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G-ATTRACTOR AND G-EXPANSIVITY OF THE G-UNIFORM LIMIT OF A SEQUENCE OF DYNAMICAL SYSTEMS

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Abstract: In this paper, we discuss the notions of G-attractor and G-expansive. It is found that if $< (X, f_n) >$ is a sequence of dynamical systems converging G-uniformly to f and if the sequence $< (X, f_n) >$ has a G-uniform attractor $Y \subset X$, then Y is also a G-attractor of f. We also show that if $< (X, f_n) >$ is a sequence of G-expansive dynamical systems with same expansivity time and expansivity constant and converging G-uniformly to f, then (X, f) is also G-expansive. We investigate the G-mixing, G-sensitive and G-shadowing property of the orbital limit f.

Keywords and Phrases: G-Attractor, G-Expansive, G-Sensitive, G-Mixing, G-Shadowing property, G-Nonwandering.

2020 Mathematics Subject Classification: 37B05, 37C85, 54A20.

1. Introduction

In modern mathematical sciences, study of dynamical systems has been an interesting field drawing attention to many mathematicians due to its interesting applications in various fields such as Physics, Biology and Economics. Let $f_n : X \to X$ be a sequence of continuous self maps on a compact metric space X. Many researchers have studied the inheritance of various dynamical notions from the sequence to the uniform limit as well as the orbital limit ([1, 5, 7, 10, 11, 13, 14, 16, 17, 22, 24]). Recently, many researchers have extended the idea to

G-uniform convergence and G-orbital convergence. In [5], it is shown that if (X, d) is a compact metric G-space with no isolated points and $f_n : X \to X$ is a sequence of G-transitive onto maps converging G-uniformly to a continuous self map f on X, then f is G-transitive if and only if there exist $x_0 \in \mathcal{D} = \cap\{tr_G(f_n) : n \in \mathbb{N}\}$ and $x_1 \in X$ such that $\overline{\{g.f_n^{k_n}(x_0) : n \in \mathbb{N}, g \in G} \cap \overline{\{g.f^n(x_1) : n \in \mathbb{N}, g \in G\}} \neq \emptyset$, for every sequence $< k_n >$ of non-negative integers. In [11], the author proved that if (X, d) is a compact metric G-space with no isolated points and $f_n : X \to X$ is a sequence of G-minimal onto maps converging G-uniformly to a continuous self map f on X, then f is G-minimal if and only if for each $x \in X$ and for each $y \in X$, $\overline{\{g.f_n^{k_n}(x) : n \in \mathbb{N}, g \in G\}} \cap \overline{O_G(y, f)} \neq \emptyset$, for every sequence $< k_n >$ of non-negative integers. In this paper, we are interested in investigating the inheritance of notions such as G-expansiveness, G-mixing e.t.c. from the sequence to the G-uniform limit and G-orbital limit.

First, we shall define some important notions of a standard dynamical system as well as group action. A dynamical system is an ordered pair (X, f), where X is a compact metric space with metric d, and f a continuous self map on X. For $n > 0$, f^n denotes the n-fold compositions of f. The set $O_f(x) = \{f^n(x) : n \in \mathbb{N}_0\}$ is called orbit of the point $x \in X$ under f, where $\mathbb{N}_0 = \mathbb{N} \cup 0$. If $\overline{O_f(x)} = X$, the point x is called a transitive point. By a G-space, we mean a triplet (X, G, θ), where X is a Hausdorff space, G a topological group and $\theta : G \times X \to X$, a continuous function (action) such that $\theta(e, x) = x, \forall\ x \in X$ and $\theta(g, \theta(h, x)) = \theta(g * h, x)$, where $*$ is the operation of the group G. We say that θ is a continuous action of G on X. Usually the action $\theta(g, x)$ is denoted by $g.x$. Henceforth $\theta(g, x)$ will be denoted by $g.x$. Let (X, G, θ) be a metric G-space and $f : X \to X$ a continuous function. The function f together with (X, G, θ) is called a dynamical system on a metric G-space. Consider $X = \mathbb{R}, G = \mathbb{Z}$, the set of integers. The action $\theta : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is defined by $\theta(n, x) = x + n$. It is well known that θ is a continuous action. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Now, f is a dynamical system on the metric G- space $(\mathbb{R}, \mathbb{Z}, \theta)$. The G-orbit of a point $x \in X$ is defined by $O_G(x) = \{g.f^n(x) : n \geq 0, g \in G\}$. A finite or infinite sequence $< x_n >_{n \geq 0}$ in X is said to be a G-δ chain, if $\forall\ n$ there exists $g_n \in G$ such that $d(g_n.f(x_n), x_{n+1}) < \delta$. In [5], the authors proved that if $< f_n >$ is a sequence of G-chain transitive maps on a metric G-space X converging uniformly to the limit f, then f is also G-chain transitive. In [3], the authors showed that G-shadowing depends on the action of G and G-shadowing does not implies shadowing nor shadowing implies G-shadowing. In [19], the authors study the dynamical properties like shadowing and transitivity of map f on G-spaces. In [6], the authors define the notion of topologically G-conjugacy and they further study G-periodic points under topological G-conjugacy

(For more details on standard notions of shadowing, transitive and chain transitive, one can see ([8, 9, 12, 15])).

In recent years chaos theory has been studied and defined for group actions ([2, 18, 23, 20]). A continuous function $f : X \to X$, where X is a metric G-space with metric d is called G-Minimal if $\overline{O_G(x, f)} = X$, for every $x \in X$ [10]. Minimality implies G-Minimality but the converse may not be true [19]. We know that minimality implies transitivity, so we can also say that G-minimality implies G-transitivity. Let X be a metric G-space and $f : X \to X$ a continuous map. Then a point $x \in X$ is said to be a G-periodic point of f if there exists an integer $n > 0$ such that $f^n(x) = gx$, for some $g \in G$. The set of all G-periodic points of f is denoted by $Per_G(f)$ [21]. It is obvious that if x is a periodic point of f then point x is also a G-periodic point. But the converse is not true.

A sequence of functions $< f_n >$ from X to X is said to converge G-uniformly to a self map f if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(g.f_n(x), k.f(x)) < \epsilon$, for all $g, k \in G$, for all $x \in X$ and for all $n \geq n_0$ [5].

A continuous self map f defined on a metric G-space X with metric d is called G-transitive if for each pair of non-empty open subsets U and V of X, there exist $n \in \mathbb{N}$ and $g \in G$ such that $g.f^n(U) \cap V \neq \emptyset$. If f is a continuous self map defined on a compact metric G-space X, then a point $x \in X$ is said to be G-transitive point if $\overline{O_G(x, f)} = X$. The set of all G-transitive points is denoted by $tr_G(f)$.

A function $f : X \to X$ is said to be G-chain transitive, if for each pair of points $x, y \in X$ and for every $\delta > 0$ there is a finite G-δ chain $x_0, \ldots, x_n$ such that $x_0 = x$ and $x_n = y$. A point $x \in X$ G-ϵ shadows a finite sequence $x_0, \ldots, x_n$ if $d(g.f^i(x), x_i) < \epsilon$, for every $i \leq n$.

In this paper, we find that if a sequence of dynamical systems $< (X, f_n) >$ converging G-uniformly to f has a common G-attractor Y, then Y is also a G-attractor of (X, f). If a sequence of G-expansive dynamical systems $< (X, f_n) >$ converges G-uniformly to f, then (X, f) is also G-expansive. We prove that if a sequence of dynamical systems $< (X, f_n) >$ is G-mixing and converges G-orbitally to a map f, then (X, f) is also G-mixing. We also show that if $< (X, f_n) >$ is a sequence G-sensitive dynamical systems converging G-orbitally to a map f, then f is also G-sensitive. If each member of dynamical systems $< (X, f_n) >$ has common G-nonwandering point, say x, then x is also a G-nonwandering point of f. We also prove that if $< (X, f_n) >$ is a sequence of dynamical systems having G-shadowing property and converging G-orbitally to a map $f : X \to X$, then f also has G-shadowing property.

A dynamical system (X, f) is said to be G-equicontinuous if for every $\epsilon > 0$ there exists $\delta > 0$ such that $d(g.f^n(x), h.f^n(y)) < \epsilon$, $\forall\ x \in X$, for all $y \in B_\delta(x)$,

for all $n \in \mathbb{N}$ and for all $g, h \in G$ [10].
In [10], the author proved that if a sequence of continuous self map $< f_n >$ on a metric G-space converges G-orbitally to a map f, then f is also G-equicontinuous. If X is a metric G-space and $f : X \to X$ is a continuous map, then a map f is said to have G-sensitive dependence on initial conditions if there is a constant $\delta > 0$, such that for every $x \in X$; and for every neighbourhood U of x, there exists a point $y \in U$ with $G(x) \neq G(y)$ satisfying $d(f^n(u), f^n(v)) > \delta$, for $n > 0$ and for all $u \in G(x)$ and for all $v \in G(y)$. Here δ is called G-sensitive constant. If f has G-sensitive dependence on initial conditions, then we say f is G-sensitive and G-sensitivity of a function f depends on the action of G [20].

2. Definitions and Results

Definition 2.1. *Let X be a metric G-space with metric d, then a function $f : X \to X$ is said to be G-continuous function if for $\epsilon > 0$ there exists $\delta > 0$ such that $d(g.x, h.y) < \delta \Rightarrow d(g.f(x), h.f(y)) < \epsilon$, for all $g, h \in G$.*

Example 2.2. Let S^1 be the unit circle. S^1 is a compact metric space with metric defined by

$$d(\alpha, \beta) = \begin{cases} |\alpha - \beta| & \text{if } \ |\alpha - \beta| \leq \pi, \\ |\alpha - \beta| - \pi & \text{if } \ |\alpha - \beta| > \pi. \end{cases}$$

$G = \left\{0, \pi, \frac{\pi}{2}, \frac{3\pi}{2}\right\}$ and $*$ is defined by

$*$	0	π	$\pi/2$	$3\pi/2$
0	0	π	$\pi/2$	$3\pi/2$
π	π	0	$3\pi/2$	$\pi/2$
$\pi/2$	$\pi/2$	$3\pi/2$	π	0
$3\pi/2$	$3\pi/2$	$\pi/2$	0	π

then G has subspace topology of $\mathbb{R}$. The action $\phi : G \times S^1 \to S^1$ is defined by $\phi(g, \theta) = g\theta = g + \theta(\text{mod } 2\pi) = \frac{n\pi}{2} + \theta(\text{mod } 2\pi)$, where $n = 0, 1, 2, 3$.
$f_\lambda : S^1 \to S^1$ is defined by $f_\lambda(\theta) = \theta + 2\pi\lambda(\text{ mod} 2\pi)$. Let $\epsilon > 0$ be given.
Suppose $d(g\theta_1, h\theta_2) < \delta$, where $\delta = \epsilon$.
Then

$$\left|\frac{n_1\pi}{2} + \theta_1 - \frac{n_2\pi}{2} - \theta_2\right| < \delta$$

$$= \left|\frac{(n_1 - n_2)\pi}{2} + \theta_1 - \theta_2\right| < \delta$$

Now

$$d(gf_\lambda(\theta_1), hf_\lambda(\theta_2)) = \left|\frac{n_1\pi}{2} + \theta_1 + 2\pi\lambda - \frac{n_2\pi}{2} - \theta_2 - 2\pi\lambda\right|$$
$$= \left|\frac{(n_1 - n_2)\pi}{2} + \theta_1 - \theta_2\right| < \delta.$$

Thus $d(gf_\lambda(\theta_1), hf_\lambda(\theta_2)) < \epsilon$, whenever $d(g\theta_1, h\theta_2) < \delta$.
Hence $f_\lambda : S^1 \to S^1$ is G-continuous.

Definition 2.3. *Let (X, d) be a metric G-space. A sequence $< x_n >$ is said to G-converges to a point x if for each $\epsilon > 0$, there exists N such that $d(g.x_n, x) < \epsilon$, $\forall\ n \geq N$ and for all $g \in G$.*

Example 2.4. Take $X = \mathbb{R}$. Let $\mathbb{Z}_2 = \{0, 1\}$, be the additive group of integer modulo 2. The action is defined by $0.x = x, 1.x = -x, \forall\ x \in \mathbb{R}$. Consider the sequence $x_n = \frac{1}{n},\ n \geq 1$. Take $\epsilon = \frac{1}{N-1}$, then $|g.x_n - 0| < \epsilon,\ \forall\ n \geq N$ and for $g \in \mathbb{Z}_2$. Therefore $< x_n >$ G-converges to 0.

Definition 2.5. [10] *A sequence of continuous self maps $< f_n >$ defined on a metric G-space (X, d) is said to converge G-orbitally to a map $f : X \to X$ if for every $\epsilon > 0$, there exists $p \in \mathbb{N}$ such that $d(g.f_n^m(x), k.f^m(x)) < \epsilon$, for all $m \in \mathbb{N}$, for all $g, k \in G$, for all $n \geq p$, and for all $x \in X$.*

Example 2.6. Take $X = [-1, 1]$. Let $\mathbb{Z}_2 = \{0, 1\}$ be the additive group of integer modulo 2, acting on X by $0.x = x, 1.x = -x, \forall x \in X$. Consider the sequence of continuous functions $f_n : X \to X$ defined by $f_n(x) = \dfrac{x}{n+1}, n \in \mathbb{N}$ and the constant function $f(x) = 0$. For $\epsilon > 0$, there exists a positive integer $p \neq 0$ such that $\dfrac{1}{1+p} < \epsilon$. Then for $n > p$, $\dfrac{1}{(1+n)^m} < \epsilon, \forall\ m > 1$. Therefore, $d(g.f_n^m(x), k.f^m(x)) = \left|g.\dfrac{x}{(n+1)^m} - k.0\right| < \epsilon$, for all $m > 1$, for all $g, k \in G$, for all $n > p$, and for all $x \in X$. Hence $< f_n >$ converges G-orbitally to f.

Definition 2.7. *Let (X, d) be a G-metric space. A set $Y \subset X$ is said to be a G-attractor, if it is non-empty, closed, $g.f(Y) = Y$, $\forall\ g \in G$ and for each $\epsilon > 0$, there exists $\delta > 0$, such that for any $x \in X$, $d(x, Y) < \delta$ implies $d(g.f^n(x), Y) < \epsilon$, for all $n \geq 0$ and $g \in G$ and $\lim_{n\to\infty} d(g.f^n(x), Y) = 0,\ \ \forall\ g \in G$.*

Example 2.8. Put $X = [-1, 1]$ and $Y = \{0\}$. $G = \{-1, 1\}$ is the multiplicative group of order 2 acting on X by $1.x = x, -1.x = -x$. It is obvious, for the function $f : X \to X$ defined by $f(x) = x^3$, $Y = \{0\}$ is a G-attractor.

Definition 2.9. *A dynamical system (X, f) is called G-expansive if there is $\epsilon > 0$ such that for each pair of distinct points $x, y \in X$, there exists a positive integer n with $d(g.f^n(x), h.f^n(y)) \geq \epsilon$, for $g, h \in G$.*
Here, ϵ is called G-expansive constant for f.

Example 2.10. Consider $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Take $G = \{-1, 1\}$, the multiplicative group of order 2 acting on S^1 by $1.z = z, -1.z = \overline{z}$. The function $f : S^1 \to S^1$ defined by $f(z) = z^n$ is G-expansive.

Example 2.11. Suppose the group $G = \{-1, 1\}$ acts on $X = [-1, 1]$ by $0.x = x$, $1.x = -x$, then the function $f : X \to X$ defined by $f(x) = x^2$ is G-expansive.

Definition 2.12. *Let (X, d) be a metric G-space. A continuous self map f defined on X is said to be G-mixing if for each pair of non-empty open subsets U and V of X, there exist $m \in \mathbb{N}$, $g \in G$, such that $g.f^n(U) \cap V \neq \emptyset, \forall\ n \geq m$.*

Example 2.13. Consider the tent map $T : X \to X$ defined by

$$T(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 2(1-x), & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases} \quad \textit{where } X = [0, 1]$$

If the multiplicative group of order 2 $G = \{-1, 1\}$ acts on X by $1.x = x$, $-1.x = 1 - x$, then the map T is G-mixing.

Definition 2.14. *A continuous self map f defined on a metric G-space X with metric d is said to be G-sensitive, if there exists $\epsilon > 0$ such that for all $x \in X$ and for all $\delta > 0$ there exists $y \in B_\delta(x)$ such that $d(g.f^n(y), h.f^n(x)) \geq \epsilon, \forall\ g, h \in G$ and for some $n \geq 0$.*
The number ϵ is called G-sensitive constant of f.

Example 2.15. As in example **2.11**, if the group $G = \{-1, 1\}$ acts on $X = S^1$ by $1.z = z$ and $-1.z = \overline{z}$. Then the function $f : X \to X$ defined by $f(z) = z^n$ is G-sensitive, where n is a positive integer.

Definition 2.16. *A dynamical system (X, f) is said to have G-shadowing property, if for any $\epsilon > 0$ there exists $\delta > 0$ such that any G-δ chain is G-ϵ shadowed by some point.*

Definition 2.17. [19] *Let X be a metric G-space with metric d and f is a continuous self map on X. Then a point $x \in X$ is said to be G-non-wandering point, if for every neighbourhood U of x, there exists $k \in \mathbb{N}$ and $g \in G$ such that $g.f^k(U) \cap U \neq \emptyset$.*

In [13], the authors proved that if a sequence of dynamical systems converges uniformly to f and each member of the sequence $< (X, f_n) >$ has a common attractor $Y \subset X$, then Y is also an attractor for (X, f). In the following theorem we are extending this result when a group G acts on the space X.

Theorem 2.18. *Let $<(X, f_n)>$ be a sequence of dynamical systems converging G-uniformly to f. If the sequence $<(X, f_n)>$ has a common G-uniform attractor $Y \subset X$. Then Y is also a G-attractor of f*

Proof. Since Y is a common G-uniform attractor of the sequence $<(X, f_n)>$. Therefore, by definition, for each $\epsilon > 0$, there exists $\delta > 0$, such that for any $x \in X$, with $d(x, Y) < \delta$, implies that $d(g.f_n^m(x), Y) < \frac{\epsilon}{2}$, $\forall\ m \geq 0$, and $\lim\limits_{m\to\infty} d(g.f_n^m(x), Y) = 0$, $\forall\ g \in G$.

Since $f_n \xrightarrow{G-uniformly} f$, we have $f_n^m \xrightarrow{G-uniformly} f^m$, therefore, for each $x \in X$ and for $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that

$$d(g.f_n^m(x), h.f^m(x)) < \frac{\epsilon}{2},\ \forall\ m \geq 0 \text{ and for all } g, h \in G.$$

By triangle inequality,

$$\begin{aligned} d(h.f^m(x), Y) &\leq d(h.f^m(x), g.f_n^m(x)) + d(g.f_n^m(x), Y), \\ &< \epsilon,\ \text{whenever } d(x, Y) < \delta. \end{aligned}$$

Therefore, $d(h.f^m(x), Y) < \epsilon$, $\forall\ m \geq 0$.

Now, $\lim\limits_{m\to\infty} d(g.f_n^m(x), Y) = 0, \forall\ m \geq 1, \forall\ n \in N$ and $\forall\ h \in G$.

Therefore, $\lim\limits_{m\to\infty} d(g.f^m(x), Y) = 0$.

Hence Y is a G-attractor for (X, f).

In [4], authors showed that G-expansivity depends on the action of G and G-expansivity does not implies expansivity nor expansivity implies G-expansivity. In the following theorem we investigate the inheritance of G-expansiveness by the uniform limit f.

Theorem 2.19. *If $<(X, f_n)>$ is a sequence of G-expansive dynamical systems with same expansivity time and expansivity constant and converging G-uniformly to f. Then (X, f) is also G-expansive.*

Proof. Suppose $<(X, f_n)>$ is a sequence of G-expansive dynamical system.

For $\epsilon > 0$ and $x, y \in X, x \neq y$, there exists a positive integer m such that $d(g.f_n^m(x), h.f_n^m(y)) \geq \frac{3\epsilon}{2}$, for $g, h \in G$.

We know that $f_n \xrightarrow{G-uniformly} f$, then $f_n^m \xrightarrow{G-uniformly} f^m$. Then there exists a positive integer N, such that $d(g.f_n^m(x), h.f^m(x)) < \frac{\epsilon}{4}$, for all $x \in X$ and for all $g, h \in G$ and $n \geq N$.

Similarly for $y \in X$, $d(g.f_n^m(y), h.f^m(y)) < \frac{\epsilon}{4}$.

By triangle inequality,

$$d(g.f_n^m(x), h.f_n^m(y)) \leq d(g.f_n^m(x), h.f^m(x)) + d(h.f^m(x), g.f^m(y)) + d(g.f^m(y), h.f_n^m(y))$$
$$\Rightarrow \frac{3\epsilon}{2} \leq \frac{\epsilon}{4} + d(g.f^m(x), h.f^m(y)) + \frac{\epsilon}{4}$$
$$\Rightarrow \epsilon \leq d(g.f^m(x), h.f^m(y)).$$

Hence (X, f) is G-expansive.

Similar to the Theorem 2.1 in [11], we establish the following result in group action.

Theorem 2.20. *Let (X, d) be a metric G-space and f a continuous self map on X. A dynamical system (X, f) is G-mixing if and only if for each pair of non-empty open sets U and V, there exists a sequence $< x_i >$ of points in U such that $g.f^{m+i}(x_i) \in V, \ \forall \ i \geq 1,$ for some $m > 0$ and for some $g \in G$.*

Proof. Suppose X is G-mixing. Let U and V be a non-empty open subsets of X. Then there exists a positive integer m and $g \in G$ such that $g.f^n(U) \cap V \neq \emptyset, \ \forall \ n \geq m$. Then for each $i \geq 1$, we have $g.f^{i+m}(U) \cap V \neq \emptyset$. Therefore there exists a point $x_i \in U$ such that $g.f^{i+m}(x_i) \in V$.

Conversely,

Suppose there exists a sequence $< x_i >$ of points in a non-empty open subset U of X such that $g.f^{m+i}(x_i) \in V, \ \forall \ i \geq 0$, for some $g \in G$ and for some $m > 0$.

This implies that $g.f^{i+m}(U) \cap V \neq \emptyset, \ \forall \ i \geq 0$ and for some $g \in G$.

Therefore $g.f^n(U) \cap V \neq \emptyset, \ \forall \ n \geq m$.

Hence X is G-mixing.

The following theorem shows that G-mixing is inherited by the $G-$orbital limit f.

Theorem 2.21. *If $< f_n >$ is a sequence of G-mixing functions converging G-orbitally to a map f. Then f is also G-mixing.*

Proof. Let u and v be two arbitrary points.

Since $f_n \xrightarrow{G-orbitally} f$, by definition, for $\epsilon > 0$, there exists a positive integer N_1 such that

$$d(g.f_n^m(x), k.f^m(x)) < \frac{\epsilon_1}{2}, \ \forall \ m \in \mathbb{N}, \forall \ g, k \in G, \forall \ n \geq N_1, \text{ and } \forall \ x \in X. \quad (1)$$

Since f_{N_1} is G- mixing, for $\epsilon_2 > 0$ there exist a sequence $x(i, N_1)$ in $B(u, \epsilon_2)$ and $m(N_1)$ such that, for some $g \in G$,

$$g.f_{N_1}^{i+m(N_1)}(x(i, N_1)) \in B(v, \frac{\epsilon_1}{2}), \text{for all } i \geq 1.$$

In (1), if we take $x = x(i, N_1)$ and $m = i + m(N_1)$ then

$$d(g.f_{N_1}^{i+m(N_1)}(x(i,N_1)), k.f^{i+m(N_1)}(x(i,N_1))) < \frac{\epsilon_1}{2}, \quad \text{for all } i \geq 1.$$

By triangle inequality,

$$\begin{aligned} d(g.f^{i+m(N_1)}(x(i,N_1)), v) &\leq d(g.f^{i+m(N_1)}(x(i,N_1)), k.f_{N_1}^{i+m(N_1)}(x(i,N_1))) \\ &+ d(k.f_{N_1}^{i+m(N_1)}(x(i,N_1), v) \\ &< \frac{\epsilon_1}{2} + \frac{\epsilon_1}{2} = \epsilon_1. \end{aligned}$$

Therefore (X, f) is G-mixing.

In the following, we want show that G-sensitivity is preserved by the G-orbital limit.

Theorem 2.22. *If a sequence of dynamical systems $< (X, f_n) >$ is G-sensitive and converges G- orbitally to a map $f : X \to X$, then (X, f) is also G-sensitive.*

Proof. Since $< f_n >$ converges G-orbitally to f. Therefore for all $x \in X$ and for $\epsilon > 0$, there exists $p \geq 0$ such that

$$d(g.f_n^m(x), h.f^m(x)) < \frac{\epsilon}{2}, \text{ for all } n \geq p \text{ for all } m \geq 0 \text{ and for all } g, h \in G.$$

In particular, we have

$$d(g.f^m(x), h.f_n^m(x)) < \frac{\epsilon}{4}, \text{ for all } m \geq 0 \text{ and for all } g.h \in G$$

$$d(g.f_n^m(y), h.f^m(y)) < \frac{\epsilon}{4}, \text{ for all } m \geq 0 \text{ and for all } g, h \in G$$

Since f_n is G-sensitive, with sensitive constant $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(g.f_n^m(y), h.f_n^m(x)) \geq \frac{3\epsilon}{2}, \text{ for some } y \in B_\delta(x) \text{ for all } m \geq 0 \text{ for all } g, h \in G.$$

By triangle inequality,

$$\begin{aligned} d(g.f_n^m(y), h.f_n^m(x)) &\leq d(g.f_n^m(y), h.f^m(y)) + d(h.f^m(y), g.f^m(x)) + d(g.f^m(x), h.f_n^m(x)) \\ &\Rightarrow \frac{3\epsilon}{2} \leq \frac{\epsilon}{4} + d(g.f^m(x), h.f^m(y)) + \frac{\epsilon}{4} \\ &\Rightarrow \epsilon \leq d(g.f^m(x), h.f^m(y)). \end{aligned}$$

Hence (X, f) is G-sensitive.

Theorem 2.23. *Let $<(X, f_n)>$ be a sequence of dynamical systems having G-shadowing property and converging G-orbitally to a limit f. Further, assume that for each $\epsilon > 0$, there exists $\delta > 0$ such that every G-δ chain is G-ϵ shadowed by some point z_n w.r.t f_n for each $n \geq 1$. Then (X, f) also has G-shadowing property.*

Proof. Each f_n has G-shadowing property. Therefore, for $\epsilon > 0$, there exists $\delta > 0$ such that every G-δ chain is G-ϵ shadowed by some point.

Consider a finite G-δ chain $\{x_0, \ldots, x_p\}$, i.e. $\forall\ m \leq p-1$ there exists $g_m \in G$ such that $d(g_m.f_n(x_m), x_{m+1}) < \delta$.

G-δ chain $(x_p)_{p\geq 0}$ is G-ϵ shadowed by some point say z_n, i.e. $\forall\ m \leq p$ there exists $h_m \in G$ such that $d(h_m.f_n^m(z_n), x_m) < \frac{\epsilon}{4}$.

We know that f_n converges G-orbitally to f, therefore, for $\epsilon > 0$ there exists a $n_0 \in N$ such that $d(g_m.f_n^m(x), h_m.f^m(x)) < \frac{\epsilon}{4}$, for all $m \in N$, for all $g_m, h_m \in G$, for all $n \geq n_0$ and for all $x \in X$.

By triangle inequality,

$d(g_m.f^m(z_n), x_m) \leq d(g_m.f^m(z_n), h_m.f_n^m(z_n)) + d(h_m.f_n^m(z_n), x_m) < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}$.

Since X is a compact metric space, the sequence $< z_n >$ G-converges to a point $z \in X$. Therefore, by G-continuity of f^m, there exists N_1, such that

$d(g_m.f^m(z_n), g_m.f^m(z)) < \frac{\epsilon}{2}, \forall\ n \geq N_1$ and $g_m, h_m \in G$.

By triangle inequality,

$d(g_m f^m(z), x_m) \leq d(g_m.f^m(z), g_m.f^m(z_n)) + d(g_m.f^m(z_n), x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

The following result shows that G-nonwandering is inherited by the G-orbital limit f.

Theorem 2.24. *If a sequence of dynamical systems $< f_n >$ converges G-orbitally and each member of the sequence has a common G-nonwandering point, say $x \in X$ with same returning times, then x is also a G-nonwandering point of f.*

Proof. Consider a nonempty open subset U of X, such that U is a neighbourhood of $x \in X$. Since the point $x \in X$ is a common G-nonwandering point of $<(X, f_n)>$, there exists $m \in \mathbb{N}$ and $g \in G$ such that $g.f_n^m(U) \cap U \neq \phi$.

That is for $\epsilon > 0$, there exists $y \in B(x, \frac{\epsilon}{2})$, such that $d(g.f_n^m(y), x) < \frac{\epsilon}{2}$.

As f_n converges G-orbitally to f, there exists $p \in \mathbb{N}$ such that

$$d(g.f_n^m(x), k.f^m(x)) < \frac{\epsilon}{2},\ \text{for all } m \in \mathbb{N},\ \text{for all } g, k \in G\ \text{ and for all } n \geq p.$$

By triangle inequality, we have

$$\begin{aligned} d(g.f^m(y), x) &\leq d(g.f^m(y), g.f_n^m(y)) + d(g.f_n^m(y), x) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence x is a G-nonwandering point of f.

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