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BULK ARRIVAL QUEUEING SYSTEM WITH STAND BY SERVER AND MULTIPLE VACATION QUEUEING SYSTEM

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Abstract: This paper deals with a bulk arrival queueing system using the concept of stand by server and multiple vacation queueing system. When the main server goes for type I or type II vacation, in order to provide continuous service, the stand-by server is used. Type I vacation is taken when all the waiting customers gets served and in the absence of main server, stand-by server provides the service whereas type II vacation is availed after returning from first vacation and still finds an empty queue. Again, in this case the stand by server is available to provide service but no customer is there to avail the service. The steady state is obtained in terms of probability generating function for the various system performance measures. Further, various performance measures are also derived for this queueing model.

Keywords and Phrases: Bulk Arrival, Stand-by Server, Multiple Vacation Queueing System, Steady State, Probability Generating Function.

2020 Mathematics Subject Classification: 68M20, 60K20, 90B22.

1. Introduction

Since several years, the concept of stand-by server is of keen interest to many researchers. Many authors have studied the queueing system with stand-by server and vacation periods. In this work, we studied bulk arrival queueing system with stand by server and multiple vacation queueing system.

Maraghi et al [6] studied the queueing system where arrivals are in batches with the concept of random breakdowns and Bernoulli schedule server vacations and obtained the steady state results for system performance measures in regard with probability generating function in explicit and closed form. Baruah et al [1] derived the steady state queue distribution for a single server queue with batch arrival and service in two fluctuating modes. They have also used the concept of reneging during vacations and breakdown. Li et al [5] obtained the probability generating function for the number of customers of an $M/M/1$ retrial queue with working vacation interruption as well as retrial policy. Khalaf et al [3] studied the batch arrival queueing system with stand-by server and obtained the various system performance measures in steady state in terms of probability generating function. They obtained the results under the condition when stand-by server is working and main server is on vacation or on repair. In this paper, we have used the concept of working of stand-by server in place of main server when the main server is on type I vacation or on type II vacation.

Niranjan et al [8] studied the performance characteristics of a batch service queueing system with multiple vacations and system failure and obtained the queue size in terms of probability generating function by using supplementary variable technique. M. Senthil Kumar [4] also analysed the performance measures of $M/G/1$ retrial queue with non-persistent calls, two phases of heterogeneous service and different vacation policies. Murugeswari [7] also studied a bulk arrival queueing system with standby server and compulsory server vacation.

The rest of the paper is arranged as follows: Section 2 gives mathematical model of the system. Section 3 deals with the notations used in the paper. Section 4 includes steady state equations ruling the system. Section 5 contains steady state queue size distribution at a random epoch. Section 6 deals with steady state mean queue length and mean waiting time followed with conclusion in section 7.

2. Mathematical Model of the System

Following assumptions are made for the formulation of the mathematical model of the system:

(i) The customers arrive in the system in batches, where batch size varies i.e.; not of fixed size, in a compound Poisson process with an arrival rate λ. Let the first order probability be λc_i dt $(i = 1, 2, 3, \ldots)$ (where c_i defined as transition probability for customers of batch size i) that means for a short span of time dt, a batch of i customers reach to get a service at the system where $0 \leq c_i \leq 1$ *and* $\sum_{i=1}^{\infty} c_i = 1$. The customers receives service on First Come First Served basis one by one. When the server is on type I or type II vacation then in that case the service is provided by the stand by server. Let $\xi > 0$ be the service rate of stand-by service which follows exponential distribution and mean stand-by service rate is $1/\xi$.

(ii) Initially, the main server is providing the service to the customers with general distribution function $A(\upsilon)$ and density function $a(\upsilon)$. For the elapsed service time x, let the conditional probability density be $\alpha(x)dx$ in the interval $(x, x+dx)$, then

$$\alpha\,(x) = \frac{a(x)}{1 - A(x)'} \tag{2.1}$$

Therefore

$$a\,(\upsilon) = \alpha\,(\upsilon)\,e^{-\int_0^{\upsilon} \alpha(x)dx} \tag{2.2}$$

(iii) After finishing the service, the main server goes for a type I vacation with a probability P and therefore 1-P is a probability for not going to a vacation where $0 \leq P \leq 1$. In this case, stand-by server is providing the service to the customers. Let the main server on type I vacation follows general distribution with distribution function $B(s)$ and density function $b(s)$. For the elapsed vacation time x, let the conditional probability density be $\beta(x)dx$ in the interval $(x, x+dx)$, then

$$\beta\,(x) = \frac{b(x)}{1 - B(x)'} \tag{2.3}$$

Therefore

$$b\,(s) = \beta\,(s)\,e^{-\int_0^{s} \beta(x)dx} \tag{2.4}$$

(iv) After returning from type I vacation, the main server finds no customer for the service and so takes another vacation called type II vacation. In this case, in absence of main server, stand-by server is available but there is no customer in the system for the service.

3. Notations

(i) $M_n\,(t, x)$: probability of $n \geq 0$ customers in the queue at time t excluding one customer in the queue served by main server, the main server is providing the service, and x be the elapsed service time for this customer.

Hence, $M_n(t) = \int_0^\infty M_n(t,x)\,dx$ denotes the probability of $n \geq 1$ customers in the queue excluding one customer in the service irrespective of the value of x.

(ii) $Y_n(t,x)$: probability of $n \geq 0$ customers in the queue at time t, main server is on type I vacation with elapsed vacation time x and one customer is served by stand-by server.

Hence, $Y_n(t) = \int_0^\infty Y_n(t,x)\,dx$ denotes the probability of $n \geq 0$ customers at time t, who are ready to get the service, main server is on type I vacation and stand-by server is available for giving service to the customers.

(iii) $I(t)$: probability that at time t, the main server is on type II vacation and stand-by server is available to provide the service but there are no customers in the system. Thus, this is an idle condition for the stand-by server though it is available in the system.

In steady state, let

$$\lim_{t\to\infty} M_n(t,x) = M_n(x) \ ,$$

$$\lim_{t\to\infty} M_n(t) = \lim_{t\to\infty} \int_0^\infty M_n(t,x) = M_n,$$

Therefore,

$$\lim_{t\to\infty} \frac{\mathrm{dM}_n(t)}{\mathrm{dt}} = 0 \tag{3.1}$$

Similarly, $\lim_{t\to\infty} \frac{\mathrm{dY}_n(t)}{\mathrm{dt}} = 0$ & $\lim_{t\to\infty} \mathrm{I}(t) = I$

4. Steady State Equations Ruling the System

As discussed above, in the short interval of time $(t,\ t+dt)$, following three states are considered in the system. The main server is providing the service in the first state, the main server is on type I vacation in the second state and stand-by server is providing the service. In the third state when the main server arrives from vacation and still not finds any customer so decides to go for another vacation called type II vacation and stand-by server is available to provide service but no customer is available. Thus, in this state the stand-by server is in idle condition. Considering the above probabilities and taking the limit $dt \to 0$, following set of differential difference equations are obtained:

$$\frac{\partial}{\partial x} M_n(x) = -\left[\lambda + \alpha(x)\right] M_n(x) + \lambda \sum_{i=1}^{n-1} c_i M_{n-i}(x)\,,\ n \geq 1 \tag{4.1}$$

$$\frac{\partial}{\partial x}M_0\left(x\right)=-\left[\lambda+\alpha\left(x\right)\right]M_0\left(x\right) \tag{4.2}$$

$$\frac{\partial}{\partial x}Y_n\left(x\right)=-\left[\lambda+\beta\left(x\right)+\xi\right]Y_n\left(x\right)+\lambda\sum_{i=1}^{n}c_iY_{n-i}\left(x\right)+\xi Y_{n+1}(x),\ \ n\geq 1 \tag{4.3}$$

$$\frac{\partial}{\partial x}Y_0\left(x\right)=-\left[\lambda+\beta\left(x\right)+\xi\right]Y_0\left(x\right)+\xi Y_1\left(x\right) \tag{4.4}$$

$$\lambda I=\left(1-P\right)\int_0^\infty M_0\left(x\right)\alpha\left(x\right)dx+\int_0^\infty Y_0\left(x\right)\beta\left(x\right)dx \tag{4.5}$$

Following the boundary conditions, the above equations are to be solved

$$M_n\left(0\right)=\left(1-P\right)\int_0^\infty M_{n+1}\left(x\right)\alpha\left(x\right)dx+\int_0^\infty Y_{n+1}\left(x\right)\beta\left(x\right)dx+\lambda c_{n+1}I,\qquad n\geq 0 \tag{4.6}$$

$$Y_n\left(0\right)=P\int_0^\infty M_n\left(x\right)\alpha\left(x\right)dx,\quad n\geq 0 \tag{4.7}$$

5. Steady State Queue Size Distribution at a Random Epoch

To obtain the steady state queue size distribution, we define the Probability generating function as follows:

$$M_q\left(x,z\right)=\sum_{n=0}^{\infty}z^nM_q(x)$$

$$Y_q\left(x,z\right)=\sum_{n=0}^{\infty}z^nY_q(x) \tag{5.1}$$

$$c\left(z\right)=\sum_{i=1}^{\infty}z^ic_i$$

Equation (4.1) is multiplied by z^n then we sum over n from 1 to ∞, adding to (4.2) then by simplifying and using equation (5.1) we get

$$\frac{\partial}{\partial x}M_q\left(x,z\right)+\left(\lambda+\alpha\left(x\right)-\lambda c\left(z\right)\right)M_0\left(x\right)M_q\left(x,z\right)=0 \tag{5.2}$$

By the similar process, from equations (4.3) and (4.4) we get

$$\frac{\partial}{\partial x}Y_q\left(x,z\right)+\left(\lambda+\beta\left(x\right)+\xi-\frac{\xi}{z}-\lambda c\left(z\right)\right)Y_q\left(x,z\right)=0 \tag{5.3}$$

In a similar manner, we will apply the operations on boundary conditions (4.6) and (4.7).

Now, we will simplify equation (4.6) with the help of generating functions defined in (5.1). For this, we will multiply equation (4.6) by z^{n+1} then we sum over n from 0 to ∞. We get

$$\text{z}M_q(0,z) = (1-p)\int_0^\infty M_q(x,z)\alpha(x)\,dx + \int_0^\infty Y_q(x,z)\beta(x)\,dx + \lambda I(c(z)-1) \tag{5.4}$$

Multiplying equation (4.7) by z^n and summing from 0 to ∞ over n, we get

$$Y_q(0,z) = p\int_0^\infty M_q(x,z)\alpha(x)\,dx \tag{5.5}$$

Now, we will integrate equations (5.2) and (5.3) w.r.t x between the limits 0 to x

$$\text{M}_q(x,z) = M_q(0,z)\,e^{-(\lambda \text{c}(z)-\lambda)x-\int_0^x \alpha(t)\text{dt}} \tag{5.6}$$

Let us denote $D = \lambda c(z) - \lambda$

$$\text{Y}_q(x,z) = Y_q(0,z)\,e^{-Qx-\int_0^x \beta(t)\text{dt}} \tag{5.7}$$

where $Q = (\lambda + \xi - \frac{\xi}{z} - \lambda c(z))$ $M_q(0,z)$ and $Y_q(0,z)$ are given by the equations (5.4) and (5.5). Integrating equations (5.6) and (5.7) w.r.t x by parts, we get

$$\text{M}_q(z) = M_q(0,z)\,[\frac{1-\overline{U}(D)}{D}] \tag{5.8}$$

where $\overline{U}(D) = \int_0^\infty e^{-(\lambda c(z)-\lambda)x}dU(x)$ is the Laplace -Stieltjes transform of the service time U(x).

Further,

$$\text{Y}_q(z) = Y_q(0,z)\,[\frac{1-\overline{W}(Q)}{Q}] \tag{5.9}$$

where $\overline{W}(Q) = \int_0^\infty e^{-\left(\lambda+\xi-\frac{\xi}{z}-\lambda c(z)\right)x}dW(x)$ is the Laplace -Stieltjes transform of the vacation time $W(x)$.

Now, we will multiply equations (5.6) and (5.7) by $\alpha(x)$ and $\beta(x)$ respectively and integrate each w.r.t. x to find the value of $\int_0^\infty M_q(x,z)\alpha(x)\,dx$ and $\int_0^\infty Y_q(x,z)\,\beta(x)\,dx$.

Hence, we get

$$\int_0^\infty M_q(x,z)\alpha(x)\,dx = M_q(0,z)\,\overline{U}(D) \tag{5.10}$$

and

$$\int_0^\infty Y_q(x,z)\beta(x)\,dx = Y_q(0,z)\,\overline{W}(Q) \tag{5.11}$$

Now, using equations (5.10) and (5.11) into equations (5.4) and (5.5) and then applying the equations (5.8) and (5.9), we get

$$\mathrm{M}_q(z) = \frac{\lambda I\ [(c(z)-1)(1-\overline{U}(D)]}{\left[z-(1-p)\,\overline{U}(D)-p\overline{U}(D)\,\overline{W}(Q)\right]D} \tag{5.12}$$

$$Y_q(z) = \frac{p(1-\overline{W}(Q))\overline{U}(D)\,\lambda I(c(z)-1)}{Q[z-(1-p)\,\overline{U}(D)-p\overline{U}(D)\,\overline{W}(Q)]} \tag{5.13}$$

Let the probability generating function of the queue size be denoted by $F_q(z)$ such that $F_q(z) = M_q(z) + Y_q(z)$.

The idle time I is determined using the normalization condition

$$I + F_q(1) = 1 \tag{5.14}$$

Now,

$$F_q(z) = \frac{\lambda I\ [(c(z)-1)(1-\overline{U}(D)]}{\left[z-(1-p)\,\overline{U}(D)-p\overline{U}(D)\,\overline{W}(Q)\right]D} + \frac{p(1-\overline{W}(Q))\overline{U}(D)\,\lambda I(c(z)-1)}{Q[z-(1-p)\,\overline{U}(D)-p\overline{U}(D)\,\overline{W}(Q)]}$$

$$\mathrm{F}_q(z) = \frac{\lambda \mathrm{I}\,(c(z)-1)\,[Q(1-\overline{U}(D))+pD\overline{U}(D)\left(1-\overline{W}(Q)\right)]}{DQ[z-(1-p)\,\overline{U}(D)-p\overline{U}(D)\,\overline{W}(Q)]} \tag{5.15}$$

If $z = 1$ then $c = 0$ $[as\ c(1) = 0]$ *and* $Q = 0$ so $F_q(1)$ is indeterminate of the 0/0 form. Therefore, to solve this we apply L-Hospital's Rule once on equation (5.15), we get

$$F_q(1) = \frac{N'(z)}{D'(z)} \tag{5.16}$$

Here numerator and denominator are denoted by $\mathrm{N}(z)\, and\ D(z)$ respectively in the right-hand side of the equation (5.15). To find the value of $F_q(1)$, we will find the derivatives of $N(z)\, and\ D(z)$ at $z = 1$.

We have

$$N'(1) = \lambda I E(X)\, p\, (\lambda - \mathbf{x})\, \overline{U}(\lambda - \mathbf{x}) \left(\mathbf{1} - \overline{W}(0)\right)$$

$$= \lambda I E(X)\, p\, (\lambda - x)\, E(S)(1 - E(V)) \tag{5.17}$$

$$D'(1) = (\lambda - x)(\xi - \lambda E(X))\,[1 - (1 - p)\, E(S) - pE(S)\, E(V)] \tag{5.18}$$

where $c'(1) = E(X)$ is the mean batch size of arriving customers, $\overline{U}(\lambda - \mathbf{x}) = E(S)$ is mean service time and $\overline{W}(0) = E(V)$ is mean vacation time.

Hence,

$$F_q(1) = \frac{\lambda \mathrm{I} \mathrm{E}(X)\, p\, (\lambda - x)\, E(S)(1 - E(V))}{(\lambda - x)(\xi - \lambda E(X))\,[1 - (1 - p)\, E(S) - pE(S)\, E(V)]} \tag{5.19}$$

As $I + F_q(1) = 1$, on putting the value of $F_q(1)$ from equation (5.19) and simplifying for the idle time I, we get

$$I = \frac{(\lambda - x)(\xi - \lambda E(X))\,[1 - (1 - p)\, E(S) - pE(S)\, E(V)]}{(\lambda - x)(\xi - \lambda E(X))\,[1 - (1 - p)\, E(S) - pE(S)\, E(V)] + \lambda E(X)\, p\, (\lambda - x)\, E(S)(1 - E(V))} \tag{5.20}$$

If we assume that, as a particular case that $\xi = 0$ i.e; there is no stand-by server then $Q = \lambda - \lambda c(z)$ and hence equations (5.15) and (5.20) reduces to

$$F_q(z) = \frac{\lambda \mathrm{I}\,(c(z) - 1)\,[(\lambda - \lambda c(z))(1 - \overline{U}(D)) + pD\overline{U}(D)\left(1 - \overline{W}(\lambda - \lambda c(z))\right)]}{D(\lambda - \lambda c(z))[z - (1 - p)\,\overline{U}(D) - p\overline{U}(D)\,\overline{W}(\lambda - \lambda c(z))]} \tag{5.21}$$

and

$$I = \frac{-(\lambda - x)(\lambda \mathrm{E}(X))\,[1 - (1 - p)\, E(S) - pE(S)\, E(V)]}{-(\lambda - x)(\lambda \mathrm{E}(X))\,[1 - (1 - p)\, E(S) - pE(S)\, E(V)] + \lambda E(X)\, p\, (\lambda - x)\, E(S)(1 - E(V))} \tag{5.22}$$

6. Steady State Mean Queue Length and Mean Waiting Time

Let the mean queue length in the steady state be denoted by L_q. Then, as we know that mean queue length is equal to the derivative of probability generating function at $z = 1$. i.e;

$$L_q = [\frac{d}{\mathrm{dz}} F_q(z)]_{z=1} \tag{6.1}$$

The above equation gives 0/0 form and hence we apply L-Hospital's Rule to find the value of L_q.

$$L_q = \frac{D^{''}(z)\, N^{'''}(z) - N^{''}(z) D^{'''}(z)}{3(D^{''})^2} \tag{6.2}$$

where

$$\begin{aligned} IN^{''}(1) &= \lambda IE(X)(\xi - \lambda)(1 - E(S)) + \lambda IE(X)(\xi - \lambda E(X))(1 - E(S)) \\ &+ \lambda Ic^{''}(1)\, p(\lambda - x)\, E(S)(1 - E(V)) + 2\lambda^2 Ip(E(X))^2 E(S)(1 - E(V)) \end{aligned} \tag{6.3}$$

$$\begin{aligned} N^{'''}(1) &= 3\lambda \mathrm{I}c^{''}(1)(\xi - \lambda E(X))(1 - E(S)) + 3\lambda IE(X)\left(-2\xi - \lambda c^{''}(1)\right)(1 - E(S)) \\ &+ \lambda Ic^{'''}(1)\ \mathrm{p}(\lambda - x)\, E(S)(1 - E(V)) + 6\lambda^2 \mathrm{IpE}(X)\, c^{''}(1)\, E(S)(1 - E(V)) \end{aligned} \tag{6.4}$$

$$\begin{aligned} D^{''}(1) &= 2\lambda\ \mathrm{E}(X)(\xi - \lambda E(X))[1 - (1 - p)E(S) - pE(S)E(V)] + (\lambda - x) \\ &\left(-2\xi - \lambda c^{''}(1)\right)[1 - (1 - p)E(S) - pE(S)E(V)] + 2(\lambda - x)(\xi - \lambda) \end{aligned} \tag{6.5}$$

$$\begin{aligned} D^{'''}(1) &= 3\lambda c^{''}(1)(\xi - \lambda E(X))[1 - (1 - p)E(S) - pE(S)E(V)] \\ &+ 3\lambda\ E(X)\left(-2\xi - \lambda c^{''}(1)\right)[1 - (1 - p)E(S) - pE(S)E(V)] \\ &+ 2\lambda E(X)(\xi - \lambda) + (\lambda - x)\left(6\xi - \lambda c^{'''}(1)\right)[1 - (1 - p)E(S) - pE(S)E(V)] \\ &+ 3(\lambda - x)\left(-2\xi - \lambda c^{''}(1)\right) + 3\lambda E(X)(\xi - \lambda E(X)) \end{aligned} \tag{6.6}$$

Substituting equations (6.3) to (6.6) into the equation (6.2), we obtain the value of L_q. With the help of L_q, other performance measures can be obtained as follows:

Average number of customers in the system $L_s = L_q + \rho$ (where $\rho = 1 - I\ \&\ I$ *is given by equation* (5.20))

Average waiting time in the queue $W_q = \frac{L_q}{\lambda}$

Average waiting time in the system $W_s = \frac{L_s}{\lambda}$

7. Conclusion

In this paper, a bulk arrival queueing system is studied with the concept of stand by server and multiple vacation queueing system. In multiple vacation queueing system, two types of vacation i.e.; type I and type II vacation are considered. Stand-by server works in absence of main server. Main server takes type I vacation after serving all the waiting customers and takes type II vacation after returning from first vacation and still finds an empty queue. Stand-by server is available in both types of vacation to provide service. Various performance measures are derived and steady state is obtained in terms of probability generating function for this queueing model.

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