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ss-EXCELLENCE IN GRAPHS

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Abstract: Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A subset S of $V(G)$ is called a semi-strong set abbreviated as ss-set if $|N[v] \cap S| \leq 1$ for all v in $V(G)$. This concept was introduced by E. Sampathkumar in the paper titled Semi-strong chromatic number of a graph. Any ss-set has hereditary property. That is, a subset of an ss-set is an ss-set. So, an ss-set is maximal iff for any $u \in (V-S)$, there exists $v \in V(G)$, $v \neq u$ such that v is adjacent with u and a vertex of S. Excellence is studied with respect to several parameters like domination. A vertex u is α-good with respect to the parameter α if u belongs to a minimum (maximum) α-set of G. A graph G is α-excellent if every vertex of G is α-good. A graph G is ss - excellent if every vertex of G is ss - good. ss - excellence and ss - just excellence are studied in this paper.

Keywords and Phrases: Semi-strong set, semi-strong partition, excellent, just-excellent.

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1. Introduction

As a generalization of strong set introduced by Claude Berge [2]. E. Sampathkumar defined semi-strong sets in a graph.In a simple graph G, a subset S

of the vertex set $V(G)$ of G is called a semi-strong set of G if $|N[v] \cap S| \leq 1$ for v in $V(G)$. A is semi-strong set has hereditary property. Hence maximum ss-set considered. A vertex u is ss-good if u belongs to a maximum ss-set of G. A graph G is said to be ss-excellent if every vertex of G is ss-good. ss-excellence and ss-just excellence are studied in this paper.

Definition 1.1. [9] *A subset S of $V(G)$ is said to be semi-strong if for every vertex $v \in V$, $|N(v) \cap S| \leq 1$ (or no two vertices of S have the same neighbour in V, that is, no two vertices of S are joined by a path of length two in V). The minimum cardinality of a semi-strong partition of G is called the semi-strong chromatic number of G and is denoted by $\chi_s(G)$.*

Definition 1.2. *A subset S of $V(G)$ is called a maximal semi-strong set of G if S is semi-strong and no proper super set of S is semi-strong. The maximum cardinality of a maximal semi-strong set of G is called semi-strong number of G and is denoted by $ss(G)$.*

Definition 1.3. *A vertex u is ss-good if u belongs to a maximum ss-set of G. A graph G is said to be ss-excellent if every vertex of G is ss-good.*

Example 1.1. (i) K_n is ss-excellent, for all $n \geq 1$.
(ii) $K_{1,n}$ is ss-excellent.
(iii) $K_{m,n}$ is ss-excellent.

Theorem 1.1. *P_n is ss-excellent if and only if $n \equiv 0$ $(mod\ 4)$.*
Proof. Let $n \equiv 0$ $(mod\ 4)$. Let $n = 4k$. $ss(P_n) = 2k$. ss-sets of P_{4k} are $\{u_1, u_2, u_5, u_6, \ldots, u_{4k-3}, u_{4k-2}\}$; $\{u_2, u_3, u_6, u_7, \ldots, u_{4k-2}, u_{4k-1}\}$; $\{u_3, u_4, u_7, u_8, \ldots, u_{4k-3}, u_{4k-2}\}$. P_{4k} is ss-excellent.

Let $n \equiv 1$ $(mod\ 4)$. Let $n = 4k+1$. $ss(P_{4k+1}) = \lceil \frac{4k+1}{2} \rceil = 2k+1$. The bad vertices are $u_3, u_7, \ldots, u_{4k-1}$.

Let $n \equiv 2$ $(mod\ 4)$. Let $n = 4k+2$. $ss(P_{4k+2}) = \frac{n}{2} + 1 = 2k+2$. The bad vertices are $u_3, u_4, u_7, u_8, \ldots, u_{4k-1}, u_{4k}$.

Let $n \equiv 3$ $(mod\ 4)$. Let $n = 4k+3$. $ss(P_{4k+3}) = \lceil \frac{4k+3}{2} \rceil = 2k+2$. The bad vertices are $u_4, u_8, \ldots, u_{4k}$.

Theorem 1.2. *C_n is ss-excellent, for every $n \geq 3$.*
Proof. Let $n \equiv 2$ $(mod\ 4)$. Let $n = 4k+2$. $ss(C_{4k+2}) = \frac{4k+2}{2} - 1 = 2k$. Let $V(C_{4k+2}) = \{u_1, u_2, \ldots, u_{4k+2}\}$, $k \geq 1$. Let $S_1 = \{u_1, u_2, u_5, u_6, \ldots, u_{4k-3}, u_{4k-2}\}$. $|S_1| = 2k$ and S_1 is a ss-set of C_{4k+2}. By rotating the vertices in S_1, it can be shown that every vertex is ss-good. Hence C_{4k+2} is ss-excellent.

Let $n \equiv 1$ $(mod\ 4)$. Let $n = 4k+1$. Then $ss(C_{4k+1}) = \frac{4k+1}{2} = 2k$. Let $S_2 = \{u_1, u_2, u_5, u_6, \ldots, u_{4k-3}, u_{4k-2}\}$. $|S_2| = 2k$ and S_2 is a ss-set of C_{4k+1}. By rotating the vertices in S_2, it can be shown that every vertex is ss-good. Hence C_{4k+1} is ss-excellent.

Let $n \equiv 3\ (mod\ 4)$. Let $n = 4k+3$. Then $ss(C_{4k+3}) = \frac{4k+3}{2} = 2k+1$. Let $S_3 = \{u_1, u_2, u_5, u_6, \ldots, u_{4k-3}, u_{4k-2}, u_{4k+1}\}$. $|S_2| = 2k$ and S_3 is a ss-set of C_{4k+3}. By rotating the vertices in S_3, it can be shown that every vertex is ss-good. Hence C_{4k+3} is ss-excellent.

Let $n \equiv 0\ (mod\ 4)$. Let $n = 4k$. Then $ss(C_{4k}) = \frac{4k}{2} = 2k$. Let $S_4 = \{u_1, u_2, u_5, u_6, \ldots, u_{4k-3}, u_{4k-2}\}$. $|S_4| = 2k$ and S_4 is a ss-set of C_{4k}. By rotating the vertices in S_4, it can be shown that every vertex is ss-good. Hence C_{4k} is ss-excellent.

Observation 1.1. *(i) W_n is ss-excellent, since $ss(W_n) = 1$.*
(ii) $K_{a_1,a_2,\ldots,a_m}$ is ss-excellent, since $ss(K_{a_1,a_2,\ldots,a_m}) = 1$.
(iii) Petersen graph P is ss-excellent, since $ss(P) = 2$.

Observation 1.2. *$ss(K_m(a_1, a_2, \ldots, a_m)) = m$. Any ss-set of $G = K_m(a_1,\ a_2,\ \ldots,\ a_m)$ consists of one pendent vertex each at every vertex of K_m. The vertices of K_m are ss-bad. Therefore for $m \geq 2$, $K_m(a_1, a_2, \ldots, a_m)$ is not ss-excellent.*

Theorem 1.3. *A vertex transitive graph is ss-excellent.*
Proof. Let G be a vertex transitive graph. Let S be a ss-set of G. Let $u \notin S$. Select any vertex v in S. As G is vertex-transitive, there exists an automorphism φ of G which maps v into u. Let $S' = \{\varphi(w) : w \in S\}$. Since S is a ss-set, S' is a ss-set of G. Since $\varphi(v) = u$, $u \in S'$. Therefore u is ss-good. That is, G is ss-excellent.

Theorem 1.4. *Suppose G has a unique ss-set. Then G is ss-excellent if and only if every component of G is either K_1 or K_2.*
Proof. Suppose G has a unique ss-set say S. If S is a proper subset of $V(G)$, then there will be ss-bad vertices. Suppose G is ss-excellent. Then $S = V(G)$ and hence $ss(G) = n$. Therefore every component of G is either K_1 or K_2.

The converse is obvious.

Theorem 1.5. *Let G be a non-ss-excellent graph. Then there exists a ss-excellent graph H such that G is an induced subgraph of H.*
Proof. Let G be a non-ss-excellent graph. Attach a P_3 with an edge at every vertex of G. Let H be the resulting graph. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$.

Let $V(H) = \{u_1, u_2, \ldots, u_n, u_{1,1}, u_{1,2}, u_{1,3}, u_{2,1}, u_{2,2}, u_{2,3}, \ldots, u_{n,1}, u_{n,2}, u_{n,3}\}$ where $u_{i,1},\ u_{i,2},\ u_{i,3}$ is a P_3 attached with u_i by an edge, $(1 \leq i \leq n)$. Then $S = \{u_{1,2},\ u_{1,3},\ u_{2,2}, u_{2,3},\ \ldots,\ u_{n,2},\ u_{n,3}\}$ is a ss-set of H and $ss(H) = 2n$. Also $S_1 = \{u_1, u_{1,3}, u_{2,2}, u_{2,3}, \ldots, u_{n,2}, u_{n,3}\}$, $S_i = \{u_i, u_{i,3}, u_{1,2},\ u_{1,3}, u_{j,2},\ u_{2,2}, u_{2,3},\ \ldots, u_{j,3},$

$\dots, u_{n,2},\ u_{n,3}\}$, $j \neq i$, $(2 \leq j \leq n)$, $(1 \leq i \leq n)$, are ss-sets of H. Therefore H is ss-excellent and G is an induced subgraph of H.

Remark 1.1. *$ss(H) = 2n$ in the above construction.*

Theorem 1.6. *Let G be a graph. Then $ss(G) = n - 1$ if and only if there exists exactly one P_3 component and other components are either K_1 or K_2.*

Proof. Let $ss(G) = n-1$. Let $V(G) = \{u_1, u_2, \dots, u_n\}$. Let $S = \{u_1, u_2, \dots, u_{n-1}\}$ be a ss-set of G. Any component of S is either K_1 or K_2. Also, $|N(u_n) \cap S| \leq 1$. If u_n is not adjacent with any vertex of S, then $S \cup \{u_n\}$ is a ss-set of G, a contradiction. If u_n is adjacent with exactly one K_1 component of S, then again $S \cup \{u_n\}$ is a ss-set of G, a contradiction. If u_n is adjacent with exactly one K_2 component of S, then $S \cup \{u_n\}$ contains exactly one P_3. Therefore every component of G is either K_1 or K_2 or P_3 (the P_3 component being unique).

The converse is obvious.

Illustration 1.1. *Let G be the graph shown in Figure 1. ss-sets of G are $\{u_1, u_2, u_4, u_5, u_6, u_7\}$, $\{u_2, u_3, u_4, u_5, u_6, u_7\}$.*

Figure 1: A graph G with $ss(G) = n - 1$

Remark 1.2. *Any graph G with $ss(G) = n - 1$ is ss-excellent.*

Theorem 1.7. *$ss(G) = n-2$ if and only if G has one of the following components:*
(i). two P_3 components
(ii). one P_4 component
(iii). one $K_{1,3}$ component
(iv). one C_4 component
(v). one P_5 component
(vi). one C_3 component
(vii). one C_3 with a pendent.

Proof. Let $ss(G) = n-2$. Let $V(G) = \{u_1, u_2, \dots, u_n\}$. Let $S = \{u_1, u_2, \dots, u_{n-2}\}$ be a ss-set of G. Then $|N(u_{n-1} \cap S| \leq 1$ and $|N(u_n) \cap S| \leq 1$. If $|N(u_{n-1} \cap S| = 0$ or $|N(u_n) \cap S| = 0$, then $S \cup \{u_{n-1}\}$ or $S \cup \{u_n\}$ is a ss-set of G, a contradiction. Therefore u_{n-1} and u_n are adjacent with one vertex of S. If u_{n-1} or u_n is adjacent with a K_1 component of S, then $ss(G) = n - 1$, a contradiction. Therefore u_{n-1} is adjacent with exactly one vertex of exactly one K_2 component. Moreover u_n

or u_{n-1}, u_n are adjacent and u_{n-1} is adjacent with a K_1 and u_n is adjacent with exactly one vertex of a K_2. That is G contains exactly one of the following: two P_3 components or exactly one P_4 component or exactly one $K_{1,3}$ component or exactly one C_3 (provided u_{n-1}, u_n are adjacent) or a triangle with a pendent vertex or C_4 or P_5.

The converse is obvious.

Illustration 1.2. *Let G_i, $(1 \leq i \leq 7)$ be the graphs given in Figure 2.*

ss-sets of G_1 are

$\{u_1, u_2, u_3, u_4, u_5, u_6, u_8, u_9\}$,
$\{u_1, u_2, u_3, u_4, u_6, u_7, u_9, u_{10}\}$,
$\{u_1, u_2, u_3, u_4, u_5, u_6, u_9, u_{10}\}$,
$\{u_1, u_2, u_3, u_4, u_6, u_7, u_8, u_9\}$.

ss-sets of G_2 are

$\{u_1, u_2, u_3, u_4, u_5, u_6\}$,
$\{u_1, u_2, u_3, u_4, u_7, u_8\}$,
$\{u_1, u_2, u_3, u_4, u_6, u_7\}$,
$\{u_1, u_2, u_3, u_4, u_5, u_8\}$.

ss-sets of G_3 are

$\{u_1, u_2, u_3, u_4, u_5, u_6\}$,
$\{u_1, u_2, u_3, u_4, u_5, u_7\}$,
$\{u_1, u_2, u_3, u_4, u_5, u_8\}$.

ss-sets of G_4 are

$\{u_1, u_2, u_3, u_4, u_5, u_6\}$,
$\{u_1, u_2, u_3, u_4, u_6, u_7\}$,
$\{u_1, u_2, u_3, u_4, u_7, u_8\}$,
$\{u_1, u_2, u_3, u_4, u_5, u_8\}$.

ss-sets of G_5 are

$\{u_1, u_2, u_3, u_4, u_5\}$,
$\{u_1, u_2, u_3, u_4, u_6\}$,
$\{u_1, u_2, u_3, u_4, u_7\}$.

ss-sets of G_6 is

$\{u_1, u_2, u_3, u_4, u_5, u_8\}$.

ss-sets of G_7 are

$\{u_1, u_2, u_3, u_4, u_5, u_8, u_9\}$,
$\{u_1, u_2, u_3, u_4, u_5, u_6, u_9\}$.

Here G_1, G_2, G_3, G_4, G_5 are ss-excellent but G_6 and G_7 are not ss-excellent.

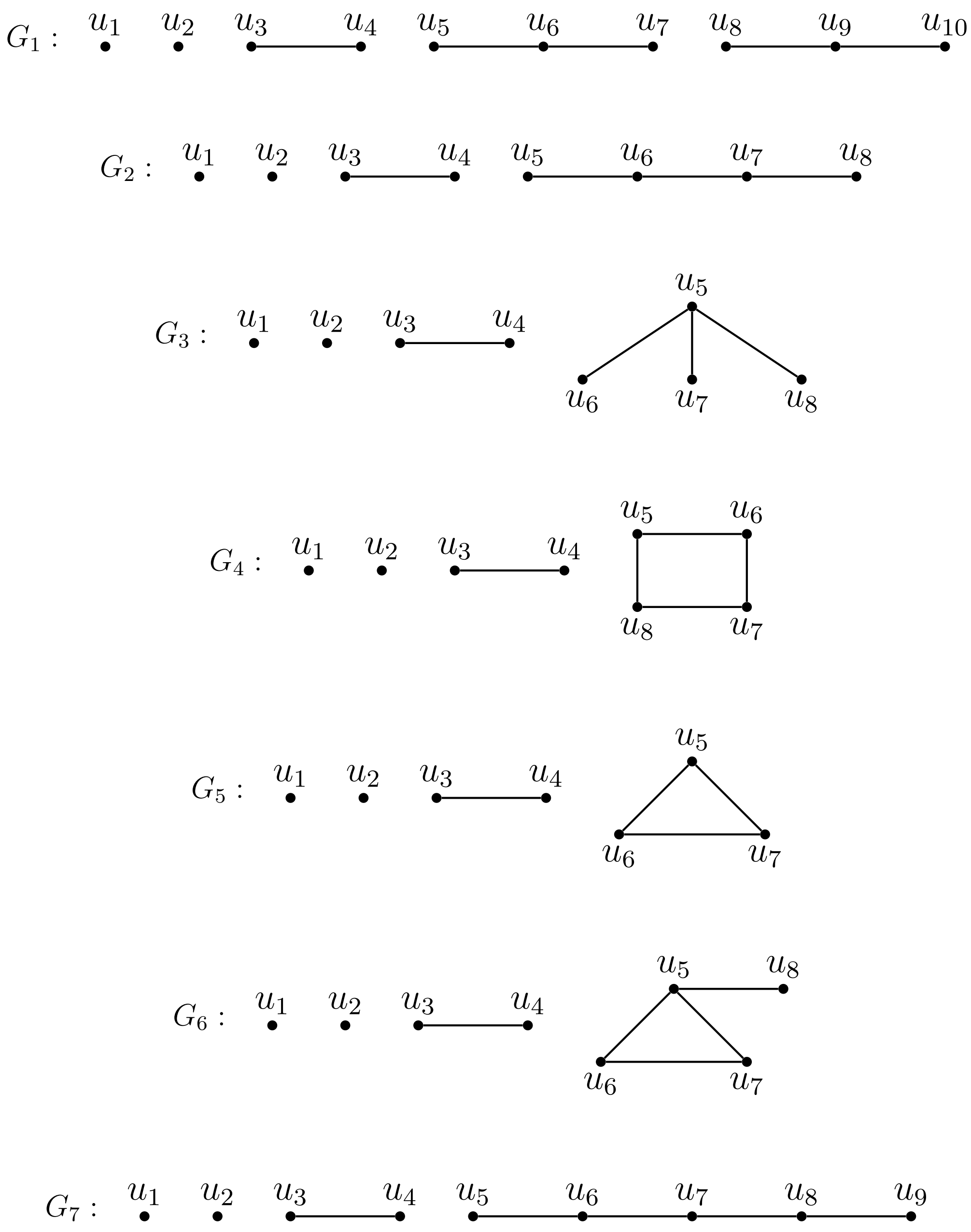

Figure 2: Set of Graphs $G_1, G_2, \ldots, G_7$ for which $ss(G) = n - 2$

Corollary 1.1. *Let G be a graph with $ss(G) = n - 2$. Then G is ss-excellent if and only if*
(i) there exists exactly two components of P_3
(ii) one P_4 component
(iii) one $K_{1,3}$ components
(iv) one C_3 component
(v) one C_4 component.
In each case the remaining components are K_1 or K_2.
If G has a P_5 component, then G is not ss-excellent or a component with C_3 with a pendent.

2. ss-excellence of Graph Operations

Theorem 2.1. *Let G, H be ss-excellent graphs with $ss(H) = |V(H)|$. Then $G\Box H$ is ss-excellent and $ss(G\Box H) = n(r - s_1) + ls_1$, where s_1 is the number of K_2 components of G and l is the number of K_1 component of H and $r = ss(G)$.*

Proof. Let G and H be ss-excellent graphs. Let $V(G) = \{u_1, u_2, \ldots, u_m\}$ and $V(H) = \{v_1, v_2, \ldots, v_n\}$. Let $S_1 = \{w_1, w_2, \ldots, w_r\}$ and $S_2 = \{v_1, v_2, \ldots, v_n\}$ be the ss-sets of G and H respectively. Let $\{w_1, w_2\}$ be a K_2 component of S_1. Without loss of generality, let $v_1, v_2, \ldots, v_l$ be the K_1 components of H. Let $\{w_1, w_2\}, \{w_3, w_4\}, \ldots, \{w_r, w_{r_1+1}\}$ be the K_2 components of S_1. The remaining vertices of S_1 are K_1 components. Let $T = \{(w_1, v_1), (w_1, v_2), \ldots, (w_1, v_n), (w_2, v_1), \ldots, (w_2, v_l), (w_3, v_1), (w_3, v_2), \ldots, (w_3, v_n), (w_4, v_1), \ldots, (w_4, v_l), \ldots, (w_{r_1}, v_1), (w_{r_1}, v_2), \ldots, (w_{r_1}, v_n), (w_{r_1+1}, v_1), \ldots, (w_{r_1+1}, v_l), (w_{r_1+2}, v_1), \ldots, (w_{r_1+2}, v_n), \ldots, (w_r, v_1), \ldots, (w_r, v_n)\}$. Then T is a semi strong set of $G\Box H$ of maximum cardinality. Let $(u_i, v_j) \in V(G\Box H)$. Since G and H are ss-excellent. u_i belongs to a ss-set D of G. If u_i belongs to a K_2 component of D, then (u_i, v_j) belongs to a ss-set of $G\Box H$. If u_i is a K_1 component of D, then also (u_i, v_j) belongs to a ss-set of $G\Box H$. $G\Box H$ is ss-excellent. Then $|T| = (n + l)s_1 + n(r - 2s_1)$, where s_1 is the number of K_2 components of S_1. That is, $|T| = nr + ls_1 - ns_1 = n(r - s_1) + ls_1$. Therefore $ss(G\Box H) = n(r - s_1) + ls_1$.

Illustration 2.1. *Let G be the graph given in Figure 3.*
$S_1 = \{u_1, u_2, u_5, u_6, u_9\}$ and $S_2 = \{v_1, v_2, \ldots, v_{10}\}$.
Then $T = \{(u_1, v_1), (u_1, v_2), \ldots, (u_1, v_{10}), (u_2, v_1), \ldots, (u_2, v_4), (u_5, v_1), (u_5, v_2), \ldots, (u_5, v_{10}), (u_6, v_1), \ldots, (u_6, v_4), (u_9, v_1), (u_9, v_2), \ldots, (u_9, v_{10})\}$ and $|T| = 10+4+10+4+10 = 38$. Here $s_1 = 2$, $n = 10$, $r = 5$, $l = 4$. $n(r - s_1) + ls_1 = 10(3) + 8 = 38$.

Corollary 2.1. *The following graphs are ss-excellent:*
$C_n\Box(tK_2 \cup sK_1)$;
$K_n\Box(tK_2 \cup sK_1)$;

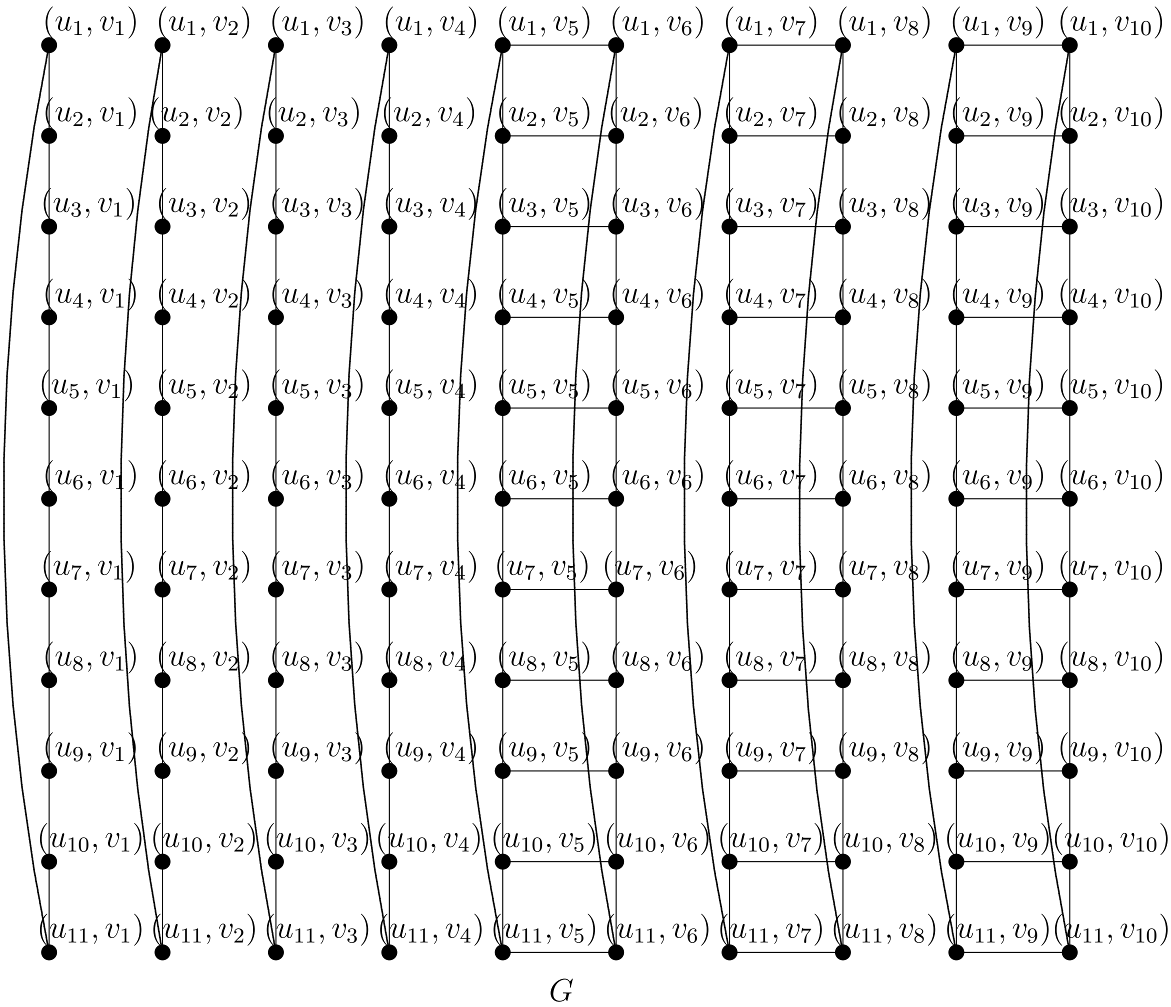

Figure 3: A graph for which $G \Box H$ is ss-excellent

$K_{1,n} \Box (tK_2 \cup sK_1)$;
$P_n \Box (tK_2 \cup sK_1)$, ($n \equiv 0 \ (mod\ 4)$);
$K_{m,n} \Box (tK_2 \cup sK_1)$;
$W_n \Box (tK_2 \cup sK_1)$;
$K_{a_1,a_2,\ldots,a_n} \Box (tK_2 \cup sK_1)$.

Theorem 2.2. *Let G and H be ss-excellent graphs. Then $G \Box H$ is ss-excellent.*
Proof. Let G and H be ss-excellent graphs. Let $V(G) = \{u_1, u_2, \ldots, u_m\}$ and $V(H) = \{v_1, v_2, \ldots, v_n\}$. Let $S_1 = \{w_1, w_2, \ldots, w_r\}$ and $S_2 = \{x_1, x_2, \ldots, x_s\}$ be

ss-sets of G and H respectively. Let $\{w_1, w_2\}, \{w_3, w_4\}, \ldots, \{w_{r_1}, w_{r_1+1}\}$ be the K_2 components of S_1 and $\{x_1, x_2\}, \{x_3, x_4\}, \ldots, \{x_{s_1}, x_{s_1+1}\}$ be the K_2 components of S_2.

Then $T = \{(w_1, x_1), (w_1, x_2), (w_1, x_3), (w_1, x_4), \ldots, (w_1, x_{s_1}), (w_1, x_{s_1+1}), (w_1, x_{s_1+2}), \ldots, (w_1, x_s), (w_3, x_1), (w_3, x_2), (w_3, x_3), (w_3, x_4), \ldots, (w_3, x_{s_1}), (w_3, x_{s_1+1}), (w_3, x_{s_1+2}), \ldots, (w_3, x_s), \ldots, (w_{r_1}, x_1), (w_{r_1}, x_2), (w_{r_1}, x_3), (w_{r_1}, x_4), \ldots, (w_{r_1}, x_{s_1}), (w_{r_1}, x_{s_1+1}), (w_{r_1}, x_{s_1+2}), \ldots, (w_{r_1}, x_s), (w_{r_1+2}, x_1), (w_{r_1+2}, x_2), \ldots, (w_{r_1+2}, x_s), (w_r, x_1), (w_r, x_2), (w_r, x_3), (w_r, x_4), \ldots, (w_r, x_s)\}$ is clearly a semi strong set of $G \Box H$ of maximum cardinality. Also any vertex (u_i, v_j) belongs to a ss-set of $G \Box H$. Therefore $G \Box H$ is ss-excellent. Also,

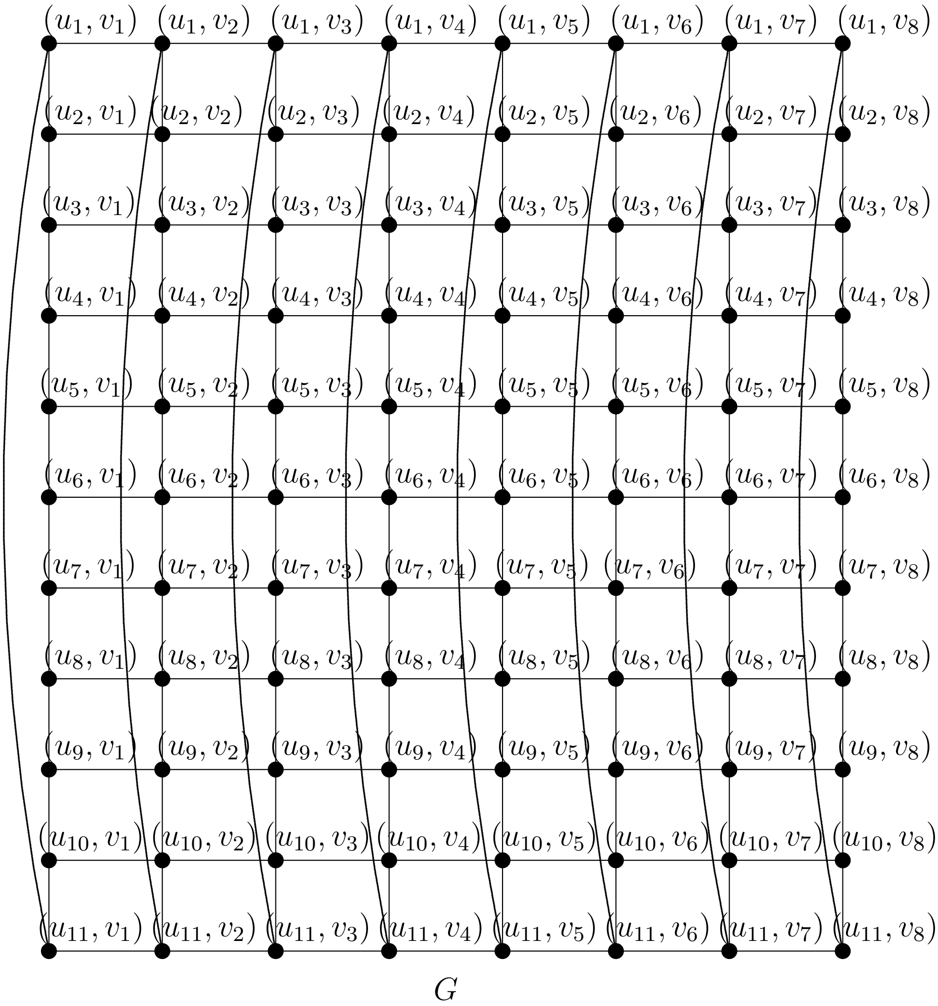

Figure 4: $G = C_{11} \Box P_8$

$$\begin{aligned} ss(G\Box H) &= \left(\frac{r_1+1}{2}\right)s + (r-(r_1+1))s \\ &= \frac{r_1 s}{2} + \frac{s}{2} + rs - r_1 s - s \\ &= rs - \frac{r_1 s}{2} - \frac{s}{2} \\ &= s\left(r - \left(\frac{r_1+1}{2}\right)\right) \end{aligned}$$

Illustration 2.2. *Let $G = C_{11}\Box P_8$ be the graph shown in Figure 4.*
$S_1 = \{u_1, u_2, u_5, u_6, u_9\}$, $S_2 = \{v_1, v_2, v_5, v_6\}$. *Let* $T = \{(u_1, v_1), (u_1, v_2), (u_1, v_5), (u_1, v_6), (u_5, v_1), (u_5, v_2), (u_5, v_5), (u_5, v_6), (u_9, v_1), (u_9, v_2), (u_9, v_5), (u_9, v_6)\}$.
Then $|T| = 12$. *Here* $r_1 = 3$, $r = 5$, $s = 4$.
Therefore, $s\left(r - \left(\frac{r_1+1}{2}\right)\right) = 4\left(5 - \left(\frac{3+1}{2}\right)\right) = 4(5-2) = 12$.

3. Just ss-excellence in Graphs

Definition 3.1. *A graph G is just ss-excellent if every vertex belongs to a unique ss-set of G.*

Example 3.1. K_n, W_n, $K_{a_1,a_2,\ldots,a_n}$, $n \geq 3$, F_n, $tK_2 \cup sK_1$ are just ss-excellent.

Remark 3.1. (i) *If $ss(G) = 1$, then G is just ss-excellent.*
(ii) *P_n, $n \geq 3$ and C_n, $n \geq 4$ are not just ss-excellent.*

Theorem 3.1. *Let G be a just ss-excellent graph. Then*
(i) *G is ss-excellent.*
(ii) *there exists a unique partition of $V(G)$ into ss-sets of G.*
(iii) $|V(G)| = \chi_s(G).ss(G)$.

Proof. Let G be a just ss-excellent graph.

(i). The result is obvious.

(ii). Let $u \in V(G)$. By hypothesis there exists a unique ss-set S_1 of G containing u. If $V - S_1 = \phi$, then the result is true. Suppose $V - S_1 \neq \phi$. Let $v \in V - S_1$. Therefore there exists a unique ss-set S_2 of G containing v. Since G is just ss-excellent, $S_1 \cap S_2 = \phi$. If $S_1 \cup S_2 = V$, then the result is true. Suppose $S_1 \cup S_2 \subsetneq V$. Then there exists $w \in V - (S_1 \cup S_2)$ and there exists a unique ss-set S_3 of G containing w. S_1, S_2, S_3 are pairwise disjoint. Proceeding in this way, after a finite number of steps, V can be partitioned into ss-sets of G. Suppose Π_1 and Π_2 are two distinct partitions of $V(G)$ into ss-sets of G. Then there exists a vertex $u \in V(G)$ which belongs to more than one ss-set of G, a contradiction. Therefore (ii) follows.

(iii). From (ii), $V = S_1 \cup S_2 \cup \ldots \cup S_k$, where each S_i is a ss-set of G and $S_1, S_2, \ldots, S_k$ are pairwise disjoint. Since $|S_i| = ss(G)$, $1 \le i \le k$, $n = ss(G)$. Therefore $\chi_s(G) \le k$. Suppose Π is a χ_s-partition of G into semi strong sets. Let $\Pi = \{T_1, T_2, \ldots, T_{\chi_s(G)}\}$. $|T_i| \le ss(G)$, $(1 \le i \le \chi_s(G))$.

Therefore $n = \bigcup_{i=1}^{\chi_s(G)} |T_i| \le \chi_s(G).ss(G)$. Therefore $\frac{n}{ss(G)} \le \chi_s(G)$. That is, $k \le \chi_s(G)$. But $\chi_s(G) \le k$. Therefore $\chi_s(G) = k$. Therefore $|V(G)| = ss(G).\chi_s(G)$.

Remark 3.2. *If $|V(G)| = \chi_s(G).ss(G)$, then G need not be just ss-excellent. For: let $G = C_6$. $ss(G) = 2$, $\chi_s(G) = 3$. Therefore $|V(G)| = \chi_s(G).ss(G)$. But C_6 is not just ss-excellent, since any vertex of C_6 belongs to two ss-sets of C_6.*

Theorem 3.2. *If G is just ss-excellent and $ss(G) < n$, then G has no isolates.*
Proof. Suppose G has an isolate. As G is just ss-excellent, $V(G)$ is an ss-set of G. Therefore $ss(G) = n$, a contradiction. Therefore G has no isolates.

Corollary 3.1. *Suppose G is just ss-excellent and $\chi_s(G) > 1$. Then G has no isolates.*
Proof. Since $n = \chi_s(G).ss(G)$ and since $\chi_s(G) > 1$, $ss(G) < n$. Therefore G has no isolates.

Corollary 3.2. *If G is just ss-excellent and G is not the union of K_1 or K_2, then G has no isolates.*
Proof. Since G is not the union of K_1 or K_2, $ss(G) < n$. Therefore G has no isolates.

Problem: Construct a connected graph G which is just ss-excellent and $ss(G) = k \ge 2$.

Theorem 3.3. *Let G be a graph without isolates. Then G is an induced subgraph of a just ss-excellent graph H.*
Proof. Let G be a graph without isolates. Add a vertex w and make w adjacent with every vertex of G. Let H be the resulting graph. Then $diam(H) \le 2$ and every edge of H is on a triangle. Therefore $N(H) = K_{n+1}$ and $ss(H) = 1$. Therefore H is a just ss-excellent graph containing G as an induced subgraph.

Illustration 3.1. *P_5 is not ss-excellent and hence not just ss-excellent, but P_5+K_1, a fan, is just ss-excellent.*

Theorem 3.4. *Let G and H be just ss-excellent graphs. $G \cup H$ is just ss-excellent if and only if every component of G and H are either K_1 or K_2.*
Proof. Suppose $G \cup H$ is just ss-excellent. Any ss-set of $G \cup H$ is of the form $S_1 \cup S_2$ where S_1 is an ss-set of G and S_2 is an ss-set of H. Since $G \cup H$ is just

ss-excellent, G and H have exactly one ss-set. Since G and H are just ss-excellent, there exists a unique partition of G (or H) into ss-sets of G (or H). Therefore $ss(G) = n$ and $ss(H) = n$. Therefore every component of G and H are either K_1 or K_2. The converse is obvious.

Theorem 3.5. *Let G and H be two graphs. $G + H$ is just ss-excellent if and only if G or H has no isolates.*

Proof. Suppose $G + H$ is just ss-excellent. Suppose $ss(G) \geq 2$. Let S_1 be a ss-set of G. Then S_1 is not a semi strong set of $G + H$. Let T be a ss-set of $G + H$. Let $T \cap V(G) = k_1$ and $T \cap V(H) = k_2$. If k_1 or $k_2 \geq 2$, then T is not a ss-set of $G + H$. Therefore $k_1 \leq 1$, $k_2 \leq 1$. Suppose G has at least two vertices. If G has no isolates, then any edge of $G + H$ will not give rise to a semi strong set. Therefore $ss(G + H) = 1$. If G has an isolate or H has an isolate, each non-isolate of G constitute a semi strong set of G and each non-isolate of G lies in a semi strong set of G of cardinality 2. Therefore $G + H$ will not be just ss-excellent. Hence G or H does not have isolates. The converse is obvious.

Illustration 3.2. *Let $G_1 = P_3 + \overline{K_2}$, $G_2 = P_3 + K_2$ be shown in Figure 5. In both the cases, P_3 has no isolates. $ss(G_1) = ss(G_2) = 1$. Therefore G_1 and G_2 are just ss-excellent.*

Figure 5: A set of just ss-excellent graph G_1 and G_2

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