$\begin{aligned}&\log^{[n-1]} T\left(r, F_n^1\right)\\ &\quad \geq \frac{1}{2}\left(\lambda_{f_{k-1}}-\varepsilon\right) T\left(\frac{r}{2^{2(n-1)}}, F_1^k\right)+O(1)\\ &\quad = \frac{1}{2}\left(\lambda_{f_{k-1}}-\varepsilon\right) T\left(\frac{r}{2^{2(n-1)}}, (1-\alpha) z+\alpha f_k\right)+O(1)\\ &\quad \geq \frac{1}{2}\left(\lambda_{f_{k-1}}-\varepsilon\right)\left\{T\left(\frac{r}{2^{2(n-1)}}, f_k\right)-T\left(\frac{r}{2^{2(n-1)}}, z\right)\right\}+O(1)\\ &\quad \geq \frac{1}{2}\left(\lambda_{f_{k-1}}-\varepsilon\right)\left\{\left(\frac{r}{2^{2(n-1)}}\right)^{\lambda_{f_k}-\frac{\varepsilon}{2}}-\log\left(\frac{r}{2^{2(n-1)}}\right)\right\}+O(1).\end{aligned}$$

Also for a sequence of values of r tending to infinity $T(r, f_1) < r^{\lambda_{f_1}+\varepsilon}$.

So,

$$\frac{\log^{[n-1]} T\left(r, F_n^1\right)}{T\left(r, f_1\right)} > \frac{\frac{1}{2}\left(\lambda_{f_{k-1}}-\varepsilon\right)\left\{\left(\frac{r}{2^{2(n-1)}}\right)^{\lambda_{f_k}-\frac{\varepsilon}{2}}-\log\left(\frac{r}{2^{2(n-1)}}\right)\right\}+O(1)}{r^{\lambda_{f_1}+\varepsilon}}.$$

Therefore,

$$\limsup_{r\to\infty}\frac{\log^{[n-1]} T\left(r, F_n^1\right)}{T\left(r, f_1\right)}=\infty.$$

Hence the proof.

Theorem 3.5. *Let $f_1, f_2, ..., f_k$ be non-constant integral functions such that ρ_{f_1}, $\rho_{f_2}, ..., \rho_{fk}$ are finite. Then for $n = km$, $m \in \mathbb{N}$*

$$\liminf_{r\to\infty} \frac{\log^{[n-1]} T(r, F_n^1)}{T(r, f_k)} \leq 3\left(1 + \rho_{f_{k-1}}\right) 2^{\rho_{f_k}}.$$

Proof. Let $\varepsilon\,(0 < \varepsilon < 1)$ be arbitrary. Also for all large values of r, $r^{\lambda_{f_k} - \frac{\varepsilon}{2}} < T(r, f_k)$.

From (3.3) we have

$$\log^{[n-1]} T\left(r, F_n^1\right) \leq \left(1 + \rho_{f_{k-1}} + \varepsilon\right)\{\log r + \log M(r, f_k)\} + O(1).$$

Hence

$$\liminf_{r\to\infty} \frac{\log^{[n-1]} T(r, F_n^1)}{T(r, f_k)} \leq \liminf_{r\to\infty} \frac{\left(1 + \rho_{f_{k-1}} + \varepsilon\right)\{\log r + \log M(r, f_k)\} + O(1)}{T(r, f_k)}$$

Therefore

$$\liminf_{r\to\infty} \frac{\log^{[n-1]} T\left(r, F_n^1\right)}{T(r, f_k)}$$

$$\leq \limsup_{r\to\infty} \frac{\left(1 + \rho_{f_{k-1}} + \varepsilon\right)\log r + O(1)}{r^{\lambda_{f_k} - \frac{\varepsilon}{2}}} + \liminf_{r\to\infty} \frac{\left(1 + \rho_{f_{k-1}} + \varepsilon\right)\log M(r, f_k)}{T(r, f_k)}$$

$$= \left(1 + \rho_{f_{k-1}}\right)\liminf_{r\to\infty} \frac{\log M(r, f_k)}{T(r, f_k)}, \text{ since } \varepsilon \text{ is arbitrary.} \tag{3.5}$$

Let $\rho_{f_k}(r)$ be a proximate order of $f_k(z)$ relative to $T(r, f_k)$. Since

$$\limsup_{r\to\infty} \frac{T(r, f_k)}{r^{\rho_{f_k}(r)}} = 1,$$

it follows that for all large values of r and given $\varepsilon\,(0 < \varepsilon < 1)$

$$T(r, f_k) < (1 + \varepsilon)\, r^{\rho_{f_k}(r)}.$$

Using Lemma 2.1 for all large values of r,

$$\log M(r, f_k) \leq 3T(2r, f_k) < 3(1 + \varepsilon)(2r)^{\rho_{f_k}(2r)}$$

and so for large values of r

$$\log M(r, f_k) < 3(1 + \varepsilon)\frac{(2r)^{\rho_{f_k} + \delta}}{(2r)^{\rho_{f_k} + \delta - \rho_{f_k}(2r)}},$$

where δ (> 0) is arbitrary.

Since $(r)^{\rho_{f_k}+\delta-\rho_{f_k}(r)}$ is an increasing function of r, it follows that for all large r

$$\log M(r, f_k) < 3(1+\varepsilon) 2^{\rho_{f_k}+\delta} r^{\rho_{f_k}(r)}. \tag{3.6}$$

Again for a sequence of values or r tending to infinity we have

$$T(r, f_k) > (1-\varepsilon) r^{\rho_{f_k}(r)}. \tag{3.7}$$

From (3.6) and (3.7) we get for a sequence of values or r tending to infinity

$$\log M(r, f_k) < 3\frac{(1+\varepsilon)}{(1-\varepsilon)} 2^{\rho_{f_k}+\delta} T(r, f_k).$$

Therefore

$$\liminf_{r\to\infty} \frac{\log M(r, f_k)}{T(r, f_k)} \leq 3\frac{(1+\varepsilon)}{(1-\varepsilon)} 2^{\rho_{f_k}+\delta}.$$

Since $\delta\,(> 0)$ and $\varepsilon\,(0 < \varepsilon < 1)$ are arbitrary, it follows that

$$\liminf_{r\to\infty} \frac{\log M(r, f_k)}{T(r, f_k)} \leq 3.2^{\rho_{f_k}}. \tag{3.8}$$

Hence from (3.5) and (3.8) we get

$$\liminf_{r\to\infty} \frac{\log^{[n-1]} T(r, F_n^1)}{T(r, f_k)} \leq 3\left(1+\rho_{f_{k-1}}\right) 2^{\rho_{f_k}}.$$

This proves the theorem.

Theorem 3.6. *Let $f_1, f_2, ..., f_k$ be non-constant integral functions such that $\rho_{f_{k-1}}$ and λ_{f_k} are finite. Also suppose that there exist integral functions $a_i(z)$ $(i = 1, 2, ..., n;\ n \leq \infty)$ such that (i) $T(r, a_i(z)) = o\{T(r, f_k)\}$ as $r \to \infty$ for $i = 1, 2, ..., n$ and (ii) $\sum_{i=1}^{n} \delta(a_i(z), f_k) = 1$. Then for $n = km, m \in \mathbb{N}$*

$$\limsup_{r\to\infty} \frac{\log^{[n-1]} T(r, F_n^1)}{T(r, f_k)} \leq \pi\left(1+\rho_{f_{k-1}}\right).$$

Proof. Let $\varepsilon\,(0 < \varepsilon < 1)$ be arbitrary. Also for all large values of r, $r^{\lambda_{f_k}-\frac{\varepsilon}{2}} < T(r, f_k)$.

From (3.3) we have

$$\log^{[n-1]} T\left(r, F_n^1\right) \leq \left(1+\rho_{f_{k-1}}+\varepsilon\right)\{\log r + \log M(r, f_k)\} + O(1).$$

Hence

$$\limsup_{r\to\infty}\frac{\log^{[n-1]}T\left(r,F_n^1\right)}{T\left(r,f_k\right)}\le\limsup_{r\to\infty}\frac{\left(1+\rho_{f_{k-1}}+\varepsilon\right)\{\log r+\log M(r,f_k)\}+O\left(1\right)}{T\left(r,f_k\right)}$$

Therefore

$\lim_{r\to\infty}\sup\frac{\log^{[n-1]}T\left(r,F_n^1\right)}{T(r,f_k)}$

$$\le\limsup_{r\to\infty}\frac{\left(1+\rho_{f_{k-1}}+\varepsilon\right)\log r+O\left(1\right)}{r^{\lambda_{f_k}-\frac{\varepsilon}{2}}}+\limsup_{r\to\infty}\frac{\left(1+\rho_{f_{k-1}}+\varepsilon\right)\log M(r,f_k)}{T\left(r,f_k\right)}$$

$$\le\left(1+\rho_{f_{k-1}}\right)\limsup_{r\to\infty}\frac{\log M(r,f_k)}{T\left(r,f_k\right)},\text{ since }\varepsilon\text{ is arbitrary.}\tag{3.9}$$

Therefore using Lemma 2.2 in (3.9) we have

$$\limsup_{r\to\infty}\frac{\log^{[n-1]}T\left(r,F_n^1\right)}{T\left(r,f_k\right)}\le\pi\left(1+\rho_{f_{k-1}}\right).$$

This proves the theorem.

Theorem 3.7. *Let $f_1,f_2,...,f_k$ be transcendental integral functions such that*
(i) $\rho_{f_k}<\infty$ and the hyper lower order of $f_k\left(z\right)$, $\overline{\lambda}_{f_k}$ is positive
(ii) $\lambda_{f_1}>0$ and
(iii) $\delta\left(0,f_1\right)<1$.
Then for $n=km,m\in\mathbb{N}$

$$\limsup_{r\to\infty}\frac{\log^{[n-1]}T\left(r,F_n^1\right)}{T\left(r,f_k\right)}=\infty.$$

Proof. Proceeding as Theorem 3.4, we have

$$\log^{[km-2]}T\left(r,F_n^1\right)>\frac{1}{2}\left(\lambda_{f_{k-2}}-\varepsilon\right)T\left(\frac{r}{2^{2(km-2)}},F_{n-(km-2)}^{k-1}\right)+O\left(1\right).$$

So, $\log^{[n-2]}T\left(r,F_n^1\right)$

$$\begin{aligned}&>\frac{1}{2}\left(\lambda_{f_{k-2}}-\varepsilon\right)T\left(\frac{r}{2^{2(n-2)}},F_2^{k-1}\right)+O\left(1\right)\\&=\frac{1}{2}\left(\lambda_{f_{k-2}}-\varepsilon\right)T\left(\frac{r}{2^{2(n-2)}},\left(1-\alpha\right)F_1^k+\alpha f_1\left(F_1^k\right)\right)+O\left(1\right)\\&\ge\frac{1}{2}\left(\lambda_{f_{k-2}}-\varepsilon\right)\left\{T\left(\frac{r}{2^{2(n-2)}},f_1\left(F_1^k\right)\right)-T\left(\frac{r}{2^{2(n-2)}},F_1^k\right)\right\}+O\left(1\right).\end{aligned}$$

Using Lemma 2.4 we have

$$\log^{[n-2]} T\left(r, F_n^1\right)$$

$$\geq \frac{1}{2}\left(\lambda_{f_{k-2}} - \varepsilon\right)[\log \frac{1}{\eta} \times \frac{N\left\{M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right), 0, f_1\right\}}{\log M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right) - O(1)} - O(1) - T\left(\frac{r}{2^{2(n-2)}}, F_1^k\right)] + O(1)$$

$$\geq \frac{1}{2}\left(\lambda_{f_{k-2}} - \varepsilon\right)[\log \frac{1}{\eta} \times \frac{N\left\{M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right), 0, f_1\right\}}{\log M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right) - O(1)} - O(1) - T\left(\frac{r}{2^{2(n-2)}}, z\right) - T\left(\frac{r}{2^{2(n-2)}}, f_k\right)] + O(1).$$

Since $\delta(0, f_1) < 1$, for given $\varepsilon > 0$ there exists a sequence of values of r tending to infinity for which $\frac{N(r,0,f_1)}{T(r,f_1)} > 1 - \delta(0, f_1) - \varepsilon > 0$.

So, $\log^{[n-2]} T\left(r, F_n^1\right)$

$$\geq \frac{1}{2}\left(\lambda_{f_{k-2}} - \varepsilon\right)[\log \frac{1}{\eta}$$

$$\times \frac{(1-\delta(0,f_1)-\varepsilon)\, T\left\{M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right), f_1\right\} - \log M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right) O(1)}{\log M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right) - O(1)}$$

$$+ \frac{O(1)}{\log M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right) - O(1)} - \log \frac{r}{2^{2(n-2)}} - T\left(\frac{r}{2^{2(n-2)}}, f_k\right)] + O(1)$$

$$= \frac{1}{2}\left(\lambda_{f_{k-2}} - \varepsilon\right)[\log \frac{1}{\eta}$$

$$\times \frac{(1-\delta(0,f_1)-\varepsilon)\, T\left\{M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right), f_1\right\} - \log M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right) O(1)}{\log M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right) - O(1)}$$

$$+ o(1) - \log \frac{r}{2^{2(n-2)}} - T\left(\frac{r}{2^{2(n-2)}}, f_k\right)] + O(1)$$

$$= \frac{1}{2}\left(\lambda_{f_{k-2}} - \varepsilon\right)[\log \frac{1}{\eta}$$

$$\times \frac{(1-\delta(0,f_1)-\varepsilon)\,T\left\{M\left(\left(\eta\frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}},F_1^k\right),f_1\right\}-\log M\left(\left(\eta\frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}},F_1^k\right)O(1)}{\log M\left(\left(\eta\frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}},F_1^k\right)-O(1)}$$

$$-\log\frac{r}{2^{2(n-2)}}-T\left(\frac{r}{2^{2(n-2)}},f_k\right)]+O(1). \tag{3.10}$$

Again

$$\begin{aligned}\log M\left(\left(\eta\frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}},F_1^k\right) &= \log M\left(\left(\eta\frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}},(1-\alpha)z+f_k\right)\\ &\le \log M\left(\left(\eta\frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}},z\right)+\log M\left(\left(\eta\frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}},f_k\right)+O(1)\\ &\le \log\left(\eta\frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}+\left(\eta\frac{r}{2^{2(n-2)}}\right)^{\frac{\rho_{f_k}+\varepsilon}{1+\alpha}}+O(1).\end{aligned} \tag{3.11}$$

Using (3.11) in (3.10) we have

$$\begin{aligned}&\log^{[n-2]}T(r,F_n^1)\\ &\ge \frac{1}{2}\left(\lambda_{f_{k-2}}-\varepsilon\right)[\log\frac{1}{\eta}\\ &\times\frac{(1-\delta(0,f_1)-\varepsilon)\,T\left\{M\left(\left(\eta\frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}},F_1^k\right),f_1\right\}-\log M\left(\left(\eta\frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}},F_1^k\right)O(1)}{\left\{\log\left(\eta\frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}+\left(\eta\frac{r}{2^{2(n-2)}}\right)^{\frac{\rho_{f_k}+\varepsilon}{1+\alpha}}\right\}\{1-o(1)\}}\\ &-\log\frac{r}{2^{2(n-2)}}-T\left(\frac{r}{2^{2(n-2)}},f_k\right)]+O(1).\end{aligned}$$

So for a sequence of values of r tending to infinity

$$\begin{aligned}&\log^{[n-1]}T(r,F_n^1)\\ &\ge \log^{[2]}\frac{1}{\eta}\\ &+\log[\frac{(1-\delta(0,f_1)-\varepsilon)\,T\left\{M\left(\left(\eta\frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}},F_1^k\right),f_1\right\}-\log M\left(\left(\eta\frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}},F_1^k\right)O(1)}{\left\{\log\left(\eta\frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}+\left(\eta\frac{r}{2^{2(n-2)}}\right)^{\frac{\rho_{f_k}+\varepsilon}{1+\alpha}}\right\}\{1-o(1)\}}\\ &-\log^{[2]}\frac{r}{2^{2(n-2)}}-\log T\left(\frac{r}{2^{2(n-2)}},f_k\right)]+O(1).\\ &=\log^{[2]}\frac{1}{\eta}+\log[\frac{(1-\delta(0,f_1)-\varepsilon)\,T\left\{M\left(\left(\eta\frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}},F_1^k\right),f_1\right\}}{\left\{\log\left(\eta\frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}+\left(\eta\frac{r}{2^{2(n-2)}}\right)^{\frac{\rho_{f_k}+\varepsilon}{1+\alpha}}\right\}}\{1\end{aligned}$$

$$
\begin{aligned}
&- \frac{\log M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right) O(1)}{(1-\delta(0,f_1)-\varepsilon)\, T\left\{M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right), f_1\right\}} \}] - \log^{[2]} \frac{r}{2^{2(n-2)}} \\
&- \log T\left(\frac{r}{2^{2(n-2)}}, f_k\right) + O(1) \\
&= \log^{[2]} \frac{1}{\eta} + \log(1-\delta(0,f_1)-\varepsilon) + \log T\left\{M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right), f_1\right\} \\
&- \log\left\{\log\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}} + \left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{\rho_{f_k}+\varepsilon}{1+\alpha}}\right\} + \log[1- \\
&\frac{\log M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right) O(1)}{(1-\delta(0,f_1)-\varepsilon)\, N_1 \log M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right)}] \\
&- \log^{[2]} \frac{r}{2^{2(n-2)}} - \log T\left(\frac{r}{2^{2(n-2)}}, f_k\right) + O(1) \\
&\geq O(\log r) + \log T\left\{M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right), f_1\right\} + \log[1- \\
&\frac{\log M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right) O(1)}{(1-\delta(0,f_1)-\varepsilon)\, T\left\{M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right), f_1\right\}}] - \log^{[2]} \frac{r}{2^{2(n-2)}} \\
&- \log T\left(\frac{r}{2^{2(n-2)}}, f_k\right) + O(1) \\
&\geq O(\log r) + \log T\left\{M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right), f_1\right\} + \log[1- \\
&\frac{\log M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right) O(1)}{(1-\delta(0,f_1)-\varepsilon)\, T\left\{M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right), f_1\right\}}] - \log \frac{r}{2^{2(n-2)}} \\
&- T\left(\frac{r}{2^{2(n-2)}}, f_k\right) + O(1) \qquad (3.12)
\end{aligned}
$$

Since f_1 is transcendental, $\lim\limits_{r\to\infty} \frac{T(r,f_1)}{\log r} = \infty$, and so for given positive number N_1, however large, and for all large values of r, $T(r,f_1) > N_1 \log r$. Therefore from (3.12) for a sequence of values of r tending to infinity we have

$$\begin{aligned}
&\log^{[n-1]} T\left(r, F_n^1\right) \\
&\geq O(1) + O(\log r) + \log T\left\{M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right), f_1\right\} + \log[1 - \\
&\frac{\log M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right) O(1)}{\left(1-\delta(0, f_1) - \varepsilon\right) N_1 \log M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right)}] - \log \frac{r}{2^{2(n-2)}} \\
&- T\left(\frac{r}{2^{2(n-2)}}, f_k\right) \\
&\geq O(1) + O(\log r) + \log T\left\{M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right), f_1\right\} + \log[1 - \\
&\frac{O(1)}{\left(1-\delta(0, f_1) - \varepsilon\right) N_1}] - \log \frac{r}{2^{2(n-2)}} - T\left(\frac{r}{2^{2(n-2)}}, f_k\right)
\end{aligned}$$

where N_1 is so large that

$$1 - \frac{O(1)}{\left(1-\delta(0, f_1) - \varepsilon\right) N_1} > 0.$$

Therefore, for a sequence of values of r tending to infinity we have

$$\begin{aligned}
&\log^{[n-1]} T\left(r, F_n^1\right) \\
&\quad \geq O(1) + O(\log r) + \log T\left\{M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right), f_1\right\} \\
&\quad - \log \frac{r}{2^{2(n-2)}} - T\left(\frac{r}{2^{2(n-2)}}, f_k\right). \qquad (3.13)
\end{aligned}$$

Again since $f_1(z)$ is of positive lower order λ_{f_1}, we get for all large values of r and for $0 < M_1 < \lambda_{f_1}$,

$$\log T(r, f_1) > M_1 \log r.$$

Hence $\log T\left\{M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right), f_1\right\}$

$$\begin{aligned}
&> M_1 \log M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right) \\
&= M_1 \log M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, (1-\alpha) z + \alpha f_k\right) \\
&\geq M_1 \left\{\log M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, f_k\right) - \log M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, z\right)\right\} + O(1) \\
&= M_1 \left\{\log M\left(\left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, f_k\right) - \log \left(\eta \frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}\right\} + O(1). \qquad (3.14)
\end{aligned}$$

Since f_k is of finite positive hyper lower order $\overline{\lambda}_{f_k}$, it follows for all large values of r that

$$\frac{\log\log\log M(r, f_k)}{\log r} > \frac{1}{2}\overline{\lambda}_{f_k}$$

$$\text{i.e., } \log M(r, f_k) > e^{r^{\frac{\overline{\lambda}_{f_k}}{2}}}.$$

Hence from (3.14) we have

$$\log T\left\{M\left(\left(\eta\frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}, F_1^k\right), f_1\right\}$$

$$> M_1\left\{e^{\left(\eta\frac{r}{2^{2(n-2)}}\right)^{\frac{\overline{\lambda}_{f_k}}{2(1+\alpha)}}} - \log\left(\eta\frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}\right\} + O(1).$$

Therefore from (3.13) we have for a sequence of values of r tending to infinity

$$\log^{[n-1]} T(r, F_n^1)$$

$$\geq O(1) + O(\log r) + M_1 e^{\left(\eta\frac{r}{2^{2(n-2)}}\right)^{\frac{\overline{\lambda}_{f_k}}{2(1+\alpha)}}} - M_1\log\left(\eta\frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}$$

$$-\log\frac{r}{2^{2(n-2)}} - T\left(\frac{r}{2^{2(n-2)}}, f_k\right).$$

Also for all large values of r, $T(r, f_k) < r^{\rho_{f_k}+\varepsilon}$.

Therefore for a sequence of values of r tending to infinity we have

$$\frac{\log^{[n-1]} T\left(r, F_n^1\right)}{T(r, f_k)}$$

$$\geq O(1) + \frac{O(\log r)}{T(r, f_k)} + M_1\frac{e^{\left(\eta\frac{r}{2^{2(n-2)}}\right)^{\frac{\overline{\lambda}_{f_k}}{2(1+\alpha)}}}}{T(r, f_k)} - M_1\frac{\log\left(\eta\frac{r}{2^{2(n-2)}}\right)^{\frac{1}{1+\alpha}}}{T(r, f_k)}$$

$$-\frac{\log\frac{r}{2^{2(n-2)}}}{T(r, f_k)} - \frac{T\left(\frac{r}{2^{2(n-2)}}, f_k\right)}{T(r, f_k)}.$$

$$\geq O(1) + M_1\frac{e^{\left(\eta\frac{r}{2^{2(n-2)}}\right)^{\frac{\overline{\lambda}_{f_k}}{2(1+\alpha)}}}}{r^{\rho_{f_k}+\varepsilon}} - 1.$$

Hence

$$\limsup_{r\to\infty}\frac{\log^{[n-1]} T(r, F_n^1)}{T(r, f_k)} = \infty.$$

This completes the proof.

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