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EXISTENCE AND UNIQUENESS OF NEW FIXED POINTS OF (ψ, ϕ) CONTRACTIONS IN RECTANGULAR METRIC SPACES

Jayesh Tiwari and Dileep Kumar Sharma*

Department of Computer Science,
Shri Vaishnav Institute of Management,
Devi Ahilya University, Indore - 452009, (M.P.), INDIA

E-mail : jayesh_tiwari2005@yahoo.com

*Department of Mathematics,
Government Polytechnic College, Barwani - 451551, (M.P.), INDIA

E-mail : dileepk.sharmadec1972@gmail.com

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Abstract: In this article, we prove existence and uniqueness of some new fixed points in rectangular metric spaces and confirm them with appropriate example. Also we give an analogue of Ehran et al. [9] fixed point result on rectangular metric spaces, which will generalize Muhammad Arshad et al. [5] fixed point results on rectangular metric spaces.

Keywords and Phrases: Metric Space, b-metric Space, rectangular metric space, fixed point, contraction mapping.

2020 Mathematics Subject Classification: 54H25, 47H10.

1. Introduction

Fixed point theory has been one of the fastest growing fields in analysis over the past few decades. In this view fixed point theorems are the most important objects. The Banach Contraction Principle is a famous and useful theory that is frequently identified in the literature.

In a large number of subjects the old concept of metric space is designed for different researchers by changing the metric part. Among these generalizations, in 2000, Branciari [8] introduced the notion of a rectangular metric spaces by substituting triangle inequality in to rectangular inequality. He involved four points instead of three points.

In this sequel Azam and Arshad [6] presented sufficient conditions for the existence and uniqueness of a distinct point of the Kannan-type map in the framework of rectangular metric spaces. Azam et al. proved the analogue of the Banach Contraction Principle in rectangular metric spaces. Since then many fixed point theorems for various contractions on rectangular metric space and extended rectangular metric space appeared such as ([1-4, 7, 9-13]).

In this article we present an analogue of Ehran et al. [9] fixed point result on rectangular metric spaces. Also we generalize Muhammad Arshad et al. [5] fixed point results on rectangular metric spaces. We give an example to verify new obtained result.

Preliminaries

Definition 2.1. [7] *Let X be a non empty set and $s \geq 1$ be a fixed real number. If a function $d : X \times X \to \mathbb{R}^+$ satisfies the following conditions:*
(bm_1) $d(u, v) = 0$ *iff* $u = v$ *for all* $u, v \in X$.
(bm_2) $d(u, v) = d(v, u)$ *for all* $u, v \in X$.
(bm_3) $d(u, v) \leq s[d(u, w) + d(w, v)]$ *for all distinct points* $u, v, w \in X$.

Then a pair (X, d) is called a b-metric space.

Definition 2.2. [8] *Let X be a non empty set . If a function $d : X \times X \to \mathbb{R}^+$ satisfies the following conditions:*
(rm_1) $d(u, v) = 0$ *iff* $u = v$ *for all* $u, v \in X$.
(rm_2) $d(u, v) = d(v, u)$ *for all* $u, v \in X$.
(rm_3) $d(u, v) \leq d(u, w) + d(w, p) + d(p, v)$ *for all distinct points* $u, v, w, p \in X$.

Then a pair (X, d) is called a rectangular metric space or generalized metric space.

Example 2.3. Let $X = N$,define $d : X \times X \to \mathbb{R}^+$ by

$$d(u,v)= \begin{cases} 0, \text{ if } u = v \\ c\lambda, \text{ if } (u, v) \in \{4, 5\} \text{ and } u \neq v \\ \lambda, \text{ if } u \text{ and } v \text{ do not belong to } \{4, 5\} \text{ and } u \neq v. \end{cases}$$

where $\lambda > 0$ and $c < 3$.

Hence (X, d) is a rectangular metric space as,

$$d(4, 5) = c\lambda < 3\lambda = d(4, 3) + d(3, 2) + d(2, 5).$$

Example 2.4. Let $X = A \cup B$, where $A = \left\{\frac{1}{n}; n \in \mathbb{N}\right\}$ and B is the set of all positive integers. Define $d : X \times X \to \mathbb{R}^+$ such that $d(x, y) = d(y, x)$ for all $x, y \in X$ and

$$d(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 2\beta, & \text{if } x, y \in A; \\ 3\beta, & \text{if } x \in A \text{ and } y \in B; \\ \beta, & \text{otherwise,} \end{cases}$$

where $\beta > 0$ is a constant. Then (X, d) is a rectangular metric space.

Definition 2.5. [5] *Let X be a non empty set. Let f and g be two self maps.*

(i) A point $x \in X$ is called a common fixed point of f and g if $x = fx = gx$.

(ii) A point $x \in X$ is said to be a point of coincidence of f and g if $fx = gx$ and if $u = fx = gx$, then u is said to be a point of coincidence of f and g.

(iii) The mapping $f, g : X \to X$ are said to be weakly compatible if they commute at their point of coincidence that is, $fgx = gfx$ whenever $gx = fx$.

Definition 2.6. [5] *Let Ψ denote the set of all functions $\phi : [0, +\infty] \to [0, +\infty)$ such that*

(i) ϕ is continuous.

(ii) $\phi(t) = 0$ if and only if $t = 0$.

Before we present our main result we write a following lemma[1].

Lemma 2.7. [5] *Let X be a non empty set. Suppose that the self mapping f and g have a unique point of coincidence u in X. If f and g are weakly compatible, then f and g have a unique common fixed point.*

3. Main Results

In this section we present our main results.

Theorem 3.1. *Let (X, d) be a Hausdorff rectangular metric space and let f, g be two self maps define onto itself such that $fX \subset gX$ and (gX, d) is a complete rectangular metric space. Suppose that the following condition holds:*

$$\psi(d(fx, fy)) \leq \psi(M(gx, gy)) - \phi(M(gx, gy)) + \psi(m(gx, gy)) - \phi(m(gx, gy)) \tag{3.1}$$

for all $x, y \in X$ and $\psi, \phi \in \Psi$, where ψ is increasing and

$M(gx, gy) = max\{d(gx, gy), d(gx, fx), d(gy, fy)\}$

$m(gx, gy) = min\{d(gx, fy), d(gy, fy), d(gx, fx), d(gy, fx)\}$

Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Firstly we show uniqueness of the point of coincidence of f and g.
Suppose that u and v be two points of coincidence of f and g. Thus there exists $x, y \in X$ such that $u = fx = gx$ and $v = fy = gy$.
Then by (3.1), we find

$$\begin{aligned}\psi(d(u,v)) =& \psi(d(fx,fy)) \leq \psi(M(gx,gy)) - \phi(M(gx,gy)) \\ &+ \psi(m(gx,gy)) - \phi(m(gx,gy))\end{aligned}$$

where

$$\begin{aligned}M(gx,gy) &= max\{d(gx,gy), d(gx,fx), d(gy,fy)\} \\ &= max\{d(gx,gy), d(fx,fx), d(fy,fy)\} \\ &= d(gx,gy) = d(u,v)\end{aligned}$$

and

$$\begin{aligned}m(gx,gy) &= min\{d(gx,fy), d(gy,fy), d(gx,fx), d(gy,fx)\} \\ &= min\{d(fx,fy), d(fy,fy), d(fx,fx), d(fy,fx)\} \\ &= 0.\end{aligned}$$

Then by (3.1)

$$\begin{aligned}\psi(d(u,v)) &\leq \psi(d(u,v)) - \phi(d(u,v)) \\ \phi(d(u,v)) &\leq 0 \\ d(u,v) &\leq 0.\end{aligned}$$

Therefore we get $u = v$. Since f and g are weakly compatible and u is the unique point of coincidence of f and g, then by lemma 2.7 the point u is the unique common fixed point of f and g.

Let x_0 be an arbitrary point in X. Since $fX \subset gX$, we define two sequence $\{x_n\}$ and $\{y_n\}$ in X as follows $y_n = fx_n = gx_{n+1}$ for $n \geq 0$.

If $\mathbf{y_n} = \mathbf{y_{n+1}}$ and take $n = 2k$, then we have

$$y_{2k} = gx_{2k+1} = fx_{2k} = gx_{2k+2} = fx_{2k+1} = y_{2k+1}.$$

$$fx_{2k+1} = gx_{2k+1}.$$

Therefore f and g have a point of coincidence x_{2k+1} in X.
If $\mathbf{y_n} \neq \mathbf{y_{n+1}}$ then $d(y_n, y_{n+1}) > 0$ and take $n = 2k$.

Now from (3.1), we have

$$\begin{aligned}\psi(d(y_{2k}, y_{2k+1})) &= \psi(d(fx_{2k}, fx_{2k+1}))\\ &\leq \psi(M(gx_{2k}, gx_{2k+1}) - \phi(M(gx_{2k}, gx_{2k+1}))\\ &+ \psi(m(gx_{2k}, gx_{2k+1})) - \phi(m(gx_{2k}, gx_{2k+1}))\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}M(gx_{2k}, gx_{2k+1}) &= max\{d(gx_{2k}, gx_{2k+1}), d(gx_{2k}, fx_{2k}), d(gx_{2k+1}, fx_{2k+1})\}\\ &= max\{d(gx_{2k}, gx_{2k+1}), d(gx_{2k}, gx_{2k+1}), d(gx_{2k+1}, gx_{2k+2})\}\\ &= max\{d(y_{2k-1}, y_{2k}), d(y_{2k-1}, y_{2k}), d(y_{2k}, y_{2k+1}\}\\ &= max\{d(y_{2k-1}, y_{2k}), d(y_{2k}, y_{2k+1}\}\end{aligned}$$

and

$$\begin{aligned}&m(gx_{2k}, gx_{2k+1})\\ &= min\{d(gx_{2k}, fx_{2k+1}), d(gx_{2k+1}, fx_{2k+1}), d(gx_{2k}, fx_{2k}), d(gx_{2k+1}, fx_{2k})\}\\ &= min\{d(gx_{2k}, gx_{2k+2}), d(gx_{2k+1}, gx_{2k+2}), d(gx_{2k}, gx_{2k+1}), d(gx_{2k+1}, gx_{2k+1})\}\\ &= min\{d(gx_{2k}, gx_{2k+2}), d(gx_{2k+1}, gx_{2k+2}), d(gx_{2k}, gx_{2k+1}), 0\}\\ &= 0.\end{aligned}$$

Case (i): If $M(gx_{2k}, gx_{2k+1}) = d(y_{2k}, y_{2k+1})$ therefore from(3.2)

$$\psi(d(y_{2k}, y_{2k+1})) \leq \psi(d(y_{2k}, y_{2k+1})) - \phi(d(y_{2k}, y_{2k+1})) + \psi(0) - \phi(0)$$

this implies

$$\phi(d(y_{2k}, y_{2k+1})) = 0.$$

$$d(y_{2k}, y_{2k+1}) = 0.$$

$$y_{2k} = y_{2k+1}.$$

which is a contradiction.

Case (ii): If $M(gx_{2k}, gx_{2k+1}) = d(y_{2k-1}, y_{2k})$ therefore from(3.2)

$$\begin{aligned}\psi(d(y_{2k}, y_{2k+1})) &\leq \psi(d(y_{2k-1}, y_{2k})) - \phi(d(y_{2k-1}, y_{2k})) + \psi(0) - \phi(0)\\ &\leq \psi(d(y_{2k-1}, y_{2k})) - \phi(d(y_{2k-1}, y_{2k}))\\ &\leq \psi(d(y_{2k-1}, y_{2k})).\end{aligned}$$

Since ψ is non decreasing, therefore

$$d(y_{2k}, y_{2k+1}) \leq d(y_{2k-1}, y_{2k}).$$

Thus the sequence $\{d(y_{2k}, y_{2k+1})\}$ is decreasing and bounded below.
Hence, it converges to a positive number, say $r > 0$.
Taking the limit as $\lim_{k\to+\infty}$, we get

$$\psi(r) \leq \psi(r) - \phi(r).$$

Which leads to $\phi(r) = 0$, and hence $r = 0$.
Thus, $\lim_{n\to+\infty} d(y_{2k}, y_{2k+1}) = 0$
Now from (3.1), we have

$$\begin{aligned}\psi(d(y_{2k}, y_{2k+2})) &= \psi(d(fx_{2k}, fx_{2k+2}))\\ &\leq \psi(M(gx_{2k}, gx_{2k+2})) - \phi(M(gx_{2k}, gx_{2k+2})) + \psi(m(gx_{2k}, gx_{2k+2}))\\ &- \phi(m(gx_{2k}, gx_{2k+2})) \end{aligned} \tag{3.3}$$

where

$$\begin{aligned}M(gx_{2k}, gx_{2k+2}) &= max\{d(gx_{2k}, gx_{2k+2}), d(gx_{2k}, fx_{2k}), d(gx_{2k+2}, fx_{2k+2})\}\\ &= max\{d(gx_{2k}, gx_{2k+2}), d(gx_{2k}, gx_{2k+1}), d(gx_{2k+2}, gx_{2k+3})\}\\ &= max\{d(y_{2k-1}, y_{2k+1}), d(y_{2k-1}, y_{2k}), d(y_{2k+1}, y_{2k+2})\}\end{aligned}$$

and

$$\begin{aligned}&m(gx_{2k}, gx_{2k+2})\\ &= min\{d(gx_{2k}, fx_{2k+2}), d(gx_{2k+2}, fx_{2k+2}), d(gx_{2k}, fx_{2k}), d(gx_{2k+2}, fx_{2k})\}\\ &= min\{d(gx_{2k}, gx_{2k+3}), d(gx_{2k+2}, gx_{2k+3}), d(gx_{2k}, gx_{2k+1}), d(gx_{2k+2}, gx_{2k+1})\}\\ &= min\{d(y_{2k-1}, y_{2k+2}), d(y_{2k+1}, y_{2k+2}), d(y_{2k-1}, y_{2k}), d(y_{2k+1}, y_{2k})\}\\ &= 0(on \lim_{k\to+\infty}).\end{aligned}$$

Case (iii): If $M(\mathrm{g}x_{2k}, \mathrm{g}x_{2k+2}) = d(y_{2k+1}, y_{2k+2})$ Then from (3.3)

$$\begin{aligned}\psi(d(y_{2k}, y_{2k+2})) &\leq \psi(d(y_{2k+1}, y_{2k+2})) - \phi(d(y_{2k+1}, y_{2k+2})) + \psi(0) - \phi(0)\\ &\leq \psi(d(y_{2k+1}, y_{2k+2})) - \phi(d(y_{2k+1}, y_{2k+2}))\end{aligned}$$

Taking $\lim_{k\to+\infty}$ we have

$$lim_{k\to\infty}\psi(d(y_{2k}, y_{2k+2})) = 0.$$

Since ψ is continuous, hence

$$lim_{n\to+\infty}d(y_{2k}, y_{2k+2}) = 0.$$

Case (iv): If $M(gx_{2k}, gx_{2k+2}) = d(y_{2k-1}, y_{2k})$ Then from (3.3)

$$\begin{aligned}\psi(d(y_{2k}, y_{2k+2})) &\leq \psi(d(y_{2k-1}, y_{2k})) - \phi(d(y_{2k-1}, y_{2k})) + \psi(0) - \phi(0)\\ &\leq \psi(d(y_{2k-1}, y_{2k})) - \phi(d(y_{2k-1}, y_{2k})).\end{aligned}$$

Taking $\lim_{k\to+\infty}$ we have

$$lim_{k\to+\infty}\psi(d(y_{2k}, y_{2k+2})) = 0.$$

Since ψ is continuous, hence

$$lim_{k\to+\infty}d(y_{2k}, y_{2k+2}) = 0.$$

Case (v): If $M(gx_{2k}, gx_{2k+2}) = d(y_{2k-1}, y_{2k+1})$ then from (3.3)

$$\begin{aligned}\psi(d(y_{2k}, y_{2k+2})) &\leq \psi(d(y_{2k-1}, y_{2k+1})) - \phi(d(y_{2k-1}, y_{2k+1})) + \psi(0) - \phi(0)\\ &\leq \psi(d(y_{2k-1}, y_{2k+1})) - \phi(d(y_{2k-1}, y_{2k+1}))\\ &\leq \psi(d(y_{2k-1}, y_{2k+1})).\end{aligned}$$

Hence $\{d(y_{2k}, y_{2k+2})\}$ is decreasing and bounded below.
Therefore, the sequence$\{d(y_{2k}, y_{2k+2})\}$ converges to a number, $s \geq 0$.
Taking $\lim_{k\to+\infty}$ we get

$$0 \leq \psi(s) \leq \psi(s) - \phi(s).$$

Which implies that $\phi(s) = 0$ and hence $s = 0$. Thus,

$$lim_{n\to+\infty}d(y_{2k}, y_{2k+2}) = 0.$$

Suppose that $y_n \neq y_m$ for all $n \neq m$.
Now we prove that $\{y_n\}$is a rectangular Cauchy sequence.
On contrary, let $\{y_n\}$ be not a rectangular Cauchy sequence.
Then there exists $\epsilon > 0$ for which, we can find sub sequences $\{y_{n_i}\}$ and $\{y_{m_i}\}$ of $\{y_n\}$ with $n_i > m_i \geq i$ such that

$$d(y_{n_i}, y_{m_i}) \geq \epsilon,$$

where n_i is the smallest integer satisfying above inequality, that is

$$d(y_{m_i}, y_{n_{i-1}}) < \epsilon.$$

Now by rectangular inequality, we have

$$\begin{aligned}\epsilon &\leq d(y_{m_i}, y_{n_i})\\ &\leq d(y_{m_i}, y_{n_{i-2}}) + d(y_{n_{i-2}}, y_{n_{i-1}}) + d(y_{n_{i-1}}, y_{n_i})\\ &\leq \epsilon + d(y_{n_{i-2}}, y_{n_{i-1}}) + d(y_{n_{i-1}}, y_{n_i}). \end{aligned} \tag{3.4}$$

Taking limit as $i \to +\infty$ we get,

$$\lim_{i\to+\infty} d(y_{m_i}, y_{n_i}) = 0.$$

Again, using the rectangular inequality, we obtain

$$d(y_{n_{i-1}}, y_{m_{i-1}}) \leq d(y_{n_{i-1}}, y_{n_i}) + d(y_{n_i}, y_{m_i}) + d(y_{m_i}, y_{m_{i-1}}).$$

Taking limit as $i \to +\infty$ we get,

$$\lim_{i\to+\infty} d(y_{n_{i-1}}, y_{m_{i-1}}) = 0.$$

Now, we substitute $x = x_{n_i}$ and $y = x_{m_i}$ in (3.1). Then we get

$$\begin{aligned}\psi(d(y_{n_i}, y_{m_i})) &= \psi(d(fx_{n_i}, fx_{m_i}))\\ &\leq \psi(M(gx_{n_i}, gx_{m_i})) - \phi(M(gx_{n_i}, gx_{m_i}))\\ &+ \psi(m(gx_{n_i}, gx_{m_i})) - \phi(m(gx_{n_i}, gx_{m_i}))\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}m(gx_{n_i}, gx_{m_i}) &= min\{d(gx_{n_i}, fx_{m_i}), d(gx_{m_i}, fx_{m_i}), d(gx_{n_i}, fx_{n_i}), d(gx_{m_i}, fx_{n_i})\}\\ &= min\{d(y_{n_{i-1}}, y_{m_i}), d(y_{m_{i-1}}, y_{m_i}), d(y_{n_{i-1}}, y_{n_i}), d(y_{m_{i-1}}, y_{n_i})\}.\end{aligned}$$

Thus, on taking $\lim_{i\to+\infty}$ we have

$$m(gx_{n_i}, gx_{m_i}) = 0$$

and

$$\begin{aligned}M(gx_{n_i}, gx_{m_i}) &= max\{d(gx_{n_i}, gx_{m_i}), d(gx_{n_i}, fx_{n_i}), d(gx_{m_i}, fx_{m_i})\}\\ &= max\{d(y_{n_{i-1}}, y_{m_{i-1}}), d(y_{n_{i-1}}, y_{n_i}), d(y_{m_{i-1}}, y_{m_i})\}.\end{aligned}$$

Thus, on taking $\lim_{i\to+\infty}$ we have

$$M(gx_{n_i}, gx_{m_i}) \to max\{\epsilon, 0, 0\} = \epsilon.$$

Now letting $\lim_{i\to+\infty}$ in (3.5). We have

$$0 \leq \psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon).$$

This implies that $\phi(\epsilon) = 0$, hence $\epsilon = 0$, which contradicts the fact that $\epsilon > 0$. Thus, $\{y_n\}$ is a rectangular Cauchy sequence.

Since (gX, d) is a complete rectangular metric space , there exists $u \in gX$ such that $\{y_n\} \to u$ as $n \to +\infty$. Let $y \in X$ such that $gy = u$.

Applying the inequality (3.1), with $x = x_n$, we obtain

$$\psi(d(fx_n, fy)) \leq \psi(M(gx_n, gy)) - \phi(M(gx_n, gy)) + \psi(m(gx_n, gy)) - \phi(m(gx_n, gy))$$

where

$$m(gx_n, gy) = min\{d(gx_n, fy), d(gy, fy), d(gx_n, gx_{n+1}), d(gy, fx_n)\}.$$

Note that $m(gx_n, gy) \to 0$ as $n \to +\infty$

$$\begin{aligned} M(gx_n, gy) &= max\{d(gx_n, gy), d(gx_n, fx_n), d(gy, fy)\} \\ &= max\{d(gx_n, gy), d(gx_n, gx_{n+1}), d(gy, fy)\} \end{aligned}$$

Now if $M(gx_n, gy) = d(gx_n, gy)$. or $M(gx_n, gy) = d(gx_n, gx_{n+1})$.

Then we have

$$d(fx_n, fy) \leq d(gx_n, gy).$$

or

$$d(fx_n, fy) \leq d(gx_n, gx_{n+1}),$$

since ψ is increasing.

In either case, taking $n \to +\infty$, we get $gx_{n+1} = fx_n \to fy$.
Since X is a Hausdorff, we deduce that $gy = fy$.

If, on the other hand

$$M(gx_n, gy) = d(gy, fy).$$

Then taking $n \to +\infty$ in

$$\psi(d(fx_n, fy)) \leq \psi(d(gy, fy)) - \phi(d(gy, fy)).$$

We get $\phi(d(gy, fy)) = 0$. Hence $d(gy, fy) = 0$, so that $gy = fy$.

Thus $u = gy = fy$. Then u is a point of coincidence of f and g.

Finally when f and g are weakly compatible, then by lemma 2.7, a well known result implies that f and g have a unique common fixed point.

Let $\wedge$ be the set of function $f : [0, +\infty) \to [0, +\infty)$ such that
(i) f is Lebesgue integral on each compact subset of $[0, +\infty)$,
(ii)$\int_0^\epsilon f(t)dt > 0$ for every $\epsilon > 0$.
For this class of functions, we can state the following result.

Theorem 3.2. *Let (X, d) be a Hausdorff rectangular metric space and let f and g be two self maps define onto itself such that $fX \subset gX$ and (gX, d) is a complete rectangular metric space. Suppose that the following condition holds:*

$$\int_0^{d(fx,fy)} f(r)dr \leq \int_0^{M(gx,gy)} f(r)dr - \int_0^{M(gx,gy)} h(r)dr + \int_0^{m(gx,gy)} f(r)dr - \int_0^{m(gx,gy)} h(r)dr$$

for all $x, y \in X$ and $f, h \in \wedge$, where

$M(gx, gy) = max\{d(gx, gy), d(gx, fx), d(gy, fy)\}$

$m(gx, gy) = min\{d(gx, fy), d(gy, fy), d(gx, fx), d(gy, fx)\}$

Then f and g have a unique common fixed point.

Proof. Let $\psi(r) = \int_0^r f(u)du$ and $\phi(r) = \int_0^r f(u)du$. Then ψ and ϕ are function in Ψ. By theorem 3.1, we get f and g have a unique common fixed point.

Example 3.3. Let $X = A \cup B$, where $A = \left\{\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}\}\right\}$ and $B = [1, 4]$. Define $d : X \times X \to [0, +\infty)$ such that $d(x, y) = d(y, x)$; for all $x, y \in X$ and

$$\begin{cases} d\left(\frac{1}{2}, \frac{3}{4}\right) & = d\left(\frac{5}{6}, \frac{7}{8}\right) = 0.3 \\ d\left(\frac{1}{2}, \frac{7}{8}\right) & = d\left(\frac{3}{4}, \frac{5}{6}\right) = 0.2 \\ d\left(\frac{1}{2}, \frac{5}{6}\right) & = d\left(\frac{7}{8}, \frac{3}{4}\right) = 0.6 \\ d\left(\frac{1}{2}, \frac{1}{2}\right) & = d\left(\frac{3}{4}, \frac{3}{4}\right) = d\left(\frac{5}{6}, \frac{5}{6}\right) = d\left(\frac{7}{8}, \frac{7}{8}\right) = 0 \end{cases}$$

and $d\big(x, y\big) = |x - y|$ if $x, y \in B$ or $x \in A, y \in B$ or $x \in B, y \in A$.

It is clear that d does not hold the triangle inequality on A. Indeed,

$$0.6 = d\left(\frac{1}{2}, \frac{5}{6}\right) \geq d\left(\frac{1}{2}, \frac{3}{4}\right) + d\left(\frac{3}{4}, \frac{5}{6}\right) = 0.3 + 0.2 = 0.5$$

Remark that d holds the rectangular inequality. Hence d is a rectangular metric.

Notice that $(X|_B, d)$ is usual metric space and hence it is Hausdorff. On the other hand, each singleton is closed and open in $(X|_A, d)$ and hence (X, d) is Hausdorff rectangular metric space.

Let $f, g : X \to X$ be defined as

$$fx = \begin{cases} \frac{7}{8}, \text{ if } x \in [1, 4] \\ \\ \frac{5}{6}, \text{ if } x \in [\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}], \end{cases}$$

$$gx = \begin{cases} \frac{3}{4}, \text{ if } x \in [1, 4] \\ \\ \frac{5}{6}, \text{ if } x \in [\frac{1}{2}, \frac{5}{6}] \\ \\ \frac{7}{8}, \text{ if } x \in [\frac{3}{4}] \\ \\ \frac{1}{2}, \text{ if } x \in [\frac{7}{8}] . \end{cases}$$

Define $\psi(t) = t$ and $\phi(t) = t/3$. Then f and g satisfy the condition of theorem (3.1) and have a unique common fixed point $x = \frac{5}{6}$ of X.

4. Conclusion

In this article we proposed some new fixed point result of (ψ, ϕ) contraction in rectangular metric spaces and proved with suitable example.

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