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SIGMA COLORING AND GRAPH OPERATIONS

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Abstract: The Sigma coloring of a graph G is an assignment of natural numbers to the vertices of G such that the color sums (the sum of the colors of the adjacent vertices) of any two adjacent vertices are different. The Sigma Chromatic number of a graph G, $\sigma(G)$, is the least number of colors used in a sigma coloring of G. In this paper, we investigate the sigma coloring and Sigma Chromatic number of some graph operations such as Tensor product of graphs, Ring sum of graphs and Joint-sum of graphs. We also obtain the sigma coloring and Sigma Chromatic number of some special graphs such as the graphs obtained by duplicating an arbitrary vertex and an arbitrary edge in cycle graphs, C_n, fusion of two vertices in cycle graphs, C_n, two copies of cycle graphs sharing a common edge.

Keywords and Phrases: $\sigma-$coloring, Sigma Chromatic number, Tensor-Product, Ring-Sum, Joint-Sum.

2020 Mathematics Subject Classification: 05C15.

1. Introduction

By a graph, we mean a finite undirected graph without loops or parallel edges. For the terms and notations not defined explicitly here, reader may refer Harary [3]. Graph coloring take a major part in Graph Theory since the rise of the famous four color conjecture. A coloring of a graph G is an assignment of colors to the vertices of G such that adjacent vertices have distinct colors. We represent the colors by natural numbers so that the function $c : V(G) \to N$ is a vertex coloring of a graph G, and $c(v)$ denote the color of a vertex v. If any two adjacent vertices u and v have $c(u) \neq c(v)$ then c is called a proper vertex coloring of G.

Consider a vertex coloring of G which is not-proper. For any $v \in V(G)$, let $\sigma(v)$ denotes the sum of colors of the vertices adjacent to v, if for any two adjacent vertices $u, v \in V(G)$, $\sigma(v) \neq \sigma(u)$. Then the coloring is called a Sigma coloring (σ-coloring) of G. The minimum number of colors used in a sigma coloring of G is called the sigma chromatic number of G and is denoted by $\sigma(G)$. The Sigma Coloring Problem is to determine the Sigma Chromatic number of a graph G.

Several types of graph coloring were investigated [1, 4] and new variations of coloring are still available recently [2, 7]. The σ- coloring was introduced by Gary Chartrand et.al. [1] in 2008 as a study project. In 2010, Gary Chartrand et.al. presented the first paper with the result to this problem [2], determining the sigma chromatic number for complete graphs, cycles and complete r-partite graph with $r \geq 2$. In the same work, it is proved that for any graph G, $\sigma(G) \leq X(G)$ where $X(G)$ is the least number of colors to a proper vertex coloring of G. To the best of our knowledge, there are few other works on the sigma coloring problem [5]. For circulant graphs, Luzon et al. [6] determined the sigma chromatic number for $C_n(1,2)$,$C_n(1,3)$, and $C_{2n}(1,n)$.

We begin by recalling some basic definitions used in this paper.

Definition 1.1. *Tensor Product of two graphs G_1 and G_2 is denoted by $G_1(T_p)G_2$ with vertex set, $V(G_1)\times V(G_2)$ and edge set, $\{(u_1,v_1)(u_2,v_2) : u_1u_2 \in E(G_1), v_1v_2 \in E(G_2)\}$.*

Definition 1.2. *Ring Sum of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph denoted by $G_1 \bigoplus G_2$ with the vertex set, $V_1 \cup V_2$ and the edge set, $E_1 \cup E_2 - (E_1 \cap E_2)$.*

Definition 1.3. *Joint Sum of a graph G is a graph obtained from two copies of G by connecting a vertex of the first copy with a vertex of the second copy by an edge.*

Definition 1.4. *Duplication of a vertex v_k of a graph G produces a new graph G_1 from G by adding a new vertex v'_k in such a way that $N(v_k) = N(v'_k)$, where $N(v)$ denote the set of all vertices of G that are adjacent to v. In other words, a vertex v'_k is said to be duplication of v_k if all the vertices which are adjacent to v_k are now adjacent to v'_k also.*

Definition 1.5. *Duplication of an edge v_iv_{i+1} of a graph G produces a new graph G_1 by adding a new edge $v'_iv'_{i+1}$ in such a way that $N(v'_i) = N(v_i) \cup \{v'_{i+1}\} - \{v_{i+1}\}$ and $N(v'_{i+1}) = N(v_{i+1}) \cup \{v'_i\} - \{v_i\}$.*

Definition 1.6. *Fusion (Identification) of two distinct vertices u, v of a graph*

G produces a new graph G_1 constructed by replacing the vertices u, v by a single vertex w such that every edge which is incident with either u or v in G is now incident with w in G_1.

Definition 1.7. *Duplication (Subdivision) of an edge $e = uv$ by a new vertex w in a graph G produces a new graph G' such that $N_{G'}(w) = \{u, v\}$.*

Definition 1.8. *Duplication of a vertex v_k by a new edge $e = v'_k v''_k$ in a graph G produces a new graph G'' such that $N_G(v'_k) = \{v_k, v''_k\}$ and $N_G(v''_k) = \{v_k, v'_k\}$.*

Definition 1.9. *The Floor function of a real number x is the largest integer less than or equal to x and it is denoted by $\lfloor x \rfloor$. The Ceil function of a real number x is the smallest integer greater than or equal to x and it is denoted by $\lceil x \rceil$.*

Definition 1.10. *Let G be a simple connected graph and $c : V(G) \to \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers, be a coloring of the vertices in G. We call $c(v)$ as the color of the vertex, v. For any $v \in V(G)$, let $\sigma(v)$ denotes the sum of colors of the vertices adjacent to v then c is called a Sigma coloring (σ-coloring) of G if for any two adjacent vertices $u, v \in V(G)$, $\sigma(v) \neq \sigma(u)$. The least number of colors used in a sigma coloring of G is called the sigma chromatic number of G and is denoted by $\sigma(G)$.*

In this paper, we investigate the σ-coloring and the Sigma Chromatic number of some graph operations such as ring sum of graphs, joint sum of graphs and tensor product of graphs. We also prove that the graphs obtained by duplicating arbitrary vertex as well as arbitrary edge in cycle C_n, fusion of two vertices in cycle C_n, two copies of cycle sharing a common edge admits sigma coloring. For the terms and definitions not explicitly defined here, reader may refer Harary [3].

2. Main Results

Theorem 3.1. *Tensor product of P_m and P_n, $m > n$, is σ-colorable and its Sigma chromatic number is*

$$\sigma(P_m(T_p)P_n) = \begin{cases} 1 & \text{if} \quad m = 3 \quad \text{and} \quad n = 2 \\ 2 & \text{if} \quad m \geq 4, \ n \geq 2. \end{cases}$$

Proof. Let P_m and P_n be two paths of length $m-1$ and $n-1$ respectively. Let $G = P_m(T_p)P_n$. Then $|V(G)| = mn, |E(G)| = 2(m-1)(n-1)$. Let the vertices of G as $u_i v_j$ $1 \leq i \leq m, 1 \leq j \leq n$.

Case 1. Let $m = 3$ and $n = 2$.

Define a coloring $c : V(G) \to \{1\}$ as follows: $c(u_i v_j) = 1$ if $1 \leq i \leq m$, $1 \leq j \leq n$. Clearly, this coloring satisfies the conditions of $\sigma-$coloring. Since the graph is non-empty, at least one color is needed so that $\sigma(P_3(T_p)P_2) = 1$.

Case 2. Let $m \geq 4$ and $n = 2$.
Define a coloring function, $c : V(G) \rightarrow \{1, 2\}$ as follows:

$$c(u_{4i-2}v_j) = 1 \text{ for all } 1 \leq i \leq \left\lfloor \frac{m+2}{4} \right\rfloor, \quad j = 1, 2$$

$$c(u_i, v_j) = 2 \quad \text{if} \quad i \neq 4i-2, \quad 1 \leq i \leq m, \ 1 \leq j \leq 2.$$

Here, c induces a $\sigma-$coloring so that $\sigma(P_3(T_p)P_2) \leq 2$. If possible, assume $\sigma(P_3(T_p)P_2) = 1$. Then the vertices u_iv_j, $2 \leq i \leq m-1$, $1 \leq j \leq 2$ are of same degree. If we color all the vertices with the same color, 1 then the adjacent vertices, $u_{i-1}v_2$, u_iv_1, $u_{i+1}v_2$ where $i = \lceil \frac{m}{2} \rceil$ will receive the same color sum, which violates the condition of $\sigma-$coloring so that $\sigma(P_3(T_p)P_2) = 2$.
Case 3. $m \geq 4$ and $n \geq 3$.
Define $c : V(G) \rightarrow \{1, 2\}$ as follows:

$$\begin{aligned}
c(u_1v_j) &= 1 \quad \text{if} \ \ 1 \leq j \leq n \\
c(u_{2i+1}, v_j) &= 1 \quad \text{if } j \text{ is odd}, \ \ 1 \leq i \leq \left\lfloor \frac{m-2}{2} \right\rfloor, \ \ 1 \leq j \leq n \\
&= 2 \quad \text{if } j \text{ is even}, \ \ 1 \leq i \leq \left\lfloor \frac{m-2}{2} \right\rfloor, \ \ 1 \leq j \leq n \\
c(u_{2i}, v_j) &= 2 \quad \text{if } j \text{ is odd}, \ \ 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor, \ \ 1 \leq j \leq n \\
&= 1 \quad \text{if } j \text{ is even}, \ \ 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor, \ \ 1 \leq j \leq n
\end{aligned}$$

$c(u_mv_j)$ can be as follows:
a) If both m and n are odd $c(u_mv_j) = 1$, $1 \leq j \leq n$.
b) If m is odd and n is even $c(u_mv_j) = 2$, $1 \leq j \leq n$
c) If m is even and n is odd $c(u_mv_j) = 2$, $1 \leq j \leq n$
d) If both m and n are even $c(u_mv_j) = \begin{cases} 1 & \text{if } j \text{ is odd} \\ 2 & \text{if } j \text{ is even} \end{cases}$

Here, c is a $\sigma-$coloring with $\sigma(P_m(T_p)P_n) \leq 2$. If possible, assume $\sigma(P_m(T_p)P_n)$ $= 1$. Since the vertices u_2v_1 and u_1v_2 are of the same degree and we color all the vertices with the same color, 1, these adjacent vertices u_2v_1 and u_1v_2 receive the same color sum, which violates the condition of $\sigma-$coloring. Hence, $\sigma(P_3(T_p)P_2) =$ 2.

Theorem 2.2. *Tensor product of Cycle graphs and Path graphs, $C_m(T_p)P_n$, $m > n$, is $\sigma-$colorable and its Sigma chromatic number is $\sigma C_m(T_p)P_n = 2$ if $m \geq 3$ and*

$n \neq 3$ and $\sigma C_m(T_p)P_3 = 1$ if $m \geq 4$.
Proof. Let the vertices of $C_m(T_p)P_n$ be denoted as $u_i v_j$ $1 \leq i \leq m$, $1 \leq j \leq n$. We note that $|V(C_m(T_p)P_n)| = mn$, $|E(C_m(T_p)P_n)| = 2m(n-1)$.
Case 1. Let $m \geq 3$ and $n = 3$.
Define a coloring function $c : V(G) \to \{1, 2\}$ as follows.

$$c(u_i v_j) = 1 \quad \text{if} \;\; 1 \leq i \leq m, \; 1 \leq j \leq 3.$$

Then, each vertex has a color sum equal to its degree and hence it satisfies the conditions of a $\sigma-$coloring. Since the graph is non-empty at least one color is needed to color G, so that $\sigma(C_m(T_p)P_3) = 1$.
Case 2. Let $m \geq 3$ and $n \neq 3$.
Define $c : V(G) \to \{1, 2\}$ as follows.

$$\begin{aligned} c(u_i v_j) &= 2 \quad \text{if } j \text{ is odd } 1 \leq i \leq m, \; 1 \leq j \leq n-1, \\ &= 1 \quad \text{if } j \text{ is even } 1 \leq i \leq m, \; 1 \leq j \leq n-1, \\ c(u_i v_n) &= 2 \quad \text{if} \;\; 1 \leq i \leq m. \end{aligned}$$

c induces a $\sigma-$coloring so that $\sigma(C_m(T_p)P_n) \leq 2$. If possible, assume $\sigma(C_m(T_p)P_n)$ $= 1$. The vertices $u_1 v_2$ and $u_m v_{n-1}$ are of same degree and are adjacent. If we color all the vertices with the same color, 1, then these adjacent vertices receives the same color sum, which will contradict the condition of $\sigma-$coloring so that $\sigma(C_m(T_p)P_n) = 2$.

Theorem 2.3. *Tensor product of two star graphs, $K_{1,n}(T_p)K_{1,n}$ is $\sigma-$colorable and its Sigma chromatic number is $\sigma(K_{1,n}(T_p)K_{1,n}) = 2$ for all $n \geq 1$.*
Proof. Let $\{u_1, u_2, u_3, ...u_n, u_{n+1}\}$, $\{v_1, v_2, v_3, ...v_n, v_{n+1}\}$ be vertices of first star, $K_{1,n}$ and the second star, $K_{1,n}$ respectively; u_1, v_1 are the roots of the stars. Let $G = K_{1,n}(T_p)K_{1,n}$. Denote the vertices of G as $u_i v_j$; $1 \leq i \leq n+1$, $1 \leq j \leq n+1$. We note that $|V(G)| = (n+1)^2$, $|E(G)| = 2n^2$.

Define a coloring function $c : V(G) \to 1, 2$ as follows:

$$\begin{aligned} c(u_i v_j) &= 1 \quad \text{if } j \text{ is odd } 1 \leq i \leq n+1, \; 1 \leq j \leq n+1, \\ c(u_i v_j) &= 2 \quad \text{if } j \text{ is even } 1 \leq i \leq n+1, \; 1 \leq j \leq n+1 \end{aligned}$$

c induces a $\sigma-$coloring so that $\sigma(K_{1,n}(T_p)K_{1,n}) \leq 2$. If $\sigma(K_{1,n}(T_p)K_{1,n}) = 1$, we color all the vertices with the same color, 1 then the two adjacent vertices $u_2 v_1$ and $u_1 v_3$ with the same degree will receive the same color sum, which violates the condition of $\sigma-$coloring so that $\sigma(K_{1,n}(T_p)K_{1,n}) = 2$.

Theorem 2.4. *Tensor product of two star graphs, $K_{1,m}(T_p)K_{1,n}$ is $\sigma-$colorable* *and its Sigma chromatic number is* $\sigma(K_{1,m}(T_p)K_{1,n}) = 1$ *for all* $m > n$.

Proof. Let $\{u_1, u_2, u_3, ...u_n, u_{n+1}\}$, $\{v_1, v_2, v_3, ...v_n, v_{n+1}\}$ be vertices of first star, $K_{1,n}$ and the second star, $K_{1,n}$ respectively; u_1, v_1 are the roots of the stars. Let $G = K_{1,m}(T_p)K_{1,n}$. Denote the vertices of G as u_iv_j; $1 \le i \le m+1$, $1 \le j \le n+1$. Then, $|V(G)| = (m+1)(n+1)$, $|E(G)| = 2mn$.
Define a coloring function $c : V(G) \to \{1\}$ as follows: $c(u_iv_j) = 1$, $1 \le i \le m+1$, $1 \le j \le n+1$. Since the graph is non-empty, at least one color is needed to color G. Clearly, c satisfies the conditions of $\sigma-$coloring for the graph G so that $\sigma(G) = 1$.

Theorem 2.5. *Ring sum of Cycle graphs and Star graphs,* $C_n \bigoplus K_{1,n}$ *is* $\sigma-$*colorable and its Sigma chromatic number is* $\sigma(C_n \bigoplus K_{1,n}) = 2$.

Proof. $V_1 = \{u_1, u_2, u_3, ...u_n\}$ be the vertex set of C_n and $V_2 = \{v, v_1, v_2, v_3, ...v_n\}$ be the vertex set of $K_{1,n}$ and let $V = V_1 \cup V_2$, where $v_1, v_2, v_3, ...v_n$ are pendent vertices.

Define a coloring function, $c : V(G) \to \{1, 2\}$ as follows: $c(v) = 1$, $c(u_1) = 1$,

$$c(v_j) = 1 \quad \text{if } 1 \le j \le n; \; c(u_{2i-1}) = 1 \;\text{ if } 1 \le i \le \left\lceil \frac{n}{2} \right\rceil; \; c(u_{2i}) = 2 \text{ if } 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor.$$

c induces a $\sigma-$coloring so that $\sigma(C_n \bigoplus K_{1,n}) \le 2$. If $\sigma(C_n \bigoplus K_{1,n}) = 1$, then the vertices u_2, u_3 are of same degree and are adjacent so that they receive the same color sum, which violates the condition of $\sigma-$coloring so that $\sigma(C_n \bigoplus K_{1,n}) = 2$.

Theorem 2.6. *The Joint sum of two copies of* C_n *is* $\sigma-$*colorable and* $\sigma(G) = 2$.

Proof. Let us denote the vertices of the first copy of C_n as $v_1, v_2, ...v_n$ and the vertices of second copy as $v_{n+1}, v_{n+2}, v_{n+3}, ..., v_{n+n}$. Let G be the resulted graph by joining an arbitrary vertex of the first copy of C_n to an arbitrary vertex of the second copy of C_n by a new edge. Without loss of generality we may assume that the new edge be v_nv_{n+1}.

Case 1. n is even.
Define $c : V(G) \to \{1, 2\}$ as follows:

$$c(v_{2i}) = 2 \;\text{ if } 1 \le i \le \frac{n}{2}, \quad c(v_{2i-1}) = 1 \;\text{ if } 1 \le i \le \frac{n}{2}$$
$$c(v_{2i}) = 1 \;\text{ if } \frac{n}{2} + 1 \le i \le n, \quad c(v_{2i-1}) = 2 \;\text{ if } \frac{n}{2} + 1 \le i \le n.$$

Case 2. n is odd.
Define $c : V(G) \to \{1, 2\}$ as follows:

$$c(v_{2i}) = 2 \;\text{ if } 1 \le i \le \frac{n-1}{2}, \quad c(v_{2i-1}) = 1 \;\text{ if } 1 \le i \le \frac{n+1}{2}, \; c(v_{n+1}) = 1.$$
$$c(v_{2i}) = 1 \;\text{ if } \frac{n}{2} + 1 \le i \le n-1, \quad c(v_{2i+1}) = 2 \;\text{ if } \frac{n}{2} + 1 \le i \le n-1, \; c(v_{2n}) = 2.$$

In both cases, c induces a $\sigma-$coloring so that $\sigma(G) \leq 2$. If $\sigma(G) = 1$, then the vertices v_n and v_{n+1} are of the same degree and are adjacent. Sine we color all the vertices with the same color, 1, these two adjacent vertices receives the same color sum, which violates the condition of $\sigma-$coloring so that $\sigma(G) = 2$.

Theorem 2.7. *The Sigma chromatic number of a graph obtained by duplication of an arbitrary vertex of cycle C_n is 2.*
Proof. Let vertices of cycle C_n be $v_1, v_2, v_3, ...v_n$. Let G be graph resulted by duplication of an arbitrary vertex. Without loss of generality let it be v_1 and duplicated vertex be v_1'.

Let us define the coloring $c : V(G) \to \{1, 2\}$. We have to consider the following three cases:
Case 1. When n is even.

$$c(v_1) = 1,\ c(v_1') = 1,\ c(v_{2i-1}) = 1 \text{ if } 1 \leq i \leq \frac{n}{2},\ c(v_{2i}) = 2 \text{ if } 1 \leq i \leq \frac{n}{2}$$

Case 2. When n is odd and the duplicate vertex is in the odd position

$$c(v_1') = 1,\ c(v_{2i-1} = 2 \text{ if } 1 \leq i \leq \frac{n-1}{2},\ c(v_{2i}) = 1 \text{ if } 1 \leq i \leq \frac{n-1}{2},\ c(v_n) = 1.$$

Case 3. When n is odd and the duplicate vertex is in the even position

$$c(v_i') = 1,\ c(v_{2i-1} = 1 \text{ if } 1 \leq i \leq \frac{n+1}{2},\ c(v_{2i}) = 2 \text{ if } 1 \leq i \leq \frac{n-1}{2},\ c(v_n) = 1.$$

c induces a $\sigma-$coloring so that is $\sigma(G) \leq 2$. If $\sigma(G) = 1$, then there exist adjacent vertices, v_3 and v_4, such that these adjacent vertices receive the same color sum, which violates the condition of $\sigma-$coloring. Hence, $\sigma(G) = 2$.

Theorem 2.8. *Fusion of two vertices v_i and v_j with $d(v_i, v_j) \geq 3$ in cycle C_n admits $\sigma-$coloring, where $d(u, v)$ is the shortest distance between two vertices u, v.*
Proof. Let $v_1, v_2, v_3, ...v_n$ be the vertices of cycle the C_n. Without loss of generality, assume that the fused vertices are v_1, v_k so that $d(v_1, v_k) \geq 3$. Let the resulting graph be G. Define a coloring function $c : V(G) \to \{1, 2\}$ as follows:

$$c(v_{2i}) = 1 \text{ if } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor,\ c(v_{2i-1}) = 2 \text{ if } 1 \leq i \leq \left\lfloor \frac{n+1}{2} \right\rfloor,\ c(v_i) = 1 = c(v_k).$$

Then c induces a $\sigma-$coloring so that is $\sigma(G) \leq 2$. If possible, assume $\sigma(G) = 1$. In the graph, G, the vertices of cycle C_n except v_1 and v_k are of the same degree and there exist two adjacent vertices v_{k+1} and v_{k+2} which will have the same color sum, which violates the condition of $\sigma-$coloring so that $\sigma(G) = 2$.

Theorem 2.9. *The Sigma chromatic number of a graph obtained from two copies of cycles C_n sharing a common edge admits $\sigma-$coloring and $\sigma(G) = 2$.*

Proof. Let $\{v_1, v_2, v_3...v_n\}$ be the vertices of the cycle, C_n. Let G be the resulting graph obtained when two copies of cycle, C_n share a common edge. Without loss of generality we can assume that the common edge of these cycles is $\{v_1v_n\}$. Let $\{v_n, v_{n+1}, v_{n+2}, v_{n+3}...v_{2n-2}, v_1\}$ be the vertices of the second copy of C_n.

Case 1. When $n = 3$.

Define $c : V(G) \to \{1, 2\}$ as follows:

Let $\{v_1, v_2, v_3\}$ be the vertices of cycle C_3. Let $\{v_1, v_3, v_4\}$ be the vertices of second copy of cycle C_3. Let the common edge be $e = v_1v_3$

$$c(v_1) = 1, c(v_2) = 2, c(v_3) = 2, c(v_4) = 1.$$

Case 2. $n \geq 5$ and n is odd.

$$c(v_{2i}) = 2 \;\text{ if } 1 \leq i \leq n-1,\; c(v_{2i-1}) = 1 \text{ if } 1 \leq i \leq n-1,\; i \neq \frac{n+1}{2},\; c(v_n) = 2.$$

Case 3. $n \geq 4$ and n is even.

$$c(v_{2i}) = 2 \;\text{ if } 1 \leq i \leq n-1,\; c(v_{2i-1}) = 1 \;\text{ if } 1 \leq i \leq n-1.$$

In all cases, c induces a $\sigma-$coloring so that is $\sigma(G) \leq 2$. If $\sigma(G) = 1$, then the two adjacent vertices v_1 and v_k are of same degree and receive the same color sum, which violates the condition of $\sigma-$coloring so that $\sigma(G) = 2$.

Theorem 2.10. *The Sigma chromatic number of a graph resulted by duplication of every vertex by an edge in cycle C_n, $n \geq 3$, is $\sigma(G) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$.*

Proof. Let $v_1, v_2, v_3, ...v_n$ be the consecutive vertices of cycle C_n. Let G be the resulted graph obtained by duplication of each of the vertices v_i in cycle C_n by new edge u_iw_i for $i = 1, 2, 3, ...n$.

Case 1. $n \geq 3$ and n is even.

Define $c : V(G) \to \{1, 2\}$ as follows:

$$\begin{aligned} c(v_{2i}) &= 2, \;\text{ if } 1 \leq i \leq \frac{n}{2}; \quad && c(v_{2i-1}) = 1 \text{ if } 1 \leq i \leq \frac{n}{2}. \\ c(u_i) &= 1 \;\text{ if } 1 \leq i \leq n && c(w_i) = 2 \text{ if } 1 \leq i < n. \end{aligned}$$

Then, c induces a σ-coloring so that is $\sigma(G) \leq 2$. If possible, assume $\sigma(G) = 1$. The vertices u_i and w_i are of same degree and are adjacent $1 \leq i \leq n$. If we color

all the vertices with the same color 1 then these two adjacent vertices will receive the same color sum, which violates the condition of σ-coloring so that $\sigma(G) = 2$.
Case 2. n is odd.
Define $c : V(G) \to \{1, 2, 3\}$ as follows:

$$c(v_{2i}) = 2, \quad \text{if } 1 \leq i \leq \frac{n-1}{2}; \qquad c(v_{2i-1}) = 1 \text{ if } 1 \leq i \leq \frac{n+1}{2}.$$
$$c(u_i) = 1 \quad \text{if } 1 \leq i \leq n \qquad c(w_i) = 2 \text{ if } 1 \leq i \leq n-1, \; c(w_n) = 3.$$

Here c induces a σ-coloring so that is $\sigma(G) \leq 3$. If $\sigma(G) = 1$, then the vertices u_i and w_i are of same degree and are adjacent $1 \leq i \leq n$ and thus will receive the same color sum, which violates the condition of σ-coloring so that is $\sigma(G) \neq 1$. If $\sigma(G) = 2$, then the vertex w_n must have the color 1 or 2. If w_n has the color 1, then $\sigma(w_n) = \sigma(u_n)$ which violates the condition of σ-coloring. If w_n has the color 2, then $\sigma(v_n) = \sigma(v_1)$ which violates the condition of σ-coloring, So, $\sigma(G) \neq 2$. Hence $\sigma(G) = 3$.

References

[1] Chartrand G. and Zhang P., Chromatic Graph Theory, Boca Raton,Chapman & Hall Press, (2008).

[2] Chartrand G., Okamoto F., Zhang P., The Sigma Chromatic Number of a Graph, Graphs and Combinatorics, 26 (2010), 755–773.

[3] Frank H., Graph Theory, Addison Wesley, Reading Mass, (1969).

[4] Gallian J. A., Dynamic Survey of Graph labelling, The Electronic Journal of Combinatorics, (2012).

[5] Gustavo L. da Soledade G., Sheila M. de Almeida, Sigma coloring on Powers of Paths and Some families of Snarks, Elsevier Science Direct Electronic Notes in Theoretical Computer Science, 346, (2019) 485-496.

[6] Luzon P. A. D., Ruiz M. J. P. and Tolentino M. A. C., The Sigma chromatic number of the Circulant Graphs $C_n(1, 2)$, $C_n(1, 3)$, $C_{2n}(1, n)$, Japanese Conference on Discrete and Computational Geometry and Graphs, (2015), 216-227.

[7] Suresh Kumar J., Graph Coloring Parameters–A survey, International Journal for Research in Applied Science and Engineering Technology, Vol. 7 (IV), (2019).