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VERTEX - EDGE DOMINATING COLORING OF GRAPHS

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Abstract: A vertex u in a graph G = (V,E) is said to ve-dominate an edge $e = vw$ if $u \in \{v, w\}$ or $uv \in E(G)$ or $uw \in E(G)$. An edge coloring is said to be a ve - dominating coloring if no two edges ve - dominated by a single vertex receive the same color. The minimum number of colors required for a ve - dominating coloring of a graph G is called ve - chromatic number of G and is denoted by $\chi_{ve}(G)$. In this paper we initiate the study of this parameter.

Keywords and Phrases: ve-Domination, ve-chromatic number.

2020 Mathematics Subject Classification: 05C15, 05C69.

1. Introduction

Let $G = (V, E)$ be a graph. Let p and q denote the number of vertices and the number of edges respectively. Let $d(v)$ denote the degree of vertex v. The minimum and maximum degree of a graph are denoted by $\delta(G)$ and $\Delta(G)$ respectively. The neighbourhood of a vertex $v \in V(G)$ is the set of all vertices adjacent to v in G and is denoted by $N(v)$.

Let $d(u, v)$ denote the length of a shortest path between two vertices u and v in G. The length of the longest path is called the *diameter* and is denoted by $diam(G)$. Let $S_1, S_2 \subseteq V(G)$. The distance between S_1 and S_2 is given by $d(S_1, S_2) =$

$\min\{d(u,v) : u \in S_1 \text{ and } v \in S_2\}$. For two edges $e_1 = u_1v_1$ and $e_2 = u_2v_2$, the distance between the two edges e_1, e_2 is defined as $d(e_1, e_2) = d(\{u_1, v_1\}, \{u_2, v_2\})$. The open 2 - neighbourhood set $N_2(v)$ of vertex $v \in V(G)$ is defined as $N_2(v) = \{u \in V(G) \mid 0 < d(u,v) \leq 2\}$. The closed 2-neighbourhood set $N_2[v]$ of v is defined as $N_2[v] = N_2(v) \cup \{v\}$.

For any subset S of vertices of V(G), the induced subgraph $\langle S \rangle$ is the maximal subgraph of G with vertex set S. The length of a shortest cycle in the graph is called the *girth* of a graph G and is denoted by $g(G)$. Graphs considered here are finite, undirected, connected, without loops and multiple edges. For definitions not defined here, the reader may refer [2, 3].

A subset S of $V(G)$ is said to be a dominating set of G if for every vertex u not in S, there is a vertex v in S such that u and v are adjacent in G. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$. A vertex v in a graph is said to *ve-dominate* an edge e = uw if either $v \in \{u, w\}$ or $vu \in E(G)$ or $vw \in E(G)$. A subset $D \subseteq V(G)$ is said to be a *ve - dominating set* of a graph G if every edge in the graph is dominated by a vertex in D. The minimum cardinality of a *ve - dominating set* of a graph is called *ve - domination number* of the graph and is denoted by $\gamma_{ve}(G)$. The study of *ve* - domination number has been initiated in [4]. An assignment of colors to the edges of a graph is said to be a proper coloring if no two adjacent edges(i.e., edges having common vertex) receive same color. The minimum number colors required for proper edge coloring is called *edge chromatic number or chromatic index* of G and is denoted by $\chi'(G)$. An edge uv is said to be dominated by the vertex u as well as the vertex v. From the definition of edge coloring, one can observe that the edges dominated by a vertex receive different colors in an edge coloring. In a similar way, when we come through *ve*-domination, one can generalize the edge coloring by means of *ve*-domination. By this observation, we define the following edge coloring. An *edge coloring* of a graph is called *ve*-dominating coloring if the edges *ve* - dominated by a single vertex receive different colors. The minimum number of colors required for a *ve* - dominating coloring of a graph G is called *ve - chromatic number* of G and is denoted by $\chi_{ve}(G)$.

2. Results

Proposition 2.1. *Two edges e_1 and e_2 receive the same color in a ve-dominating coloring if and only if $d(e_1, e_2) \geq 3$.*

Proof. Let G be a graph and $e_1, e_2 \in E(G)$. Suppose that, in a *ve*-dominated coloring, e_1 and e_2 receive same color. Then both e_1 and e_2 are not dominated by a vertex. Therefore $d(e_1, e_2) \geq 3$. Conversely, let $d(e_1, e_2) \geq 3$. Then e_1 and e_2 can not be dominated by a vertex. Then we can give a *ve*-dominating coloring to G so that e_1 and e_2 receive the same color.

Proposition 2.2. *Let G be a (p, q) graph. Then $\chi_{ve}(G) = q$ if and only if the distance between any two edges is less than or equal to two.*

Proof. Assume that $\chi_{ve}(G) = q$. Then any two edges e_1 and e_2 are ve-dominated by a vertex in G. Then $d(e_1, e_2) \leq 2$. Conversely, if $\chi_{ve}(G) < q$, then there are edges e_1 and e_2 receiving same color in a ve-coloring of G. Then the edges e_1 and e_2 can not be ve-dominated by a vertex. Therefore, $d(e_1, e_2) \geq 3$.

Corollary 2.1. *If G is a (p, q) graph with diameter less than or equal to two, then $\chi_{ve}(G) = q$.*

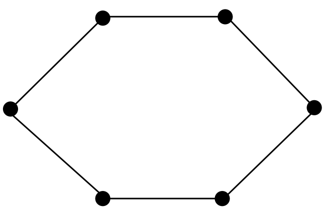
Corollary 2.2. $\chi_{ve}(K_{r,s}) = rs$.

Proof. The diameter of every complete bipartite graph G is less than or equal to two, by above corollary $\chi_{ve}(G) = q = rs$.

Remark 2.1. *If G is a (p, q) graph with $diam(G) = 3$ or 4, then $\chi_{ve}(G)$ need not be equal to q.*

Example 2.1.

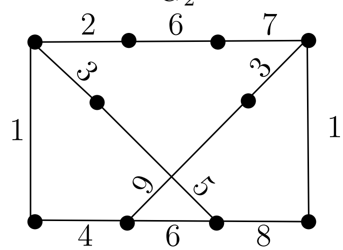
G_1



$diam(G_1) = 3$

$\chi_{ve}(G_1) = q(G_1) = 6$

G_2



$diam(G_2) = 3$, but

$\chi_{ve}(G_2) = 9 < 12 = q(G_2)$

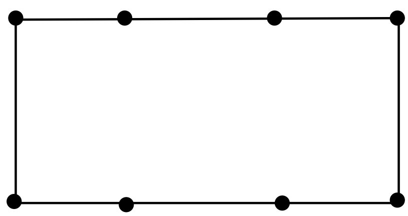
Example 2.2.

G_3

$diam(G_3) = 4$

$\chi_{ve}(G_3) = 4 = q(G_3)$

G_4



$diam(G_4) = 4$

$\chi_{ve}(G_4) = 4 < 8 = q(G_4)$

Remark 2.2. *If $diam(G) \geq 5$, then $\chi_{ve}(G) < q$.*

Definition 2.1. *For a vertex v of a graph G, the ve - degree of v is defined as the number of edges ve - dominated by the vertex v and is denoted by $deg_{ve}(v)$. The minimum and maximum ve - degrees of the graph are defined as $\delta_{ve}(G) = \min\{deg_{ve}(v)|v \in V(G)\}$ and $\Delta_{ve}(G) = \max\{deg_{ve}(v)|v \in V(G)\}$ respectively.*

Note 2.1. $\delta_{ve}(G) \geq \binom{\delta(G)+1}{2}$

Theorem 2.1. *For any (p, q)-graph G, $\Delta_{ve}(G) \leq \chi_{ve}(G) \leq q$. Moreover, $\chi_{ve}(G) = \Delta_{ve}(G)$ if there exists a γ_{ve} - set D of G such that $|N_2(v) \cap N_2(u)| \leq 1$ for all $u, v \in D$.*

Proof. Suppose that the theorem is not true. Let v be a vertex in G and $deg_{ve}(v) = \Delta_{ve}$. The number of edges dominated by v exceeds the number of colors. Then some edges receive same color, which is not a ve - dominating coloring. Therefore $\Delta_{ve}(G) \leq \chi_{ve}(G)$. By giving different colors to all the edges, we get a trivial ve-coloring and hence $\chi_{ve}(G) \leq q$.

Let $D = \{v_1, v_2, \cdots, v_{\gamma_{ve}}\}$ be a γ_{ve} - set of G. Then $\bigcup_{i=1}^{\gamma_{ve}} E(\langle N_2(v_i)\rangle) = E(G)$. If $|N_2(v_i) \cap N_2(v_j)| \leq 1$, $v_i \neq v_j \in D$. $E(\langle N_2(v_i)\rangle) \cap E(\langle N_2(v_j)\rangle) = \phi$. Therefore, $\{E(\langle N_2(v_1)\rangle), E(\langle N_2(v_2)\rangle), \cdots, E(\langle N_2(v_{\gamma_{ve}})\rangle\}$ is a partition of E(G). Since $|E(\langle N_2(v_i)\rangle)| \leq \Delta_{ve}$, $\chi_{ve}(\langle N_2(v_i)\rangle) \leq \Delta_{ve}$ for all i. Hence, $\chi_{ve}(G) \leq \Delta_{ve}$. Since, $\chi_{ve}(G) \geq \Delta_{ve}$, $\chi_{ve}(G) = \Delta_{ve}(G)$.

Theorem 2.2. *For any graph G with maximum degree Δ,*

$$\chi_{ve}(G) \leq 2\Delta((\Delta-1)^2+1)-1$$

Proof. Let the edge xy be assigned by a color red. Then the color red can not be assigned to the edges ux or vy. Since u and v ve-dominate xy, red can not be assigned to the edges vv_1, uu_1 and v_1v_2, u_1u_2. The coloring of the edge xy affects the coloring of at most $2(\Delta-1)$ adjacent edges of the edge xy, the coloring of at most $2(\Delta-1)^2$ second neighbours of xy and the coloring of $2(\Delta-1)^3$ third neighbours of xy. Therefore,

$$\begin{aligned}
\chi_{ve}(G) &\leq 2(\Delta-1)+2(\Delta-1)^2+2(\Delta-1)^3+1 \\
&= 2(\Delta-1)\left(1+\Delta-1+\Delta^2-2\Delta+1\right)+1 \\
&= 2\Delta((\Delta-1)^2+1)-1
\end{aligned}$$

Proposition 2.3. *For any graph G,*

$$\Delta_{ve}(G) = \max_{v\in V(G)}\left\{d(v) + \sum_{u\in N(v)}(d(u)-1) - |E(\langle N(v)\rangle)|\right\} \text{ and}$$

$$\delta_{ve}(G) = \min_{v\in V(G)}\left\{d(v) + \sum_{u\in N(v)}(d(u)-1) - |E(\langle N(v)\rangle)|\right\}$$

Proof. Every $v \in V(G)$ ve-dominate all two distance edges. For each $u \in N(v)$, the vertex v ve-dominates $d(v)$ edges incident with v and $d(u)-1$ edges incident with u other than v. An edge $e = uw \in E(\langle N(v)\rangle)$ is counted twice, for u as well as for w in the $deg_{ve}(v)$. Therefore,

$$deg_{ve}(v) = d(v) + \sum_{u\in N(v)} d(u) - 1 - |E(\langle N(v)\rangle)|$$

Hence,

$$\Delta_{ve}(G) = \max_{v\in V(G)}\left\{d(v) + \sum_{u\in N(v)}(d(u)-1) - |E(\langle N(v)\rangle)|\right\}$$

and

$$\delta_{ve}(G) = \min_{v\in V(G)}\left\{d(v) + \sum_{u\in N(v)}(d(u)-1) - |E(\langle N(v)\rangle)|\right\}$$

Corollary 2.3. *For any graph G with maximum degree Δ, $\Delta_{ve}(G) \leq \Delta^2$*

Proof. Since for any vertex v, $d(v) \leq \Delta$, $\sum_{u\in N(v)}(d(u)-1) \leq \Delta^2 - \Delta$. Hence $\Delta_{ve} \leq \Delta^2$.

Corollary 2.4. *For a graph G with girth $g(G) \geq 4$,*

$$\Delta_{ve}(G) = \max_{v\in V(G)}\left\{d(v) + \sum_{u\in N(v)}(d(u)-1)\right\}.$$

For a bipartite graph $g(G) \geq 4$, but not the converse. So, as a special case we have the following.

Corollary 2.5. *For a bipartite graph G,* $\Delta_{ve}(G) = \max_{v\in V(G)}\left\{d(v) + \sum_{u\in N(v)}(d(u)-1)\right\}.$

Corollary 2.6. $\Delta_{ve}(P_p) = 4$, $p \geq 5$.

Proposition 2.4. *If G is an r - regular graph, then* $\Delta_{ve}(G) = r^2 - \min_{v\in V(G)}\{E(\langle N(v)\rangle)\}$.

Proof. Let G be an r-regular graph. Then from Proposition 2.3,

$$\begin{aligned}\Delta_{ve}(G) &= \max_{v\in V(G)}\left\{r+\sum_{u\in N(v)}(r-1)-|E(\langle N(v)\rangle)|\right\}\\ &= \max_{v\in V(G)}\{r+r(r-1)-|E(\langle N(v)\rangle)|\}\\ &= r^2-\min_{v\in V(G)}\{E(\langle N(v)\rangle)\}\end{aligned}$$

Corollary 2.7. *If G is an r - regular graph and girth atleast 4, then $\Delta_{ve}(G)=r^2$.*

Corollary 2.8. $\Delta_{ve}(C_p)=4$, $p\geq 4$.

Corollary 2.9. $\Delta_{ve}(K_p)=\frac{p(p-1)}{2}$, $p\geq 2$.

Corollary 2.10. *If G is an r - regular bipartite graph, then $\Delta_{ve}(G)=r^2$.*

Proposition 2.5. *Let P_p be a path on p vertices. Then, $\chi_{ve}(P_p)=4$ for $p\geq 5$.*
Proof. Let $P_p: v_1v_2 \cdots v_p$ be a path on p vertices. Let $p=4k+r$. Let $M_1=\{v_1v_2, v_5v_6,\cdots,v_{4k-3}v_{4k-2}\}$, $M_2=\{v_2v_3, v_6v_7,\cdots,v_{4k-2}v_{4k-1}\}$, $M_3=\{v_3v_4, v_7v_8,\cdots,v_{4k-1}v_{4k}\}$, $M_4=\{v_4v_5, v_8v_9,\cdots,v_{4(k-1)}v_{4k-3}\}$.
If r = 0, then M_1, M_2, M_3 and M_4 are ve-color classes.
If r = 1, then M_1, M_2, M_3 and $M_4\cup\{v_{4k}v_{4k+1}\}$ are ve-color classes.
If r = 2, then $M_1\cup\{v_{4k+1}v_{4k+2}\}, M_2, M_3$ and $M_4\cup\{v_{4k}v_{4k+1}\}$ are ve-color classes.
If r = 3, then $M_1\cup\{v_{4k+1}v_{4k+2}\}, M_2\cup\{v_{4k+2}v_{4k+3}\}, M_3$ and $M_4\cup\{v_{4k}v_{4k+1}\}$ are ve-color classes.
Hence, $\chi_{ve}(P_p)\leq 4$. Since $\chi_{ve}(P_p)\geq\Delta_{ve}(P_p)=4$, $\chi_{ve}(P_p)=4$ for $p\geq 5$.

Definition 2.2. *An edge subset S of E is said to be 3 - distance edge set if $d(e_i,e_j)\geq 3$ for all $e_i,e_j\in S$. The maximum cardinality of a 3 - distance edge set of a graph is denoted by $\beta_{3e}(G)$.*

Proposition 2.6.

i) $\beta_{3e}(P_p)=\left\lceil\frac{p-1}{4}\right\rceil$

ii) $\beta_{3e}(C_p)=\left\lfloor\frac{p}{4}\right\rfloor$, if $p\geq 4$

Proposition 2.7. *If G is a graph of size $q\geq 1$, then $\chi_{ve}(G)\geq\dfrac{q}{\beta_{3e}(G)}$.*

Proof. Suppose that $\chi_{ve}(G)=k$ and that $E_1,E_2,\cdots,E_k$ are the ve-color classes in a k-edge coloring in G. Thus $|E_i|\leq\beta_{3e}(G)$ for each $i(1\leq i\leq k)$. Hence

$q = |E(G)| = \sum_{i=1}^{k} |E_i| \leq k\beta_{3e}(G)$ and $\chi_{ve}(G) = k \geq \dfrac{q}{\beta_{3e}(G)}$.

Proposition 2.8. *Let C_p be a cycle on p vertices. Then,*

$$\chi_{ve}(C_p) = \begin{cases} 4, & \text{if } p \equiv\ 0(mod\ 4) \\ p, & \text{if } 3 \leq p \leq 7 \\ 6, & \text{if } p = 11 \\ 5, & \text{otherwise} \end{cases}$$

Proof. Let $C_p : v_1v_2 \cdots v_pv_1$ be a cycle on p vertices.
Case 1: Let $p \equiv 0\ (mod\ 4)$. Then $\{v_1v_2, v_5v_6, \cdots, v_{4k-3}v_{4k-2}\}$, $\{v_2v_3, v_6v_7, \cdots, v_{4k-2}\ v_{4k-1}\}$, $\{v_3v_4, v_7v_8, \cdots, v_{4k-1}v_{4k}\}$ and $\{v_4v_5, v_8v_9, \cdots, v_{4k)}v_1\}$ are the four *ve*-domination color classes of C_{4k}. Hence $\chi_{ve}(C_{4k}) \leq 4$. But $\chi_{ve}(C_{4k}) \geq \Delta_{ve}(C_{4k}) = 4$. Therefore, $\chi_{ve}(C_{4k}) = 4$.
Case 2: Let $p \not\equiv 0\ (mod\ 4)$. When $3 \leq p \leq 7$, $d(e, f) \leq 2$ for all $e, f \in E(C_p)$. Therefore, $\chi_{ve}(C_p) = p$ for $3 \leq p \leq 7$.
Let $p \geq 9$. Let $p = 4k + r$, where $0 < r \leq 3$. By lemma, $\chi_{ve}(C_p) \geq \left\lceil \dfrac{|E(C_p)|}{\beta_{3e}(C_p)} \right\rceil = \left\lceil \dfrac{4k+r}{k} \right\rceil = \left\lceil 4 + \dfrac{r}{k} \right\rceil$. If $k \geq r$, then $\left\lceil 4 + \dfrac{r}{k} \right\rceil = 5$. Therefore, $\chi_{ve}(C_p) \geq 5$.
Sub-case 2.1: Let $p \equiv 1\ (mod\ 4)$. Let $M_1 = \{v_1v_2\} \cup \{v_6v_7, v_{10}v_{11}, \cdots, v_{4k-2}v_{4k-1}\}$; $M_2 = \{v_2v_3\} \cup \{v_7v_8, v_{11}v_{12}, \cdots, v_{4k-1}v_{4k}\}$; $M_3 = \{v_4v_5, v_8v_9, \cdots, v_{4k}v_{4k+1}\}$; $M_4 = \{v_5v_6, v_9v_{10} \cdots, v_{4k+1}v_1\}$ and $M_5 = \{v_3v_4\}$.
Sub-case 2.2: Let $p \equiv 2\ (mod\ 4)$.
Let $M_1 = \{v_1v_2, v_6v_7\} \cup \{v_{11}v_{12}, v_{15}v_{16} \cdots, v_{4k-1}v_{4k}\}$;
$M_2 = \{v_2v_3, v_7v_8\} \cup \{v_{12}v_{13}, v_{16}v_{17} \cdots, v_{4k}v_{4k+1}\}$;
$M_3 = \{v_4v_5\} \cup \{v_9v_{10}, v_{13}v_{14}, \cdots, v_{4k+1}v_{4k+2}\}$; $M_4 = \{v_5v_6, v_{10}v_{11}, v_{14}v_{15} \cdots, v_{4k+2}v_1\}$ and $M_5 = \{v_3v_4, v_8v_9\}$.
Sub-case 2.3: Let $p \equiv 3\ (mod\ 4)$.
Let $M_1 = \{v_1v_2, v_6v_7, v_{11}v_{12}\} \cup \{v_{16}v_{17}, v_{20}v_{21} \cdots, v_{4k}v_{4k+1}\}$;
$M_2 = \{v_2v_3, v_7v_8, v_{12}v_{13}\} \cup \{v_{17}v_{18}, v_{21}v_{22} \cdots, v_{4k+1}v_{4k+2}\}$;
$M_3 = \{v_4v_5, v_9v_{10}\} \cup \{v_{14}v_{15}, v_{18}v_{19}, \cdots, v_{4k+2}v_{4k+3}\}$;
$M_4 = \{v_5v_6, v_{10}v_{11}\} \cup \{v_{15}v_{16}, v_{19}v_{20}, \cdots, v_{4k+3}v_1\}$ and $M_5 = \{v_3v_4, v_8v_9, v_{13}v_{14}\}$. In all cases M_1, M_2, M_3, M_4, M_5 are the *ve*-color classes of C_p. Therefore, $\chi_{ve}(C_p) \leq 5$. Thus if $k > r$, $\chi_{ve}(C_p) = 5$.
Let $k < r$. Since $k \geq 2$ and $r \leq 3$, $2 \leq k < r \leq 3$. Hence, $k = 2$ and $r = 3$, therefore $p = 11$. Then, $\left\lceil 4 + \dfrac{r}{k} \right\rceil = 6$. Therefore, $\chi_{ve}(C_{11}) \geq 6$. Now, $M_1 = \{v_1v_2, v_7v_8\}$;

$M_2 = \{v_2v_3, v_8v_9\}$; $M_3 = \{v_3v_4, v_9v_{10}\}$;$M_4 = \{v_4v_5\}$; $M_5 = \{v_5v_6, v_{10}v_{11}\}$; $M_6 = \{v_6v_7, v_{11}v_1\}$ are the color classes of C_{11}. Therefore, $\chi_{ve}(C_{11}) \leq 6$. Hence, $\chi_{ve}(C_{11}) = 6$.

3. χ_{ve} of tree

Proposition 3.1. *For any tree T,* $\chi_{ve}(T) = \Delta_{ve}(T)$.

Proof. Let T be a tree. Let v be a vertex of maximum ve-degree Δ_{ve}. Root at v. Color all the edges joining v to the vertives ve - dominated by v. Let v_1 be the descendent of v and v_1 have r_1 descendents. Let v_2 be a descendent of v_1 and v_2 have r_2 descendents. The line joins v_2 to its r_2 descendents are not yet colored. If $r_2 > \Delta_{ve} - (r_1 + 1)$, then $r_1 + r_2 + 1 > \Delta_{ve}$. The ve - degree of $v_2 \geq r_1 + r_2 + 1 > \Delta_{ve}$. Therefore, $deg_{ve}(v_2) > \Delta_{ve}$, a contradiction. Thus, $r_2 \leq \Delta_{ve} - (r_1 + 1)$. But all the lines joining these r_1 descendents of v_1 to v_1 and the line vv_1 are colored with $r_1 + 1$ colors. Also the remaining $\Delta_{ve} - (r_1 + 1)$ ve-neighbors of v are not dominated by v_2. By using these $\Delta_{ve} - (r_1 + 1)$ colors, color the edges joining v_2 to its r_2 descendents,since $r_2 \leq \Delta_{ve} - (r_1 + 1)$. Thus all the lines joining the descendents v_{1_j} of v_1 to the descendents of v_{1_j}. Hence all the edges dominated by v_j is colored with atmost Δ_{ve} edges. Hence $\chi_{ve}(T) \leq \Delta_{ve}(T)$. Since $\chi_{ve}(T) \geq \Delta_{ve}(T)$, $\chi_{ve}(T) = \Delta_{ve}(T)$.

4. Conclusion

In this paper, some basic results and some characterization theorems on *ve*-chromatic number have been studied. In the forthcoming papers, the *ve*-chromatic number of some more special types of graphs and the relation connecting *ve*-chromatic number and the domination, chromatic index will be studied.

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