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# A note on Bailey pairs and q-series identities

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**Abstract:** In this paper, making use of Barley pairs and a general transformation formula we have established interesting q-series identities.
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## 1. Introduction

As usual, we employ the notations,

$$(a;q)_n = (1-a)(1-aq)...(1-aq^{n-1}), \qquad n \geq 1,$$

$$(a;q)_0 = 1, \qquad (a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^r)$$

and

$$(a_1, a_2, ..., a_r; q)_n = (a-1;q)_n (a_2;q)_n ... (a_r;q)_n.$$

An ${}_r\Phi_s$ basic hypergeometric series

$${}_r\Phi_s\left[\begin{array}{l} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s \end{array}\right] = \sum_{n=0}^{\infty} \frac{[a_1, a_2, ..., a_r; q]_n z^n}{[q, b_1, b_2, ..., b_s; q]_n}, \qquad |z| < 1$$

Bailey is 1947 showed that

If $\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{[q;q]_{n-r}[aq;q]_{n+r}}$ and $\gamma_n = \sum_{r=n}^{\infty} \frac{\delta_r}{[q;q]_{r-n}[aq;q]_{r+n}}$

then

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n. \tag{1.1}$$

If we take $\delta_r = [\delta_1; q]_r [\delta_2; q]_r \left(\frac{aq}{\delta_1\delta_2}\right)^r$ Then we find.

$$\gamma_n = \frac{1}{[aq;q]_{2n}} \sum_{r=0}^{\infty} \frac{[\delta_1;q]_{2+n}[\delta;q]_{r+n}\left(\frac{aq}{\delta_1\delta_2}\right)^{2+n}}{[q;q]_r[aq^{2n+1};q]_r}$$

$$= \frac{[\delta_1;q]_n[\delta_r;q]_n\left(\frac{aq}{\delta_1\delta_2}\right)^n}{[aq;q]_{2n}} {}_2\Phi_1\left[\begin{array}{l} \delta_1 q^n, \delta_r q^n; q\frac{aq}{\delta_1\delta_2} \\ aq^{2n+1} \end{array}\right] \quad (1.2)$$

Now, using the q-analogue of Gauss summation formula, viz.,

$${}_2\Phi_1\left[\begin{array}{l} a, b; q; c/ab \\ c \end{array}\right] = \frac{[c/a, c/b; q]_\infty}{[c, c/ab; q]_\infty}, \quad (1.3)$$

we have

$$\gamma_n = \frac{[\delta_1;q]_n[\delta_2;q]_n\left(\frac{aq}{\delta_1\delta_2}\right)^n}{\left[\frac{aq}{\delta_1}, \frac{ar}{\delta_2}; q\right]_n} \times \frac{\left[\frac{aq}{\delta_1}, \frac{ar}{\delta_2}; q\right]_\infty}{\left[aq, \frac{ar}{\delta_1\delta_2}; q\right]_\infty}$$

Now, putting these values of $\delta_n$ and $\gamma_n$ in (1.1) we have
If

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{[q;q]_{n-r}[aq;q]_{n+r}} \quad (1.4)$$

Then

$$\sum_{n=0}^{\infty}[\delta_1, \delta_2; q]_n \left(\frac{aq}{\delta_1\delta_2}\right)^n \beta_n = \frac{\left[\frac{aq}{\delta_1}, \frac{aq}{\delta_r}; q\right]_\infty}{\left[aq, \frac{ar}{\delta_1\delta_2}; q\right]_\infty} \times$$

$$\times \sum_{n=0}^{\infty} \frac{[\delta_1, \delta_r; q]_n \left(\frac{aq}{\delta_1\delta_2}\right)^n}{\left[\frac{aq}{\delta_1}, \frac{ar}{\delta_2}; q\right]_n} \alpha_n. \quad (1.5)$$

The sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying (1.4) are called Bailey pair.
It was pointed out by Bailey (1949) that if the following summation formula is used

$${}_2\Phi_1\left[\begin{array}{l} a, b; q; c/ab \\ cq \end{array}\right] = \frac{[cq/a, cq/b; q]_\infty}{[cq, cq/ab; q]_\infty} \left\{\frac{ab(1+c) - c(a+b)}{ab - c}\right\}, \quad (1.6)$$

instead of (1.3) we get the identity,

If $\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{[q;q]_{n-r}[aq;q]_{n+r}}$

Then

$$\sum_{n=0}^{\infty}[\delta_1,\delta_r;q]_n\left(\frac{a}{\delta_1\delta_r}\right)^n\beta_n = \frac{\left[\frac{aq}{\delta_1},\frac{aq}{\delta_2};q\right]_\infty}{\left[aq,\frac{aq}{\delta_1\delta_2};q\right]_\infty}\times$$

$$\times\sum_{n=0}^{\infty}\frac{[\delta_1,\delta_r;q]_n}{\left[\frac{aq}{\delta_1},\frac{aq}{\delta_2};q\right]_n}\left(\frac{a}{\delta_1\delta_r}\right)^n\left[\frac{\delta_1\delta_2(1+aq^{2n})-aq^n(\delta_1+\delta_2)}{\delta_1\delta_2-a}\right]\alpha_n. \tag{1.8}$$

we shall make use of the following Bailey pairs in our analysis.

**(i)** The simplest Bailey pair is

$$\beta_n = \begin{matrix} 1 & if \quad n=0 \\ 0 & if \quad n>0 \end{matrix} \tag{1.9}$$

$$\alpha_n = \frac{(1-aq^{2n})[a;q]_n(-)^n q^{n(n-1)/2}}{(1-a)[q;q]_n} \quad , n\geq 0. \tag{1.10}$$

**(ii)** If a =1 then

$$\alpha_n = \begin{matrix} (-)^n\left\{z^n q^{n(n-1)/2}+z^{-n}q^{n(n+1)/2}\right\} \quad n>0 \\ 1, n=0 \end{matrix} \tag{1.11}$$

and

$$\beta_n = \frac{[z,q/z;q]_n}{[q;q]_{2n}}, \qquad n\geq 0, \tag{1.12}$$

are Bailey pair.

**(iii)** Replacing q by $q^2$ and a by $q^2$ in (1.4) we obtain

$$\alpha_n = \frac{(-)^{n-1}\{z^{n+1}q^{n^2+n}-z^{-n}q^{n^2+n}\}}{(1-q^2)}, \quad n\geq 0, \tag{1.13}$$

and

$$\beta_n = \frac{[z;q^2]_{n+1}[q^2/z;q^2]_n}{[q^2;q^2]_{2n+1}}, \quad n\geq 0, \tag{1.14}$$

are Bailey pair. **(iii)** Bailey pair due to Bailey are;
**(a)** If

$$\alpha_0 = 1, \alpha_n = (-)^n(1 - aq^{2n})\frac{[aq;q]_{n-1}}{[q;q]_n}a^n q^{n(3n-1)/2} \quad for \quad n \geq 1, \tag{1.15}$$

then

$$\beta_n = \frac{1}{[q;q]_n}. \tag{1.16}$$

**(b)** If

$$\alpha_0 = 1, \alpha_n = (1 - aq^{2n})\frac{[aq;q]_{n-1}[b;q]_n a^n q^{n^2}}{[q;q]_n[aq/b;q]_n b^n} \quad for \quad n \geq 1, \tag{1.17}$$

then

$$\beta_n = \frac{[-aq^{n+1}/b;q]_n}{[q^2;q^2]_n[-aq;q]_{2n}[aq/b;q]_n}. \tag{1.18}$$

**(c)** If

$$\alpha_0 = 1, \alpha_n = (-)^n(1 - aq^{2n})\frac{[aq;q]_{n-1}}{[q;q]_n}a^n q^{n(3n-1)/2} \quad for \quad n \geq 1, \tag{1.19}$$

then

$$\beta_n = \frac{[aq;q]_{3n}}{[q^3;q^3]_n[a^3q^3;q^3]_n}, \qquad n \geq 1. \tag{1.20}$$

**(d)** If

$$\alpha_0 = 1, \alpha_{2n} = [aq^2;q^2]_{n+1}[f;q^2](1 - aq^{4n})(a/f)^n q^{2n^2}, \quad \alpha_{2n-1} = 0 \tag{1.21}$$

for $n \geq 1$, then

$$\beta_n = \frac{[aq/f;q^2]_n}{[q;q^2]_n[aq;q^2]_n[aq/f;q]_n}. \tag{1.22}$$

Following is the Bailey pair due to Slater.
**(e)** If

$$\alpha_0 = 1, \;\; \alpha_{3n-1} = -q^{6n^2-5n+1}, \;\; \alpha_{3n-2} = -q^{6n^2-7n+2}, \;\; \alpha_{3n} = q^{6n^2-n} + q^{6n^2+n} \tag{1.23}$$

for $n \geq 1$, then

$$\beta_n = \frac{1}{[q;q]_{2n}} \tag{1.24}$$

## 2. Main Results

**(i)** Using Bailey pair (1.9) and (1.10) in (1.8) we get,

$$\sum_{n=0}^{\infty} \frac{[\delta_1, \delta_2; q]_n}{\left[\frac{aq}{\delta_1}, \frac{aq}{\delta_2}; q\right]_n} \left(\frac{a}{\delta_1\delta_2}\right) \left[\frac{\delta_1\delta_2(1+aq^{2n}) - aq^n(\delta_1\delta_2)}{\delta_1\delta_2 - a}\right] \times$$

$$\times(-)^n \frac{[a;q]_n(1-aq^{2n})q^{n(n-1)/2}}{(1-a)[q;q]_n} = \frac{\left[aq, \frac{aq}{\delta_1\delta_2}; q\right]_\infty}{\left[\frac{aq}{\delta_1}, \frac{aq}{\delta_2}; q\right]_\infty}, \tag{2.1}$$

As $\delta_1\delta_2 \to \infty$, (2.1) yields,

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)}a^n(1+aq^{2n})(1-aq^{2n})(-)^n[a;q]_n q^{n(n-1)/2}}{(1-a)[q;q]_n} = [aq;q]_\infty. \tag{2.2}$$

Taking $a \to 1$ in (2.2) we find

$$2 + \sum_{n=1}^{\infty} (-)^n q^{3n(n-1)/2}(1+q^{2n})(1+q^n) = [q;q]_\infty. \tag{2.3}$$

Again, taking $a = -1$ in (2.2) we have

$$\sum_{n=0}^{\infty} q^{3n(n-1)/2} \frac{(1-q^{4n})[-1;q]_n}{[q;q]_n} = 2[-q;q]_\infty. \tag{2.4}$$

For a=q, (2.2) gives

$$\sum_{n=0}^{\infty} q^{n^2}(1-q^{4n+2})(-)^n q^{n(n-1)/2} = [q;q]_\infty$$

which can be written as

$$\sum_{n=0}^{\infty}(-)^n q^{(3n^2-n)/2} - q^2 \sum_{n=0}^{\infty}(-)^n q^{(3n^2+7n)/2} = [q;q]_\infty \tag{2.5}$$

**(ii)** Taking Bailey pair (1.11), (1.12) and putting them in (1.8) after replacing a by we get,

$$\sum_{n=0}^{\infty} [\delta_1, \delta_2; q]_n \left(\frac{1}{\delta_1\delta_2}\right)^n \frac{[z, q/z; q]_n}{[q;q]_{2n}} = \frac{\left[\frac{q}{\delta_1}, \frac{q}{\delta_2}; q\right]_\infty}{\left[q, \frac{q}{\delta_1\delta_2}; q\right]_\infty} \times$$

$$\Big[\sum_{n=1}^{\infty}\frac{[\delta_1,\delta_2;q]_n}{[\left[\frac{q}{\delta_1},\frac{q}{\delta_2};q\right]_n}\left(\frac{1}{\delta_1\delta_2}\right)^n\left\{\frac{\delta_1\delta_2(1+q^{2n})-q^4(\delta_1+\delta_2)}{\delta_1\delta_2-1}\right\}$$

$$(-)^n\left\{z^nq^{n(n-1)/2}+z^{-n}q^{n(n+1)/2}\right\}+\frac{2\delta_1\delta_2-(\delta_1+\delta_2)}{\delta_1\delta_2-1}\Big]. \tag{2.6}$$

As $\delta_1\delta_2\to\infty$, (2.6) yields,

$$\sum_{n=0}^{\infty}\frac{[z,q/z;q]_n}{[q;q]_{2n}}q^{n(n-1)}=\frac{1}{[q;q]_\infty}\times$$

$$\left[2+\sum_{n=1}^{\infty}(-)^nq^{n(n-1)}(1+q^{2n})\left\{z^nq^{n(n-1)/2}+z^{-n}q^{n(n+1)/2}\right\}\right] \tag{2.7}$$

Applying Jacobi's triple product identity in (2.7) we get,

$$\sum_{n=0}^{\infty}\frac{[z,q/z;q]_nq^{n(n-1)}}{[q;q]_{2n}}=\frac{1}{[q;q]_\infty}\left\{[q^3,z,q^3/z;q^3]_\infty+[q^3,zq^2,q/z;q^3]_\infty\right\}$$

$$=\frac{[q^3;q^3]_\infty}{[q;q]_\infty}\left\{[z,q^3/z;q^3]_\infty+[zq^2,q/z;q^3]_\infty\right\} \tag{2.8}$$

If we take z=-1 in (2.8) we find,

$$\sum_{n=0}^{\infty}\frac{[-1,-q;q]_nq^{n(n-1)}}{[q;q]_{2n}}=\frac{[q^3;q^3]_\infty}{[q;q]_\infty}\left\{\left[-1,-q^3;q^3\right]_\infty+[-q,-q^2;q^3]_\infty\right\}$$

$$=\frac{[q^3;q^3]_\infty}{[q;q]_\infty}\left\{2(-q^3;q^3)_\infty^2+[-q,-q^2;q^3]_\infty\right\}. \tag{2.9}$$

**(iii)** Using Bailey pair (1.15) and (1.16) in (1.8) we get,

$$\sum_{n=0}^{\infty}\frac{[\delta_1,\delta_2;q]_n}{\left[\frac{aq}{\delta_1},\frac{aq}{\delta_2};q\right]_n}\left(\frac{a}{\delta_1\delta_2}\right)^n\left\{\frac{\delta_1\delta_2(1+aq^{2n})-aq^n(\delta_1+\delta_2)}{\delta_1\delta_2-a}\right\}\times$$

$$(-)^n\frac{(1-aq^{2n})}{(1-aq^n)}\frac{[aq;q]_n}{[q;q]_n}a^nq^{n(3n-1)/2}=\sum_{n=0}^{\infty}\frac{[\delta_1,\delta_2;q]_n\left(\frac{a}{\delta_1\delta_2}\right)}{[q;q]_n}. \tag{2.10}$$

Taking $\delta_1\delta_2 \to \infty$ in (2.10) we obtain

$$\sum_{n=0}^{\infty} \frac{a^n q^{n(n-1)}}{[q;q]_n}$$

$$= \frac{1}{[aq;q]_\infty}\left[(1+a)+\sum_{n=1}^{\infty} a^n q^{n(n-1)}(1+aq^{2n})(-)^n(1-aq^{2n})\frac{[aq;q]_n a^n q^{n(3n-1)/2}}{[q;q]_n(1-aq^n)}\right]. \tag{2.11}$$

For a=q, (2.11) yields,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{[q;q]_n} = \frac{1}{[q;q]_\infty}\left[(1-q^2)+\sum_{n=1}^{\infty} q^{(5n^2+n)/2}(1-q^{4n+2})\right]$$

$$= \frac{1}{[q;q]_\infty}\left[\sum_{n=0}^{\infty} q^{n(5n+1)/2}(1-q^{4n+2})\right]. \tag{2.12}$$

We know that $\sum_{n=0}^{\infty} \frac{q^{n^2}}{[q;q]_\infty} = \frac{1}{[q,q^4;q^5]_\infty}$, so we get from (2.12)

$$\sum_{n=0}^{\infty} q^{n(5n+1)/2}(1-q^{4n+2}) = [q^2,q^3,q^5;q^5]_\infty. \tag{2.13}$$

For $a = q^2$, (2.11) yields

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{[q;q]_\infty} = \frac{1}{[q^3;q]_\infty}\left[(1+q^2)+\sum_{n=0}^{\infty} \frac{q^{5n(n+1)/2}(1-q^{4n+4})(-)^n(1-q^{n+1})}{(1-q)(1-q^2)}\right]$$

$$= \frac{1}{[q;q]_\infty}\left[(1-q)(1-q^4)+\sum_{n=1}^{\infty}(-)^n q^{5n(n-1)/2}(1-q^{n+1})(1-q^{4n+4})\right]$$

$$[q,q^4,q^5;q^5]_\infty = \sum_{n=0}^{\infty}(-)^n q^{5n(n+1)/2}(1-q^{n+1})(1-q^{4n+4}). \tag{2.14}$$

Taking a=1 in (2.11) we find,

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)}}{[q;q]_n} = \frac{1}{[q;q]_\infty}\left[2+\sum_{n=1}^{\infty}(-)^n q^{n(5n-3)/2}\frac{(1-q^{4n})}{(1-q^n)}\right]$$

$$= \frac{1}{[q;q]_\infty}\left[2+\sum_{n=1}^{\infty}(-)^n q^{n(5n-3)/2}(1+q^{2n})(1+q^n)\right]$$

$$= \frac{1}{[q;q]_\infty}\left[\sum_{n=-\infty}^{\infty} q^{n(5n-3)/2}+\sum_{n=-\infty}^{\infty} q^{n(5n-1)/2}\right]$$

$$= \frac{1}{[q;q]_\infty}\left\{[q,q^4,q^5;q^5]_\infty+[q^2,q^3,q^5;q^5]_\infty\right\}$$

$$\sum_{n=0}^{\infty}\frac{q^{n(n-1)}}{[q;q]_n} = \frac{1}{[q^2,q^3;q^5]_\infty}+\frac{1}{[q,q^4;q^5]_\infty}. \tag{2.15}$$

(2.15) can be written as

$$\sum_{n=0}^{\infty}\frac{q^{n(n-1)}}{[q;q]_n} = \sum_{n=0}^{\infty}\frac{q^{n^2}}{[q;q]_n}+\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{[q;q]_n} = \sum_{n=0}^{\infty}\frac{q^{n^2}(1+q^n)}{[q;q]_n} \tag{2.16}$$

Using Bailey pair (1.17) and (1.18) in (1.8) we get,

$$\sum_{n=0}^{\infty}[\delta_1,\delta_2;q]_n\left(\frac{a}{\delta_1\delta_2}\right)^n\frac{[-aq^{n+1}/b;q]_n}{[q^2;q^2]_n[-aq;q]_{2n}[aq/b;q]_n}$$

$$= \frac{\left[\frac{aq}{\delta_1},\frac{aq}{\delta_2};q\right]_\infty}{\left[aq,\frac{aq}{\delta_1\delta_2};q\right]_\infty}\Big[\frac{\delta_1\delta_2(1+a)-a(\delta_1+\delta_2)}{\delta_1\delta_2-a}+\sum_{n=1}^{\infty}\frac{[\delta_1\delta_2;q]_n\left(\frac{a}{\delta_1\delta_2}\right)^n}{\left[\frac{aq}{\delta_1},\frac{aq}{\delta_2};q\right]_n}\times$$

$$\frac{\delta_1\delta_2(1+aq^{2n})-aq^n(\delta_1+\delta_2)}{\delta_1\delta_2-a}\frac{(1-aq^{2n})[aq;q]_{n-1}[b;q]_n a^n q^{n^2}}{[q;q]_n[aq/b;q]_n b^n}\Big] \tag{2.17}$$

As $\delta_1\delta_2\to\infty$, (2.17) yields,

$$= \sum_{n=1}^{\infty}a^n q^{n(n-1)}(1+aq^{2n})(1-aq^{2n})\frac{[aq;q]_n[b;q]_n a^n q^{n^2}}{(1-aq^n)[q;q]_n[aq/b;q]_n b^n}. \tag{2.18}$$

For a=1, (2.18) yields

$$\sum_{n=0}^{\infty}\frac{q^{n(n-1)}[-q^{n+1}/b;q]_n}{[q^2;q^2]_n[-q;q]_{2n}[q/b;q]_n}$$

$$= \frac{1}{[q;q]_\infty}\left[2+\sum_{n=1}^{\infty}q^{n(n-1)}(1+q^{2n})(1+q^n)\frac{[b;q]_n q^{n^2}}{[q/b;q]_n b^n}\right]. \tag{2.19}$$

(2.19) can be written as,

$$\sum_{n=0}^{\infty}\frac{[-q^{n+1}/b;q]_n q^{n(n-1)}}{[q^2;q^2]_n[-q;q]_{2n}[q/b;q]_n}=\frac{1}{[q;q]_\infty}+\frac{1}{[q;q]_\infty}\sum_{n=-\infty}^{\infty}(1-q^n)q^{2n^2-n}\frac{[b;q]_n}{[q/b;q]_n b^n}$$

$$=\frac{1}{[q;q]_\infty}\left\{1+\sum_{n=-\infty}^{\infty}(1+q^n)\frac{[b;q]_n q^{2n^2-n}}{[q/b;q]_n b^n}\right\}. \tag{2.20}$$

As $b\to\infty$, (2.20) gives

$$\sum_{n=0}^{\infty}\frac{q^{n(n-1)}}{[q^2;q^2]_n[-q;q]_{2n}}=\frac{1}{[q;q]_\infty}+\frac{1}{[q^2,q^3;q^5]_\infty}+\frac{1}{[q,q^4;q^5]_\infty}. \tag{2.21}$$

Using Bailey pair (1.19) and (1.20) in (1.8) we get,

$$\sum_{n=0}^{\infty}[\delta_1,\delta_2;q]_n\left(\frac{a}{\delta_1\delta_2}\right)^n\frac{[aq;q]_{3n}}{[q^3;q^3]_n[a^3q^3;q^3]_n}=\frac{\left[\frac{aq}{\delta_1},\frac{aq}{\delta_2};q\right]_\infty}{\left[aq,\frac{aq}{\delta_1\delta_2};q\right]_\infty}\times$$

$$\sum_{n=0}^{\infty}\frac{[\delta_1,\delta_2;q]_n}{\left[\frac{aq}{\delta_1},\frac{aq}{\delta_2};q\right]_n}\left(\frac{a}{\delta_1\delta_2}\right)^n\left[\frac{\delta_1\delta_2(1+aq^2n)-aq^n(\delta_1+\delta_2)}{\delta_1\delta_2-a}\right]\times$$

$$(-)^n(1-aq^{2n})\frac{[aq;q]_n a^n q^{n(3n-1)/2}}{[q;q]_n(1-aq^n)} \tag{2.22}$$

Taking $\delta_1\delta_2\to\infty$ in (2.22) we obtain

$$\sum_{n=0}^{\infty}q^{n(n-1)}a^n\frac{[aq;q]_{3n}}{[q^3;q^3]_n[a^3q^3;q^3]_n}=\frac{1}{[aq;q]_\infty}\times$$

$$\left[(1+a)+\sum_{n=1}^{\infty}a^nq^{n(n-1)}(1+aq^{2n})(-)^n\frac{(1-aq^{2n})[aq;q]_n}{(1-aq^n)[q;q]_n}a^nq^{n(3n-1)/2}\right] \tag{2.23}$$

Taking a=1 in (2.23) we find,

$$\sum_{n=0}^{\infty}q^{n(n-1)}\frac{[q,q^2;q^3]_n}{[q^3;q^3]_n}=\frac{1}{[q;q]_\infty}\left[2+\sum_{n=1}^{\infty}q^{n(n-1)}(1+q^{2n})(1+q^n)q^{n(3n-1)/2}(-)^n\right]$$

$$= \frac{1}{[q;q]_\infty}\left[2+\sum_{n=1}^{\infty}(-)^n(1+q^{2n})(1+q^n)q^{(5n^2-3n)/2}\right]$$

$$= \frac{1}{[q;q]_\infty}\left[\sum_{n=-\infty}^{\infty}q^{(5n^2-3n)/2}+\sum_{n=-\infty}^{\infty}q^{(5n^2-1)/2}\right]$$

$$= \frac{1}{[q;q]_\infty}\left[[q^5,q^4,q;q^5]_\infty+[q^5,q^3,q^2;q^5]_\infty\right]$$

$$= \frac{1}{[q^2,q^3;q^5]_\infty}+\frac{1}{[q,q^4;q^5]_\infty}.$$

Thus we have

$$\sum_{n=0}^{\infty}\frac{[q,q^2;q^3]_n q^{n(n-1)}}{[q^3;q^3]_n}=\frac{1}{[q,q^4;q^5]_\infty}+\frac{1}{[q^2,q^3;q^5]_\infty}. \tag{2.24}$$

**Part II**
**WP Bailey pair and q-series identities**

**3. In this section we shall use following results**
**(i)** If A(n,r) is an arbitrary sequence involving integers n and r then

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n}A(n,r)=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}A(n+r,r), \tag{3.1}$$

[Srivastava & Karlsson 4; lemma 1(2) p. 100]

**(ii)**

$${}_6\Phi_5\left[\begin{array}{l} a,q\sqrt{a},-q\sqrt{a},a^2/k,kq^n,q^{-n};q;q/a \\ \sqrt{a},-\sqrt{a},kq/a,aq^{1-n}/k,aq^{1+n}\end{array}\right]=\frac{[a,aq;q]_n}{[k/a,kq/a;q]_n}\left(\frac{k}{a^2}\right)^n, \tag{3.2}$$

which can be deduced from [Gasper & Rahman 2; App. II (II.21)] by putting $c=kq^n$ and $b=a^2/k$ in it.
If we take $c=aq^{1+n}$ in [Gasper & Rahman 2; App. II (II.21)] we get
**(iii)**

$${}_4\Phi_3\left[\begin{array}{l} a,q\sqrt{a},-q\sqrt{a},b;q;1/b \\ \sqrt{a},-\sqrt{a},aq/b\end{array}\right]=\frac{[aq,bq;q]_n}{[q,aq/b;q]_n b^n}. \tag{3.3}$$

**(iv)**

$${}_6\Phi_5\left[\begin{array}{l} a,q\sqrt{a},-q\sqrt{a},a/k,kq^n,q^{-n};q;q \\ \sqrt{a},-\sqrt{a},kq,aq^{1-n}/k,aq^{1+n}\end{array}\right]=\left\{\begin{array}{ll} 1, & if\ n=0 \\ 0 & if\ n>0\end{array}\right\} \tag{3.4}$$

[Gasper & Rahman 2; App. II (II.21)]

**(v)**

$$
{}_8\Phi_7\left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, a^2q/bck, kq^n, q^{-n}; q; q \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, bck/a, aq^{1-n}/k, aq^{1+n} \end{array}\right] = \frac{[aq, aq/bc, bk/a, ck/a; q]_n}{[aq/b, aq/c, bck/a, k/a; q]_n}, \tag{3.5}
$$

which can be deduced from [Gasper & Rahman 2; App. II (II.22)] by putting $e = kq^n, d = a^2q/bck$

**(vi)**

$$
{}_6\Phi_5\left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, b, c, a/bc; q; q \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, bcq \end{array}\right] = \frac{[aq, aq/bc, bq, cq; q]_n}{[aq/b, aq/c, bcq, q; q]_n} \tag{3.6}
$$

which can be deduced from (3.5) by taking k=aq.

**(vii)**

$$
\sum_{n=0}^{\infty}(-a)^n q^{n(n+1)/2} = \frac{1}{1+}\,\frac{aq}{1+}\,\frac{a(q^2-q)}{1+}\,\frac{aq^3}{1+}\,\frac{a(q^4-q^2)}{1+\ldots} \tag{3.6(a)}
$$

[Andrews & Berndt 1 (6.2.29) p. 152]

A WP Bailey pair is pair of sequences $\{\alpha_n(a,k,q), \beta_n(a,k,q)\}$ satisfying $\{\alpha_0(a,k,q) = \beta_0(a,k,q) = 1\}$ and

$$
\beta_n(a,k,q) = \sum_{r=0}^{n} \frac{[k/a;q]_{n-r}[k;q]_{n+r}}{[q;q]_{n-r}[aq;q]_{n+r}}
$$

$$
= \frac{[k,k/a;q]_n}{[q,aq;q]_n}\sum_{r=0}^{n}\frac{[q^{-n},kq^n;q]_r}{[aq^{1+n},aq^{1-n}/k;q]_r}\left(\frac{aq}{k}\right)^r \alpha_r(a,k,q). \tag{3.7}
$$

[Laughlin 3; (1.1)]

Multiplying both sides of (3.7) by $\Omega_n z^n$, summing over n from zero to infinity and applying the identity (3.1) on the right hand side we get,

$$
\sum_{n=0}^{\infty}\Omega_n\beta_n z^n = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{[k/a;q]_n[k;q]_{n+2r}}{[q;q]_n[aq;q]_{n+2r}}\Omega_{n+r}\alpha_r z^{n+r} \tag{3.8}
$$

Taking k=aq in (3.7) and (3.8) we find,

If

$$
\beta_n(a;q) = \sum_{r=0}^{n}\alpha_r(a,q) \tag{3.9}
$$

Then

$$\sum_{n=0}^{\infty} \Omega_n \beta_n z^n = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \Omega_{n+r} \alpha_r z^{n+r} \tag{3.10}$$

We shall make use of (3.7), (3.8) and (3.9), (3.10) in order to establish our main transformation formulae.

## 4. Main Results

In this part we shall use (3.7) and (3.8) in order to establish our main results.
Taking $\Omega_n = 1$ and $z = a^2q/k^2$ in (3.8) we get,

$$\sum_{n=0}^{\infty} \beta_n \left(\frac{a^2q}{k^2}\right)^n = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{[k;q]_{2r}}{[aq;q]_{2r}} \left(\frac{a^2q}{k^2}\right)^r \frac{[k/a, kq^{2r};q]_n}{[q, aq^{1+2r};q]_n} \left(\frac{a^2q}{k^2}\right)^n \alpha_r$$

$$= \sum_{r=0}^{\infty} \frac{[k;q]_{2r}}{[aq;q]_{2r}} \left(\frac{a^2q}{k^2}\right)^r {}_2\Phi_1 \left[\begin{matrix} k/a, kq^{2r}; q; a^2q/k^2 \\ aq^{1+2r} \end{matrix}\right] \alpha_r. \tag{4.1}$$

Summing the inner ${}_2\Phi_1$ series on the right hand side of (4.1) by basic analogue of Gauss summation formula [Gasper & Rahman; App. II (II.8)] we get

$$\sum_{n=0}^{\infty} \beta_n \left(\frac{a^2q}{k^2}\right)^n = \frac{[aq/k, a^2q/k;q]_\infty}{[aq, a^2q/k^2;q]_\infty} \sum_{r=0}^{\infty} \frac{[k, kq;q^2]_r \left(\frac{a^2q}{k^2}\right)^r}{[a^2q/k, a^2q^2/k^2;q^2]_r} \alpha_r, \tag{4.2}$$

where $\{\alpha_n, \beta_n\}$ are WP-Bailey pair defined in (3.7).

**(A)** Choosing $\alpha_r = \alpha_r(a,k,q) = \frac{[a, q/\sqrt{(a)}, -q\sqrt{a}, a^2/k;q]_r}{[q, \sqrt{a}, -\sqrt{a}, kq/a;q]_r} \left(\frac{k}{a^2}\right)^r$ in (3.7) and using (3.2) we get,

$$\beta_n = \beta_n(a,k,q) = \frac{[k, k/a;q]_n [a, aq;q]_n}{[q, aq;q]_n [k/a, kq/a;q]_n} \left(\frac{k}{a^2}\right)^n$$

$$= \frac{[a, k;q]_n}{[q, kq/a;q]_n} \left(\frac{k}{a^2}\right)^n.$$

Putting these values in (4.2) we find,

$${}_8\Phi_7 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}, \frac{a^2}{k}; q; q/k \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{\sqrt{k}}, -\frac{aq}{\sqrt{k}}, a\sqrt{\frac{q}{k}}, -a\sqrt{\frac{q}{k}}, \frac{kq}{a} \end{matrix}\right]$$

$$= \frac{[aq, a^2q/k^2]_\infty}{[aq/k, a^2q/k; q]_\infty} \, {}_2\Phi_1 \left[ \begin{matrix} a, k; q; q/k \\ kq/a \end{matrix} \right]. \tag{4.3}$$

**(B)** Choosing $\alpha_r = \alpha_r(a,k,q) = \dfrac{[a, q/\sqrt{(a)}, -q\sqrt{a}, b, c, a^2q/bck; q]_r}{[q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, bck/a; q]_r} \left(\dfrac{k}{a}\right)^r$ in (4.1) and using (3.5) we get,

$$\beta_n(a,k,q) = \frac{[aq/bc, bk/a, ck/a, k; q]_n}{[aq/b, aq/c, bck/a, q; q]_n}.$$

Putting these values in (4.2) we have;

$${}_{10}\Phi_9 \left[ \begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}, b, c, \dfrac{a^2q}{bck}; q; aq/k \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq}{\sqrt{k}}, -\dfrac{aq}{\sqrt{k}}, a\sqrt{\dfrac{q}{k}}, -a\sqrt{\dfrac{q}{k}}, \dfrac{aq}{b}, \dfrac{aq}{c}, \dfrac{bck}{a} \end{matrix} \right]$$

$$= \frac{[aq, a^2q/k^2; q]_\infty}{[aq/k, a^2q/k; q]_\infty} \, {}_4\Phi_3 \left[ \begin{matrix} k, aq/bc, bk/a, ck/a; q; a^2q/k^2 \\ aq/b, aq/c, bck/a \end{matrix} \right], \tag{4.4}$$

where $|q| < 1$ and $|a/k| < 1$.
Taking k=a/b in (4.4) we find,

$${}_8\Phi_7 \left[ \begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, \sqrt{a/b}, -\sqrt{a/b}, \sqrt{aq/b}, -\sqrt{aq/b}, b; q; bq \\ \sqrt{a}, -\sqrt{a}, q\sqrt{ab}, -q\sqrt{ab}, \sqrt{abq}, -\sqrt{abq}, \dfrac{aq}{b} \end{matrix} \right]$$

$$= \frac{[aq, b^2q; q]_\infty}{[bq, abq; q]_\infty}, \qquad |bq| < 1. \tag{4.5}$$

**(C)** Choosing $\alpha_r = \alpha_r(a,k,q) = \dfrac{[a, q/\sqrt{(a)}, -q\sqrt{a}, a/k; q]_r}{[q, \sqrt{a}, -\sqrt{a}, kq; q]_r} \left(\dfrac{k}{a}\right)^r$ in (4.1) and using (3.4) we get,
$\beta_n = \beta_n(a,k,q) = 1$, if n=0 and $\beta_n = \beta_n(a,k,q) = 0$, if $n > 0$ Putting these values in (4.2) we find,

$${}_8\Phi_7 \left[ \begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}, \dfrac{a}{k}; q; aq/k \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq}{\sqrt{k}}, -\dfrac{aq}{\sqrt{k}}, a\sqrt{\dfrac{q}{k}}, -a\sqrt{\dfrac{q}{k}}, kq \end{matrix} \right]$$

$$= \frac{[aq, a^2q/k^2]_\infty}{[aq/k, a^2q/k; q]_\infty}, \qquad |aq/k| < 1. \tag{4.6}$$

which is same as (4.5).
**(D)** If we take

$$\alpha_r(a,k,q) = \left\{ \begin{array}{ll} 1, & for \;\; r=0 \\ 0, & for \;\; r>0 \end{array} \right\}$$

in (3.7) we find,

$$\beta_n = \beta_n(a,k,q) = \frac{[k, k/a; q]_n}{[q, aq; q]_n}.$$

Putting these values in (4.2) we find,

$${}_2\Phi_1 \left[ \begin{array}{l} k, k/a; q; a^2q/k^2 \\ aq \end{array} \right] = \frac{[aq/k, a^2q/k; q]_\infty}{[aq, a^2q/k^2; q]_\infty} \tag{4.7}$$

which is basic analogue of Gauss summation formula.
**(E)** Putting the values of $\alpha_n(a,k,q)$ and $\beta_n(a,k,q)$ given in (A) in equation (3.8) we get,

$$\sum_{n=0}^{\infty} \Omega_n \frac{[a,k;q]_n}{[q,kq/a;q]_n} \left(\frac{kz}{a^2}\right)^n$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{[k/a;q]_n [k;q]_{n+2r}}{[q;q]_n [aq;q]_{n+2r}} \Omega_{n+r} \frac{[a, a^2/k; q]_r (1-aq^{2r})}{[q, kq/a; q]_r (1-a)} z^{n+r} \left(\frac{k}{a^2}\right)^r$$

$$= \sum_{r=0}^{\infty} \frac{[k;q]_{2r} [a, a^2/k; q]_r (1-aq^{2r}) (zk/a^2)^r}{[aq;q]_{2r} [q, kq/a; q]_r (1-a)} \sum_{n=0}^{\infty} \frac{[k/a, kq^{2r}; q]_n z^n \Omega_{n+r}}{[q, aq^{1+2r}; q]_n} \tag{4.8}$$

**(F)** Taking $\Omega_n = \dfrac{[q, kq/a; q]_n q^{n(n+1)/2}}{[a, k; q]_n}$ in (4.8) and using (3.6(a)) we find,

$$\sum_{r=0}^{\infty} \frac{[k;q]_{2r} [a^2/k; q]_r (1-aq^{2r}) q^{r(r+1)/2} \left(\dfrac{zk}{a^2}\right)^r}{[aq;q]_{2r} [k;q]_r (1-a)} \times$$

$$\times {}_4\Phi_3 \left[ \begin{array}{l} k/a, kq^{2r}, q^{1+r}, kq^{1+r}/a; q; zq^{1+r} \\ aq^{1+2r}, aq^r, kq^r; q \end{array} \right]$$

$$= \frac{1}{1-} \frac{kzq/a^2}{1+} \frac{kzq/a^2(1-q)}{1-} \frac{kzq^3/a^2}{1+} \frac{kzq^2/a^2(1-q^2)}{1-\ldots} \tag{4.9}$$

Taking $\Omega_n = \dfrac{[kq/a;q]_n}{[k;q]_n}$ in (4.8) we have

$$\sum_{r=0}^{\infty} \frac{[k;q]_{2r}[a,a^2/k;q]_r(1-aq^{2r})\left(\frac{zk}{a^2}\right)^r}{[aq;q]_{2r}[q,k;q]_r(1-a)} \;{}_3\Phi_2\left[\begin{matrix} k/a, kq^{2r}, kq^{1+r}/a; q; z \\ aq^{1+2r}, kq^r \end{matrix}\right]$$

$$= \frac{[kz/a;q]_\infty}{[kz/a^2;q]_\infty}. \tag{4.10}$$

Taking k=aq in (4.10) we have,

$$\sum_{r=0}^{\infty} \frac{[a,a/q;q]_r}{[q,aq;q]_r}\left(\frac{1-aq^{2r}}{1-a}\right)(zq/a)^r \;{}_2\Phi_1\left[\begin{matrix} q, q^{r+2}; q; z \\ aq^{r+1} \end{matrix}\right]$$

$$= \frac{[aq;q]_\infty}{[zq/a;q]_\infty}. \tag{4.11}$$

Now we shall make use of (3.9) and (3.10) in order to establish certain transformation formulae.

**(a)** Choosing $\alpha_r = \dfrac{[a, q\sqrt{a}, -q\sqrt{a}, b;q]_r}{[q, \sqrt{a}, -\sqrt{a}, aq/b;q]_r b^r}$ in (3.9) and using (3.3) we get,

$$\beta_n = \frac{[aq,bq;q]_n}{[q,aq/b;q]_n b^n}.$$

Putting these values in (3.10) we find,

$$\sum_{n=0}^{\infty} \frac{[aq,bq;q]_n}{[q,aq/b;q]_n}\left(\frac{z}{b}\right)\Omega_n$$

$$= \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{[a;q]_r[b;q]_r(1-aq^{2r})}{[q,aq/b;q]_r(1-a)}\left(\frac{z}{b}\right)^r z^n \Omega_{n+r}. \tag{4.12}$$

**(b)** Now, taking $\Omega_n = q_1^{n(n+1)/2}$ we have

$$\sum_{n=0}^{\infty} \frac{[aq,bq;q]_n q_1^{n(n+1)/2}}{[q,aq/b;q]_n}\left(\frac{z}{b}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{[a;q]_r[b;q]_r(1-aq^{2r})q_1^{r(r+1)/2}}{[q,aq/b;q]_r(1-a)}\left(\frac{z}{b}\right)^r \sum_{n=0}^{\infty}(zq_1^r)^n q_1^{n(n+1)/2}. \tag{4.13}$$

**(c)** Again using (3.6(a)) in (4.13) we get,

$$\sum_{n=0}^{\infty}\frac{[aq,bq;q]_n q_1^{n(n+1)/2}}{[q,aq/b;q]_n}\left(\frac{z}{b}\right)^n$$

$$=\sum_{r=0}^{\infty}\frac{[a;q]_r[b;q]_r(1-aq^{2r})q_1^{r(r+1)/2}}{[q,aq/b;q]_r(1-a)}\left(\frac{z}{b}\right)^r\times$$

$$\times\left\{\frac{1}{1-}\frac{zq_1^{r+1}}{1+}\frac{zq_1^{r+1}(1-q_1)}{1-}\frac{zq_1^{r+3}}{1+}\frac{zq_1^{r+2}(1-q_1^2)}{1-...}\right\}. \tag{4.14}$$

**(d)** As $b\to\infty$, (4.14) yields

$$\sum_{n=0}^{\infty}\frac{[aq;q]_n(qq_1)^{n(n+1)/2}z^n(-)^n}{[q;q]_n}$$

$$=\sum_{r=0}^{\infty}\frac{[a;q]_r(1-aq^{2r})(qq_1)^{r(r+1)/2}(-)^r}{[q;q]_r(1-a)}\left(\frac{z}{q}\right)^r\times$$

$$\times\left\{\frac{1}{1-}\frac{zq_1^{r+1}}{1+}\frac{zq_1^{r+1}(1-q_1)}{1-}\frac{zq_1^{r+3}}{1+}\frac{zq_1^{r+2}(1-q_1^2)}{1-...}\right\}. \tag{4.15}$$

**(e)** Taking a=1 in (4.15) and using (3.6(a)) again on the left hand side of it we get,

$$\frac{1}{1+}\frac{zqq_1}{1+}\frac{zqq_1(qq_1-1)}{1+}\frac{zq^3q_1^3}{1+}\frac{zq^2q_1^2(q_1^2-1)}{1+...}$$

$$=\left\{\frac{1}{1-}\frac{zq_1}{1+}\frac{zq_1(1-q_1)}{1-}\frac{zq_1^3}{1+}\frac{zq_1^2(1-q_1^2)}{1-...}\right\}$$

$$+\sum_{r=1}^{\infty}(1+q^r)(qq_1)^{r(r+1)/2}(-z/q)^r\times$$

$$\times\left\{\frac{1}{1-}\frac{zq_1^{r+1}}{1+}\frac{zq_1^{r+1}(1-q_1)}{1-}\frac{zq_1^{r+3}}{1+}\frac{zq_1^{r+2}(1-q_1^2)}{1-...}\right\}. \tag{4.16}$$

**(f)** Taking a=0 in (4.15) we find,

$$\sum_{n=0}^{\infty}\frac{(qq_1)^{n(n+1)/2}z^n(-)^n}{[q;q]_n}=\sum_{r=0}^{\infty}\frac{(-z/q)^r(qq_1)^{r(r+1)/2}}{[q;q]_r}\times$$

$$\times \left\{ \frac{1}{1-} \frac{zq_1^{r+1}}{1+} \frac{zq_1^{r+1}(1-q_1)}{1-} \frac{zq_1^{r+3}}{1+} \frac{zq_1^{r+2}(1-q_1^2)}{1-\ldots} \right\}. \tag{4.17}$$

**(g)** For $q_1 = q$, (4.17) yields,

$$\sum_{n=0}^{\infty} \frac{(q)^{n(n+1)}(-z)^n}{[q;q]_n} = \sum_{r=0}^{\infty} \frac{(-z/q)^r (q)^{r(r+1)}}{[q;q]_r} \times$$

$$\times \left\{ \frac{1}{1-} \frac{zq^{r+1}}{1+} \frac{zq^{r+1}(1-q)}{1-} \frac{zq^{r+3}}{1+} \frac{zq^{r+2}(1-q^2)}{1-\ldots} \right\}. \tag{4.18}$$

**(h)** Taking z=-1 in (4.18) we find,

$$\sum_{r=0}^{\infty} \frac{q^{r^2}}{[q;q]_r} \left\{ \frac{1}{1-} \frac{zq^{r+1}}{1+} \frac{zq^{r+1}(1-q)}{1-} \frac{zq^{r+3}}{1+} \frac{zq^{r+2}(1-q^2)}{1-\ldots} \right\}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{[q;q]_n} = \frac{1}{[q^2,q^3;q^5]_\infty}. \tag{4.19}$$

**(i)** Again taking $z = -1/q$ in (4.18) we have

$$\sum_{r=0}^{\infty} \frac{q^{r^2-r}}{[q;q]_r} \left\{ \frac{1}{1+} \frac{zq^{r}}{1-} \frac{zq^{r}(1-q)}{1+} \frac{zq^{r+2}}{1-} \frac{zq^{r+1}(1-q^2)}{1+\ldots} \right\}$$

$$\sum_{n=0}^{\infty} \frac{q^2}{[q;q]_n} = \frac{1}{[q,q^4;q^5]_\infty}. \tag{4.20}$$

**(j)** Taking $\alpha_r = \dfrac{[a, q\sqrt{a}, -q\sqrt{a}, b, c, a/bc; q]_r q^r}{[q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, bcq; q]_r}$ in (3.9) and using (3.6) we get,

$$\beta_n = \frac{[aq, bq, cq, aq/bc; q]_n}{[q, aq/b, aq/c, bcq; q]_n}.$$

Putting these values in (3.10) we have

$$\sum_{n=0}^{\infty} \frac{[aq, bq, cq, aq/bc; q]_n z^n \Omega_n}{[q, aq/b, aq/c, bcq; q]_n}$$

$$= \sum_{r=0}^{\infty} \frac{[a, b, c, a/bc; q]_r (1-aq^{2r})(zq)^r}{[q, aq/b, aq/c, bcq; q]_r (1-a)} \sum_{n=0}^{\infty} \Omega_{n+r} z^n. \tag{4.21}$$

For $\Omega_n = 1$, (4.21) yields

$${}_6\Phi_5\left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, b, c, a/bc; q; zq \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, bcq \end{array}\right]$$

$$= (1-z)\ {}_4\Phi_3\left[\begin{array}{l} aq, bq, cq, aq/bc; q; z \\ aq/b, aq/c, bcq \end{array}\right]. \quad (4.22)$$

As $a \to 1$ in (4.21) we get

$$\sum_{n=0}^{\infty} \frac{[bq, cq, q/bc; q]_n z^n \Omega_n}{[q/b, q/c, bcq; q]_n}$$

$$= \sum_{n=0}^{\infty} \Omega_n z^n + \sum_{r=1}^{\infty} (1+q^r) \frac{[b, c, 1/bc; q]_r (zq)^r}{[q/b, q/c, bcq; q]_r} \sum_{n=0}^{\infty} \Omega_{n+r} z^n. \quad (4.23)$$

Taking $z = 1, \Omega_n = q^{n(n+1)/2}$ in (4.23) e get,

$$\sum_{n=0}^{\infty} \frac{[bq, cq, q/bc; q]_n q^{n(n+1)/2}}{[q/b, q/c, bcq; q]_n} = \frac{[q^2; q^2]_\infty}{[q; q^2]_\infty}$$

$$+ \sum_{r=1}^{\infty} (1+q^r) \frac{[b, c, 1/bc; q]_r (zq)^r q^{r(r+1)/2}}{[q/b, q/c, bcq; q]_r} \sum_{n=0}^{\infty} q^{n(n+1)/2} q^{nr}. \quad (4.24)$$

Now using (3.6(a)) in (4.24) we get,

$${}_4\Phi_3\left[\begin{array}{l} bq, cq, q/bc, q; q; q \\ q/b, q/c, bcq; q \end{array}\right] = \frac{[q^2; q^2]_\infty}{[q; q^2]_\infty}$$

$$+ \sum_{r=1}^{\infty} (1+q^r) \frac{[b, c, 1/bc; q]_r (zq)^r q^{r(r+1)/2}}{[q/b, q/c, bcq; q]_r}$$

$$\times \left\{ \frac{1}{1+} \frac{zq^r}{1-} \frac{zq^r(1-q)}{1+} \frac{zq^{r+2}}{1-} \frac{zq^{r+1}(1-q^2)}{1+\ldots} \right\} \quad (4.25)$$

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