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NEW OPERATOR VIA STRONG FORM OF NANO OPEN SETS

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Abstract: The aim of this paper is to introduce an operator γ, a function from N-BO(U) $\rightarrow$ P(U) in nano topological space (U,τ_R(X)). Here we have also characterised various properties based on the operation γ. Moreover we have made an attempt to develop some new spaces through Nγ_b open sets in nano topological space and finally we have also examined the relationship among them.

Keywords and Phrases: Nγ_b-open set, Nγ_b-Cl, Nγ_b-g-closed set, Nγ_b-T $_{1/2}$ space.

2020 Mathematics Subject Classification: 54A05, 54C20, 54D10.

1. Introduction

Lellis Thivagar et al initiated nano topological space with respect to a subset X of an universe which is defined in terms of lower and upper approximations of X. The elements of a nano topological space are called the nano-open sets. But certain nano terms are satisfied simply to mean “very small”. It originates from the

Greek word 'Nanos' which means 'dwarf' in its modern scientific sense, an order to magnititude - one billionth of something. Nano car is an example. The topology recommended here is named so because of its size, since it has atmost five elements in it. Ogata defined the notion γ-open set in a topological space. Here we use this new operator γ to define the notion of new form of nano open sets namely Nγ_b-open sets and some of its characterisations were studied. Further we have defined N-bCl$_\gamma$ and Nγ_b-g-closed in nano topological space (U,τ_R(X)) and also studied its properties. Finally we have developed some new spaces via Nγ_b open sets in nano topological space and also examined the relationship among them.

2. Preliminaries

In this section we recall some preliminary definition which are required in the sequel of our work.

Definition 2.1. [9] *Let U be a nonempty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation.Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U,R) is said to be the approximation space. Let $X \subseteq U$.*

(i) The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and its is denoted by $L_R(X)$. That is, $L_R(X) = \cup_{x \in U}\{R(x) : R(x) \subseteq X\}$.

(ii) The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and its is denoted by $U_R(X)$. That is, $U_R(X) = \cup_{x \in U}\{R(x) : R(x) \cap X \neq \phi\}$.

(iii) The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X)$=$U_R(X) - L_R(X)$.

Definition 2.2. [9] *Let U be the universe, R be an eqivalence relation on U and $\tau_R(X) = \{\ \phi,\ U,\ L_R(X),\ U_R(X),\ B_R(X)\}$ where $X \subseteq U$.*
Then,$\tau_R(X)$ satisfies the following axioms:

(i) U and $\phi \in \tau_R(X)$.

(ii) The union of the elements of any subcollection of $\tau_R(X)$ is in $\tau_R(X)$.

(iii) The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.

That is, $\tau_R(X)$ is a topology on U called the Nano topology on U with respect to X. We call (U,$\tau_R(X)$) as the Nano topological space. The elements of $\tau_R(X)$ are called as Nano open sets. A subset E of U is Nano closed if its complement is Nano open.

Definition 2.3. [3] *A subset A of a nano topological space (U,$\tau_R(X)$) is said to be N_b-open if A $\subseteq$ N-Cl(N-Int(A)) $\cup$ N-Int(N-Cl(A)).*
The collection of N_b-open is denoted by N-BO(U).

Definition 2.4. [3] *Let (X,τ) be a topological space. An operation γ on the topology τ is a function from BO(X) into power set P(X) such that $V \subset V^\gamma$ for each $V \in \tau$, Where V^γ denote the value of γ at V and BO(X) denote the collection of b-open sets. It is denoted by γ : BO(X) $\rightarrow$ P(X).*

Definition 2.5. [5] *A subset A of a topological space (X,τ) is called b-γ-open set if, for each $x \in A$, there exists an b-open set V such that $x \in V$ and $V^\gamma \subset A$. γBO(X) denotes the set of all b-γ-open sets in (X,τ). The complement of a b-γ-open set is called γ-b-closed.*

Definition 2.6. [5] *Let A be subset of a topological space (X,τ) and γ : BO(X,τ) $\rightarrow$ P(X) an operation on BO(X,τ). Then the τ_γ-b-closure of A is defined as the intersection of all b-γ-closed sets containing A. That is, τ_γ-bCl(A) = $\cap$ $\{F : F$ is b-γ-closed and $A \subseteq F\}$.*

Definition 2.7. [3] *A topological space (X,τ) is said to be b-γ-regular, where γ is an operation on BO(X,τ), if for each $x \in X$ and for each b-open neighborhood V of x, there exists an b-open neighborhood H of x such that H^γ contained in V.*

Definition 2.8. [3] *An operation γ on τ is said to be regular, if for any open neighborhoods U, V of $x \in X$, there exists an open neighborhood W of x such that $W^\gamma \subseteq U^\gamma \cap V^\gamma$.*

Theorem 2.9. [3] *If A is a b-γ-open sets in (X,τ), then it is b-open set.*

Remark 2.10. [3] *γBO(X) is not topological space.*
That is, finite intersection need not be open.

Theorem 2.11. [3] *If A and B are two b-γ-open subsets and γ is a regular operator, then $A \cap B$ is b-γ-open.*

Definition 2.12. [3] *Let A be subset of a topological space (X,τ) and γ : BO(X) $\rightarrow$ P(X) an operation on BO(X). A point $x \in X$ is in the bCl_γ -closure of a set A if $U^\gamma \cap A = \phi$ for each b-open set U containing x. The bCl_γ -closure of A is denoted by $bCl_\gamma(A)$.*

Definition 2.13. [5] *A topological space* (X,τ) *is said to be*

(i) b-γ-T_0 *if for any two distinct points* x , $y \in X$ *there exists b-open set* U *such that either* $x \in U$ *and* $y \notin U^\gamma$ *or* $y \in U$ *and* $x \notin U^\gamma$.

(ii) b-γ-T_1 *if for any two distinct points* x , $y \in X$ *there exist two b-open sets* U *and* V *containing* x *and* y *respectively such that* $y \notin U^\gamma$ *and* $x \notin V^\gamma$.

(iii) b-γ-T_2 *if for any two distinct points* x , $y \in X$ *there exist two b-open sets* U *and* V *containing* x *and* y *respectively such that* $U^\gamma \cap V^\gamma = \phi$.

Definition 2.14. [5] *Let* A *be a subset of a topological space* (X,τ) *with an operation* γ *on* τ*. The b-*γ*-kernel of* A*, denoted by* $\gamma ker_b(A)$ *is defined to be the set* $\gamma ker_b(A) = \cap \{V \in \gamma\text{-}BO(X) : A \subseteq V\}$.

3. Nγ_b-Open Sets

In this section we introduce a new operator γ to define the notion of new form of nano open sets namely Nγ_b-open sets and some of its characterisations were studied.

Definition 3.1. *Let* $(U,\tau_R(X))$ *be a nano topological space. An operation* γ *is a function from N-BO(U)* $\rightarrow$ *P(U) such that* $V \subseteq V^\gamma$ *for each* $V \in$ *N-BO(U), Where* V^γ *denote the value of* γ *at* V *and N-BO(U) denote the collection of* N_b*-open sets.*

Definition 3.2. *A subset* A *of a nano topological space* $(U,\tau_R(X))$ *is called* $N\gamma_b$*-open set if, for each* $x \in A$*, there exists an* N_b*-open set* V *such that* $x \in V$ *and* $V^\gamma \subseteq A$*.* $N\gamma$*-BO(U) denotes the set of all* $N\gamma_b$*-open sets in* $(U,\tau_R(X))$*. The complement of a* $N\gamma_b$*-open set is called* $N\gamma_b$*-closed.*

Example 3.3. Let U = {a,b,c} be a universe and U / R = { {a}, {b,c}} be a equivalence relation. Let X = {a,b} $\subseteq$ U, $\tau_R(X)$ = { ϕ, U, {a}, {b,c}}.
N-BO(U) = { ϕ, U, {a}, {b}, {c}, {a,b}, {b,c}, {c,a}}.
Then the operation γ : N-BO(U) $\rightarrow$ P(U) defined by

$$\gamma(A) = \begin{cases} A & \text{if } a \in A \\ N_b\text{-}Cl(A) & \text{if } a \notin A \end{cases}$$

for A $\in$ N-BO(U). Then Nγ-BO(U) = { ϕ, U, {a}, {b}, {c}, {a,b}, {b,c}, {c,a}}, where Nγ-BO(U) denote the collection of Nγ_b-open sets.

Theorem 3.4. *Let* γ : *N-BO(U)* $\rightarrow$ *P(U) be any operation on N-BO(U). Then* *(i) Every $N\gamma_b$-open set of $(U,\tau_R(X))$ is N_b-open in $(U,\tau_R(X))$.*

(ii) An arbitrary union of $N\gamma_b$-open sets is $N\gamma_b$-open set.

Proof. (i). Suppose that A $\in$ Nγ-BO(U). Let $x \in$ A. Then, there exists a N_b-open set U(x) containing x such that $U(x)^\gamma \subseteq$ A. Then,
$\cup$ { U(x) : x $\in$ A} $\subseteq \cup$ { $U(x)^\gamma$: $x \in$ A} $\subseteq$ A and so
A = $\cup$ { U(x) : $x \in$ A} $\in$ N-BO(U). Hence A is N_b-open set.
(ii). Let $x \in \cup$ { A_i : i $\in$ J}, where J is any index set, then $x \in A_i$ for some i $\in$ J. Since A_i is $N\gamma_b$-open set, there exists a N_b-open set U containing x such that U^γ $\subseteq A_i \subseteq \cup$ {A_i : i $\in$ J}.
Hence $\cup$ {A_i : i $\in$ J} is a $N\gamma_b$-open set.

Remark 3.5. *The following example shows that converse of the above Theorem (i), does not be hold.*

Example 3.6. Let U = { a, b, c, d} be a universe and U / R={ {a}, {b}, {c}, {d}} be a equivalence relation. Let X = {a,b} $\subseteq$ U,
$\tau_R(X)$ = { ϕ, U, {a,b}}.
N-BO(U) = { ϕ, U, {a}, {b}, {a,b}, {b,c}, {c,a}, {a,d}, {b,d}, {a,b,c}, {b,c,d}, {c,d,a}, {d,a,b}}.
Then the operation γ : N-BO(U) $\to$ P(U) defined by, γ(A) = A $\cup$ {a}, for A $\in$ N-BO(U).
Nγ-BO(U) = { ϕ, U, {a}, {a,b}, {a,b,c}, {a,c}, {a,d}, {c,d,a}, {d,a,b}}.
Then {b,d} is N_b-open set but not $N\gamma_b$-open set.

Remark 3.7. *The following Example Shows that, the intersection of $N\gamma_b$-open sets need not be $N\gamma_b$-open set.*

Example 3.8. Let U = {a,b,c,d} be an universe and U / R = { {a}, {b,c}, {d}} be a partition of U and X = {a,b}. Then $\tau_R(X)$ = { ϕ, U, {a}, {b,c}, {a,b,c}} and N-BO(U) = { ϕ, U, {a}, {b}, {c}, {a,b}, {b,c}, {c,a}, {a,d}, {a,b,c}, {b,c,d}, {c,d,a}, {d,a,b}}.
Define γ : N-BO(U) $\to$ P(U) by $\gamma(A)$ = A for A $\in$ N-BO(U).
Nγ-BO(U) = { ϕ, U, {a}, {b}, {c}, {a,b}, {b,c}, {c,a}, {a,d}, {a,b,c}, {b,c,d}, {c,d,a}, {d,a,b}}.
Let A = {a,d} and B = {b,c,d} where A and B are $N\gamma_b$-open sets
but A $\cap$ B = {d} is not $N\gamma_b$-open set.

Definition 3.9. *A nano topological space $(U,\tau_R(X))$ is said to be $N\gamma_b$-regular, where γ is an operation on BO(U), if for each $x \in U$ and for each N_b-open neighborhood V of x, there exists an N_b-open neighborhood H of x such that H^γ contained* *in V.*

Theorem 3.10. *Let (U,τ_R(X)) be a nano topological space and γ : N-BO(U) $\rightarrow$ P(U) an operation on N-BO(U). Then the following statements are equivalent:*

(i) N-BO(U) = Nγ-BO(U).

(ii) (U,τ_R) is a Nγ_b-regular space.

(iii) For every $x \in$ U and for every N$_b$-open set V of (U,τ_R(X)) containing x, there exists a Nγ_b-open set W of (U,τ_R(X)) such that $x \in$ W and W $\subseteq$ V.

Proof. (i) $\Rightarrow$ (ii) Let $x \in$ U and V be a N$_b$-open set containing x. By assumption, V is a Nγ_b-open set.This implies that there exists a N$_b$-open set W containing x such that $W^\gamma \subseteq$ V. Hence (U,τ_R(X)) is Nγ_b-regular.
(ii) $\Rightarrow$ (iii) Let $x \in$ U and V be a N$_b$-open set containing x. Then by (ii), there is a N$_b$-open set W containing x and W $\subseteq W^\gamma \subseteq$ V. By using (ii), it is shown that W is Nγ_b-open. Hence W is a Nγ_b-open set containing x such that W $\subseteq$ V.
(iii) $\Rightarrow$ (i) We Know that "Nγ-BO(U) $\subseteq$ N-BO(U)". Since W is Nγ_b-open, then there exist a N$_b$-open S such that $S^\gamma \subseteq$ W $\subseteq$ V.Then V is a Nγ_b-open set. Therefore, N-BO(U) $\subseteq$ Nγ-BO(U).

Definition 3.11. *An operation γ : N-BO(U) $\rightarrow$ P(U) is called N$_b$-regular if for each point $x \in$ U and for every pair of N$_b$-open sets say, W and V containing $x \in$ U, there exists a N$_b$-open set S such that $x \in$ S and $S^\gamma \subseteq W^\gamma \cap V^\gamma$.*

Example 3.12. The operation γ defined in Example 3.3. is N$_b$-regular.

Theorem 3.13. *Let γ : N-BO(U) $\rightarrow$ P(U) be a N$_b$-regular operation. If A and B are Nγ_b-open sets in (U,τ_R(X)), then A $\cap$ B is also Nγ_b-open in (U,τ_R(X)).*
Proof. Let $x \in$ A $\cap$ B. Since A and B are Nγ_b-open sets, there exist N$_b$-open sets W, V such that $x \in$ W, $x \in$ V and W $\subseteq$ A and V $\subseteq$ B. By N$_b$-regularity of γ, there exists a N$_b$-open set S containing x such that $S^\gamma \subseteq W^\gamma \cap V^\gamma \subseteq$ A $\cap$ B. Therefore, A $\cap$ B is a Nγ_b-open set

Remark 3.14. *By Theorem 3.4., (ii) and 3.13, Nγ-BO(U) is topological space but not nano topological space.*

4. N-bCl$_\gamma$-closure

In this section we have defined the Nγ_b-closure and some of its properties were studied.

Definition 4.1. *Let A be subset of a nano topological space (U,τ_R(X)) and γ :*

N-BO(U) $\rightarrow$ P(U) an operation on N-BO(U). Then the $N\gamma_b$-closure of A is defined as the intersection of all $N\gamma_b$-closed sets containing A. That is, $N\gamma_b$-Cl(A) = $\cap$ {F : F is $N\gamma_b$-closed and A $\subseteq$ F}.

Example 4.2. In Example 3.6. Let A = {b,c,d}, then $N\gamma_b$-Cl(A) = A.

Definition 4.3. *Let A be subset of a nano topological space (U,τ_R(X)) and γ : N-BO(U) $\rightarrow$ P(U) an operation on N-BO(U). A point $x \in$ U is in the N-bCl_γ-closure of a set A if $V^\gamma \cap A \neq \phi$ for each N_b-open set V containing x. The N-bCl_γ-closure of A is denoted by N-bCl_γ(A).*

Example 4.4. In Example 3.3., A = {a,b}.Then N-bCl_γ(A) = {a,b}.

Example 4.5. In Example 3.6., A = {a,b,c}. Then N-bCl_γ(A) = U.

Theorem 4.6. *For a point $x \in U$, $x \in N\gamma_b$-Cl(A) if and only if for all $N\gamma_b$-open set V of U containing x, $V \cap A \neq \phi$.*
Proof. Let F be the set of all $y \in$ U such that V $\cap$ A $\neq \phi$ for every V $\in$ Nγ-BO(U) and y $\in$ V. To prove that F = $N\gamma_b$-Cl(A). Let $x \in N\gamma_b$-Cl(A). Let us assume $x \notin$ F, then there exists a $N\gamma_b$-open set W of x such that W $\cap$ A = ϕ. This implies A $\subseteq$ W^c. Therefore $x \notin N\gamma_b$-Cl(A) Which is a contradiction. Hence $N\gamma_b$-Cl(A) $\subseteq$ F. Conversely, let $x \notin N\gamma_b$-Cl(A). Then there exists a $N\gamma_b$-closed set E such that A $\subseteq$ E and $x \notin$ E. Then we have that $x \in$ U / E, U / E $\in$ Nγ-BO(U) and (U / E) $\cap$ A = ϕ. This implies that $x \notin$ F. Hence F $\subseteq N\gamma_b$-Cl(A). Therefore, we have that F = $N\gamma_b$-Cl(A).

Theorem 4.7. *For a nano topological space (U,τ_R(X)), N_b-Cl(A) $\subseteq$ N-bCl_γ(A).*
Proof. Let $a \in N_b$-Cl(A).Then by above theorem, for all N_b-open set W of U containing a, W $\cap$ A $\neq \phi$. We know that W $\subseteq W^\gamma$. Then $W^\gamma \cap$ A $\neq \phi$. Therefore, $a \in$ N-bCl_γ(A).

Corollary 4.8. *If (U,τ_R(X)) is $N\gamma_b$-regular, then $N\gamma_b$-Cl(A) $\subseteq$ N-bCl_γ(A).*
Proof. We Know that "If (U,τ_R) is a $N\gamma_b$-regular space, then N-BO(U) = Nγ-BO(U) ". Then $N\gamma_b$-Cl(A) = N_b-Cl(A).
By above theorem, $N\gamma_b$-Cl(A) $\subseteq$ N-bCl_γ(A).

Theorem 4.9. *Let A and B be subsets of a nano topological space (U,τ_R(X)) and γ : N-BO(U) $\rightarrow$ P(U) an operation on N-BO(U). Then we have the following properties:*

(i) *The set N-bCl_γ(A) is a N_b-closed set of (U,τ_R(X)) and A $\subseteq$ N-bCl_γ(A).*

(ii) *A is N_b-closed if and only if N-bCl_γ(A) = A.*

(iii) *If A $\subseteq$ B, then N-bCl$_\gamma$(A) $\subseteq$ N-bCl$_\gamma$(B).*

(iv) *N-bCl$_\gamma$(A) $\cup$ N-bCl$_\gamma$(B) $\subseteq$ N-bCl$_\gamma$(A $\cup$ B).*

(v) *If γ : N-BO(U) $\rightarrow$ P(U) is N$_b$-regular, then N-bCl$_\gamma$(A $\cup$ B) = N-bCl$_\gamma$(A) $\cup$ N-bCl$_\gamma$(B) holds.*

Proof. (i). For each point $x \in$ U / N-bCl$_\gamma$(A), then $x \notin$ N-bCl$_\gamma$(A). Then, there exists a N$_b$-open set V(x) containing x such that $V(x)^\gamma \cap$ A = ϕ. Let w = $\cup$ {V(x) : V(x) $\in$ N-BO(U), $x \in$ U / N-bCl$_\gamma$(A)}. Then it is shown that W = U / N-bCl$_\gamma$(A) holds. For a point $y \in$ W , there exists V(x) $\in$ N-BO(U) such that $y \in$ V(x) and $V(x)^\gamma \cap$ A = ϕ. This shows that $y \notin$ N-bCl$_\gamma$(A) and so W $\subseteq$ U / N-bCl$_\gamma$(A). Conversely, let $y \in$ U / N-bCl$_\gamma$(A), then $y \notin$ N-bClγ(A). Then there exists V(y) $\in$ N-BO(U) such that $y \in$ V(y) and $V(y)^\gamma \cap$ A = ϕ and so $y \in$ V(y) $\subseteq$ W . Thus, we conclude that U / N-bCl$_\gamma$(A) $\subseteq$ W ; It follows that W = U / N-bCl$_\gamma$(A). since W $\in$ N-BO(U), N-bCl$_\gamma$(A) is N$_b$-closed in (U,τ_R(X)) . Clearly, we have A $\subseteq$ N-bCl$_\gamma$(A).
(ii). Suppose that A is N$_b$-closed. Then U / A is N$_b$-open in (U,τ_R(X)). We claim that N-bCl$_\gamma$(A) $\subseteq$ A. Let $x \notin$ A. There exists a N$_b$-open set V containing x such that $V^\gamma \subseteq$ U / A, that is, $V^\gamma \cap$ A = ϕ. Hence we have that $x \notin$ N-bCl$_\gamma$(A) and so N-bCl$_\gamma$(A) $\subseteq$ A. By (i), N-bCl$_\gamma$(A) = A. Conversely, suppose that N-bCl$_\gamma$(A) = A. Let $x \in$ U / A. Since $x \notin$ N-bCl$_\gamma$(A), there exists a N$_b$-open set W containing x such that $W^\gamma \cap$ A = ϕ, that is
$W^\gamma \subseteq$ U / A. Therefore, A is N$_b$-closed.
(iii). By (i), A $\subseteq$ N-bCl$_\gamma$(A). Since A $\subseteq$ B, N-bCl$_\gamma$(A) $\subseteq$ B.
Then N-bCl$_\gamma$(A) $\subseteq$ N-bCl$_\gamma$(B).
(iv). By (i), A $\subseteq$ N-bCl$_\gamma$(A). Since A $\subseteq$ A $\cup$ B, N-bCl$_\gamma$(A) $\subseteq$ N-bCl$_\gamma$(A $\cup$ B). Since B $\subseteq$ A $\cup$ B, N-bCl$_\gamma$(A) $\cup$ N-bCl$_\gamma$(B) $\subseteq$ N-bCl$_\gamma$(A $\cup$ B).
(v). Let $x \notin$ N-bCl$_\gamma$(A) $\cup$ N-bCl$_\gamma$(B), then there exist two N$_b$-open sets W and V containing x such that $W^\gamma \cap$ A = ϕ and $V^\gamma \cap$ B = ϕ.
Since γ is a N$_b$-regular operator, there exists a N$_b$-open set S containing x such that $S^\gamma \subseteq W^\gamma \cap V^\gamma$. Thus, we have $S^\gamma \cap$ (A $\cap$ B) $\subseteq$ ($W^\gamma \cap V^\gamma$) $\cap$ (A $\cup$ B) $\subseteq$ (($W^\gamma \cap V^\gamma$) $\cup$ A) $\cap$ (($W^\gamma \cap V^\gamma$) $\cap$ B) $\subseteq$ ($W^\gamma \cap$ A) $\cup$ ($V^\gamma \cap$ B) = ϕ, that is, W^γ $\cap$ (A $\cup$ B) = ϕ. Hence, $x \notin$ N-bCl$_\gamma$ (A $\cup$ B).
This shows that N-bCl$_\gamma$(A) $\cup$ N-bCl$_\gamma$(B) $\subseteq$ N-bCl$_\gamma$(A $\cup$ B).

Theorem 4.10. *Let A and B be any two subsets of a topological space (U,τ) and γ : BO(U) $\rightarrow$ P(U) an operation on BO(U). Then bCl$_\gamma$(A $\cap$ B) = bCl$_\gamma$(A) $\cap$ bCl$_\gamma$(B) is true for discrete topological space with γ(A) = A for A $\in$ BO(X).*

Remark 4.11. *If A and B are any two subsets of a nano topological space, then*

N-bCl_γ(A ∩ B) = N-bCl_γ(A) ∩ N-bCl_γ(B), when $\tau_R(X) = \{\phi, U, L_R(X), B_R(X)\}$ where $U_R(X) = U$, with $\gamma(A) = A$ for $A \in$ N-BO(U) and equality does not holds for other types of nano topological space which can be shown by the following example.

Example 4.12. In Example 3.6., A = {a} and B = {b,c} and A ∩ B = ϕ. Then N-bCl_γ(A) = U and N-bCl_γ(B) = B $\implies$ N-bCl_γ(A) ∩ N-bCl_γ(B) = B but N-bCl_γ(A ∩ B) = ϕ. Therefore, N-bCl_γ(A ∩ B) ≠ N-bCl_γ(A) ∩ N-bCl_γ(B).

Theorem 4.13. *Let A and B be subsets of a nano topological space (U,$\tau_R(X)$) and γ : N-BO(U) → P(U) an operation on N-BO(U). Then we have the following properties:*

(i) *The set Nγ_b-Cl(A) is a Nγ_b-closed set of (U,$\tau_R(X)$) and A ⊆ Nγ_b-Cl(A).*

(ii) *A is Nγ_b-closed if and only if Nγ_b-Cl(A) = A.*

(iii) *If A ⊆ B, Nγ_b-Cl(A) ⊆ Nγ_b-Cl(B).*

(iv) *Nγ_b-Cl(A) ∪ Nγ_b-Cl(B) ⊆ Nγ_b-Cl(A ∪ B).*

(v) *If γ : N-BO(U) → P(U) is N_b-regular, then Nγ_b-Cl(A ∪ B) = Nγ_b-Cl(A) ∪ Nγ_b-Cl(B) holds.*

(vi) *Nγ_b-Cl(A ∩ B) ⊆ Nγ_b-Cl(A) ∩ Nγ_b-Cl(B).*

Proof. It follows that the above theorem.

Theorem 4.14. *For a subset A of a nano topological space (U,$\tau_R(X)$) and any operation γ : N-BO(U) → P(U), then N-bCl_γ(A) ⊆ Nγ_b-Cl(A).*

Proof. We Know that "A ⊆ Nγ_b-Cl(A)". Then N-bCl_γ(A) ⊆ N-bCl_γ(Nγ_b-Cl(A)). But A ⊆ N-bCl_γ(A), then N-bCl_γ(A) ⊆ Nγ_b-Cl(A).

Theorem 4.15. *Let A be a subset of a nano topological space (U,$\tau_R(X)$) and (U,$\tau_R(X)$) is Nγ_b-regular and γ : N-BO(U) → P(U) an operation on N-BO(U). The following properties are equivalent:*

(i) *A subset A is a Nγ_b-open in (U,$\tau_R(X)$).*

(ii) *N-bCl_γ(U / A) = U / A*

Proof. (i) ⇒ (ii) Let A be a Nγ_b-open. We Know that "Every Nγ_b-open set of (U,τ_R(X)) is N_b-open in (U,τ_R(X))". Then U / A is Nγ_b-closed. Therefore, N-bCl_γ(U / A) = U / A.

(ii) ⇒ (i) Since N-bCl$_\gamma$(U / A) = U / A, U / A is N$_b$-closed. Then A is N$_b$-open. Therefore, A is Nγ_b-open.

5. Nγ_b-g-closed

In this section we have defined Nγ_b-ker and also a new form of nano closed set namely Nγ_b-g-closed set and some of its properties were anlaysed with respect to its Nγ_b-Cl.

Definition 5.1. *Let A be a subset of a nano topological space (U,$\tau_R(X)$) with an operation γ on N-BO(U).The Nγ_b-kernel of A, denoted by Nγ_b-ker(A) is defined to be the set Nγ_b-ker(A) = ∩ {$V \in$ Nγ-BO(U,$\tau_R(X)$) : $A \subseteq V$}.*

Example 5.2. Let V = {a,b,c} be a universe and V / R = { {a}, {b,c}} be a equivalence relation.Y = {a,b} ⊆ V, $\tau_R(Y)$ = { ϕ, V, {a}, {b,c}}.
N-BO(U) = { ϕ, U, {a}, {b}, {c}, {a,b}, {b,c}, {c,a}}. Then the operation γ : N-BO(U) → P(V) defined by γ(A) = A, for every A ∈ N-BO(U). Nγ-BO(U) = { ϕ, U, {a}, {b}, {c}, {a,b}, {b,c}, {c,a}}. Let A = {a,b}. Then Nγ_b-ker(A) = {a,b} = A.

Remark 5.3. *For any subset A of a nano topological space (U,$\tau_R(X)$),*
A ⊆ Nγ_b-ker(A).

Theorem 5.4. *If A is Nγ_b-open set in (U,$\tau_R(X)$), then A = Nγ_b-ker(A).*
Proof. We know that A ⊆ Nγ_b-ker(A).
It suffices to prove that Nγ_b-ker(A) ⊆ A.
Let x ∈ Nγ_b-ker(A). Then x ∈ V for each x and V is Nγ_b-open set.
But A itself Nγ_b-open set, x ∈ A.
∴ A = Nγ_b-ker(A).

Remark 5.5. *The converse of the above theorem need not be true.*

Example 5.6. In Example 3.8, A = {c,d}
Then Nγ_b-ker(A) = {d,b,c} ∩ {b,c,d} = {c,d} = A.
But not Nγ_b-open set.

Remark 5.7. *If A ⊆ B, then Nγ_b-ker(A) ⊆ Nγ_b-ker(B).*
Proof. We Know that A ⊆ Nγ_b-ker(A).
But A ⊆ B, Nγ_b-ker(A) ⊆ B.
∴ Nγ_b-ker(A) ⊆ Nγ_b-ker(B).

Theorem 5.8. *Let (U,$\tau_R(X)$) be a nano topological space with an operation γ on N-BO(U) and $x \in$ U.Then $y \in$ Nγ_b-ker({x}) if and only if*
$x \in$ Nγ_b-Cl({y}).
Proof. Suppose that $y \notin$ Nγ_b-ker({x}).

Then there exists a Nγ_b-open set V containing x such that $y \notin$ V.
$\therefore$ we have $x \notin$ Nγ_b-Cl($\{y\}$).
The converse proof follows from the reverse.

Lemma 5.9. *Let (U,$\tau_R(X)$) be a nano topological space and A be a subset of U. Then, Nγ_b-ker(A) = \{ x $\in$U : Nγ_b-Cl($\{x\}$) $\cap$ A $\neq \phi$\}.*
Proof. Let $x \in$ Nγ_b-ker(A) and suppose that Nγ_b-Cl($\{x\}$) $\cap$ A = ϕ.
Hence $x \notin$ U / Nγ_b-Cl($\{x\}$) which is a Nγ_b-open set containing A.
This is impossible, since $x \in$ Nγ_b-ker(A).
$\therefore$ Nγ_b-Cl($\{x\}$) $\cap$ A $\neq \phi$.
Conversely, let $x \in$ U such that Nγ_b-Cl($\{x\}$) $\cap$ A $\neq \phi$ and suppose that $x \notin$ Nγ_b-ker(A).
Then, there exists a Nγ_b-open set V containing A and $x \notin$ V.
Let $y \in$ Nγ_b-Cl($\{x\}$) $\cap$ A.
Hence, V is a Nγ_b-neighborhood of y which does not contain x.
By this contradiction, we have $x \in$ Nγ_b-ker(A).
$\therefore$ Nγ_b-ker(A) = \{ x $\in$ U : Nγ_b-Cl($\{x\}$) $\cap$ A $\neq \phi$\}.

Definition 5.10. *A subset A of a nano topological space (U,$\tau_R(X)$) is said to be Nγ_b-generalized closed (namely, Nγ_b-g-closed) in (U,$\tau_R(X)$) if N-bCl$_\gamma$(A) $\subseteq$ V whenever A $\subseteq$ V and V is a Nγ_b-open set of (U,$\tau_R(X)$). The complement of a Nγ_b-g-closed set is called a Nγ_b-g-open set.*

Example 5.11. In Example 5.2, N-BO(U) = Nγ-BO(U) = \{ ϕ, U, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}\}. Therefore, \{a\} is Nγ_b-g-closed set.

Theorem 5.12. *Let γ : N-BO(U) $\to$ P(U) be an operation on N-BO(U) and A a subset of a nano topological space (U,$\tau_R(X)$). Then the following statements are equivalent:*

(i) A subset A is Nγ_b-g-closed in (U,$\tau_R(X)$).

(ii) Nγ_b-Cl($\{x\}$) $\cap$ A $\neq \phi$ for every $x \in$ N-bCl$_\gamma$(A).

(iii) N-bCl$_\gamma$(A) $\subseteq$ Nγ_b-ker(A) holds for any subset A of a nano topological space (U,$\tau_R(X)$).

Proof. (i) $\Rightarrow$ (ii) Let A be a Nγ_b-g-closed set of (U,$\tau_R(X)$). Suppose that there exists a point $x \in$ N-bCl$_\gamma$(A) such that Nγ_b-Cl($\{x\}$) $\cap$ A = ϕ. By Theorem 4.13.(i), Nγ_b-Cl($\{x\}$) is a Nγ_b-closed. Put V = U / Nγ_b-Cl($\{x\}$). Then, we have that A $\subseteq$ V, $x \in$ V and V is a Nγ_b-open set of (U,τ_R(X)). Since A is a Nγ_b-g-closed set, N-bCl$_\gamma$(A) $\subseteq$ V. Thus, we have $x \notin$ N-bCl$_\gamma$(A). This is a contradiction.

(ii) $\Rightarrow$ (iii) Let $x \in$ N-bCl$_\gamma$(A). By (ii), Nγ_b-Cl(x) $\cap$ A $\neq \phi$.
Lemma 5.9, $x \in$ Nγ_b-ker(A).
(iii) $\Rightarrow$ (i) Let V be any Nγ_b-open set such that A $\subseteq$ V. Let x be a point such that x $\in$ N-bCl$_\gamma$(A). By (iii), $x \in$ Nγ_b-ker(A).Since U is Nγ_b-open set, then $x \in$ U.

Theorem 5.13. *Let A be a subset of a nano topological space (U,τ_R(X)). If A is Nγ_b-g-closed in (U,τ_R(X)), then N-bCl$_\gamma$(A) / A does not contain any nonempty Nγ_b-closed set.*

Proof. Suppose that there exists a Nγ_b-g-closed set F such that F $\subseteq$ N-bCl$_\gamma$(A) / A. Then, we have that A $\subseteq$ U / F and U / F is Nγ_b-open. It follows from assumption that N-bCl$_\gamma$(A) $\subseteq$ U / F and so
F $\subseteq$ (N-bCl$_\gamma$(A) / A) $\cap$ (U / N-bCl$_\gamma$(A)). Therefore, we have that F = ϕ.

6. Operator approach Separation Axioms

In this section we have made an attempt to develop some new spaces via Nγ_b open sets in nano topological space and finally we have also examined the relationship among them.

Definition 6.1. *A nano topological space (U,τ_R(X)) is said to be a Nγ_b-$T_{1/2}$ space if every Nγ_b-g-closed set of (U,τ_R(X)) is Nγ_b-closed.*

Example 6.2. In Example 5.2, (U,τ_R(X)) is Nγ_b-T$_{1/2}$ space.

Theorem 6.3. *Let A be a subset of a nano topological space (U,τ_R(X)) and γ : N-BO(U) $\rightarrow$ P(U) be an operation on N-BO(U), then for each point $x \in$ U, $\{x\}$ is Nγ_b-closed or $\{x\}$ is Nγ_b-g-open set of (U,τ_R(X)).*

Proof. Suppose that $\{x\}$ is not Nγ_b-closed set. Then $\{x\}$ is not Nγ_b-open set. Let V be any Nγ_b-open set such that U / $\{x\}$ Then V = U and so we have that N-bCl$_\gamma$(U / $\{x\}$)$\subseteq$ V. Therefore, U / $\{x\}$ is a Nγ_b-g-closed set in (U,τ_R(X).

Theorem 6.4. *A nano topological space (U,τ_R(X)) is said to be a Nγ_b-$T_{1/2}$ space if and only if for each point $x \in$ U, $\{x\}$ is Nγ_b-open or Nγ_b-closed in (U,τ_R(X)). Then for each point $x \in$ U, $\{x\}$ is N$_b$-open or N$_b$-closed in (U,τ_R(X)).*

Proof. Suppose that $\{x\}$ is not Nγ_b-closed set, by Theorem 6.3, U / $\{x\}$ is a Nγ_b-g-closed. Since (U,τ_R(X)) is a Nγ_b-T$_{1/2}$ space, U / $\{x\}$ is Nγ_b-closed. Hence $\{x\}$ is a Nγ_b-open set. Conversely, let F be a Nγ_b-g-closed set in (U,τ_R(X)). We shall prove that N-bCl$_\gamma$(F) = F. It is sufficient to show that N-bCl$_\gamma$(F)$\subseteq$ F. Assume that there exists a point x such that $x \in$ N-bCl$_\gamma$(F) / F. Then by assumption, $\{x\}$ is Nγ_b-closed or Nγ_b-open.
Case (i): $\{x\}$ is Nγ_b-closed set. For this case, we have a Nγ_b-closed set $\{x\}$ such that $\{x\} \subseteq$ N-bCl$_\gamma$(F) / F. This is a contradiction to Theorem 5.13. Case (ii): $\{x\}$ is Nγ_b-open set. Then we have $x \in$ N-bCl$_\gamma$(F). Since $\{x\}$ is Nγ_b-open, $\{x\} \cap$

F = ϕ. This is a contradiction. Thus, we have that, N-bCl$_\gamma$(F) = F and so F is Nγ_b-closed set.

Definition 6.5. *A nano topological space (U,$\tau_R(X)$) is said to be*

(i) Nγ_b-T$_0$ if for any two distinct points x,$y \in$ U there exists N$_b$-open set W such that either $x \in$ W and $y \notin W^\gamma$ or $y \in$ W and $x \notin W^\gamma$.

(ii) Nγ_b-T$_1$ if for any two distinct points x,$y \in$ U there exist two N$_b$-open sets W and V containing x and y respectively such that $y \notin W^\gamma$ and $x \notin V^\gamma$.

(iii) Nγ_b-T$_2$ if for any two distinct points x,$y \in$ U there exist two N$_b$-open sets W and V containing x and y respectively such that $W^\gamma \cap V^\gamma = \phi$.

Example 6.6. In Example 5.2., (U,τ_R(X)) is Nγ_b-T$_i$, i=0,1,2.

Theorem 6.7. *A nano topological space (U,$\tau_R(X)$) with an operation γ on N-BO(U) is Nγ_b-T$_0$ if and only if for each pair of distinct points $x, y \in$ U, N-bCl$_\gamma$($\{x\}$) $\neq$ N-bCl$_\gamma$($\{y\}$).*

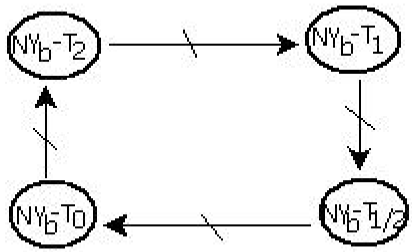
Proof. Suppose that (U,$\tau_R(X)$) is Nγ_b-T$_0$. Let $x, y \in$ U and $x \neq y$.
Then there exists a N$_b$-open set V in U containing x such that $y \notin U^\gamma$ or a N$_b$-open set W in U containing y such that $x \notin W^\gamma$ respectively.
Hence $x \notin$ N-bCl$_\gamma$($\{y\}$) or $y \notin$ N-bCl$_\gamma$($\{x\}$).
$\therefore$ N-bCl$_\gamma$($\{x\}$) $\neq$ N-bCl$_\gamma$($\{y\}$).
Conversely, Suppose that N-bCl$_\gamma$($\{x\}$) $\neq$ N-bCl$_\gamma$($\{y\}$).
Suppose that $x \neq y$ for any $x, y \in$ U. Then we have that, N-bCl$_\gamma$($\{x\}$)) $\neq$ N-bCl$_\gamma$($\{y\}$). Thus, we assume that there exists $z \in$ N-bCl$_\gamma$($\{x\}$) but $z \notin$ N-bCl$_\gamma$($\{y\}$). If $x \in$ N-bCl$_\gamma$($\{y\}$), then we get N-bCl$_\gamma$($\{x\}$) $\subseteq$ N-bCl$_\gamma$($\{y\}$). This implies that $z \in$ N-bCl$_\gamma$($\{y\}$). This contradiction shows that $x \in$ N-bCl$_\gamma$($\{y\}$). Then there exists a N$_b$-open set W such that $x \in$ W and $W^\gamma \cap \{y\} = \phi$. Thus, we have that $x \in$ W and $y \notin W^\gamma$. Hence, (U,τ_R(X)) is a Nγ_b-T$_0$ space.

Theorem 6.8. *A space (U,$\tau_R(X)$) is Nγ_b-T$_1$ if and only if every singleton set of U is Nγ_b-closed.*

Proof. Let (U,$\tau_R(X)$) be Nγ_b-T$_1$ and x be any point of U.
Suppose $y \in$ U/$\{x\}$, then $x \neq y$ and so there exists a Nγ_b-open set V such that $y \in$ V but $x \notin V^\gamma$. Since $y \in$ V $\subseteq$ U/$\{x\}$,
i.e., U/$\{x\}$ = $\cup$ { V : $y \in$ U/$\{x\}$} which is Nγ_b-open set. Therefore $\{x\}$ is Nγ_b-closed set. Since x is arbitrary, $\{x\}$ is Nγ_b-closed set.
Conversely, Suppose $\{p\}$ is Nγ_b-closed set for every $p \in$ U.
Let $x, y \in$ U with $x \neq y$.

Now $x \neq y$ implies $y \in$ U/$\{x\}$.
Hence U/$\{x\}$ is a Nγ_b-open set contains y but not x.
By definition, there exist a N$_b$-open set W such that $y \in$ W but $x \notin W^\gamma$. Similarly U/$\{y\}$ is a Nγ_b-open set contains x but not y and we have S is N$_b$-open, $x \in$ S but $y \notin S^\gamma$.
$\therefore$ (U,τ_R(X)) is a Nγ_b-T$_1$ space.

Remark 6.9. *It is clear that $N\gamma_b$-$T_2 \implies N\gamma_b$-$T_1 \implies N\gamma_b$-$T_{1/2} \implies N\gamma_b$-$T_0$. But the reverse does not hold which is shown in the following figure.*



Example 6.10. Let U = {a,b,c,d} be a universe and U / R = { {a}, {b,c}, {d}} be a equivalence relation. Let X = {a,b} $\subseteq$ U, $\tau_R(X)$ = {ϕ, U, {a}, {b,c},{a,b,c}}. N-BO(U) = { ϕ, U, {a}, {b}, {c}, {a,b}, {b,c}, {c,a}, {d,a}, {a,b,c}, {b,c,d}, {c,d,a}, {d,a,b}}.
Then the operation γ : N-BO(U) $\rightarrow$ P(U) defined by

$$\gamma(A) = \begin{cases} A & \text{if } b \in A \\ N\text{-}Cl(A) & \text{if } b \notin A \end{cases}$$

for A $\in$ N-BO(U). Then Nγ-BO(U) = { ϕ, U, {b}, {a,b}, {b,c}, {c,a}, {a,b,c}, {b,c,d}, {c,d,a}}, where Nγ-BO(U) denoted the collection of Nγ_b-open sets. Clearly, (U,τ_R(X)) is Nγ_b-T$_0$ but not Nγ_b-T$_{1/2}$.

Example 6.11. Let U = {a,b,c} be a universe and U / R = {{a}, {b,c}} be a equivalence relation. Let X = {a,b} $\subseteq$ U, $\tau_R(X)$ = { ϕ, U, {a}, {b,c}}.
N-BO(U) = { ϕ, U, {a}, {b}, {c}, {a,b}, {b,c}, {c,a}}.
Then the operation γ : N-BO(U) $\rightarrow$ P(U) defined by

$$\gamma(A) = \begin{cases} A \cup \{b,c\} & \text{if } A = \{a\}\, or\, \{a,c\}\, or\, \{c\} \\ A & \text{if } A \neq \{a\}, \{a,c\}, \{c\} \end{cases}$$

for A $\in$ N-BO(U). Nγ-BO(U) = {ϕ, U, {b}, {a,b}, {b,c}}.
Clearly, (U,τ_R(X)) is Nγ_b-T$_{1/2}$ but not Nγ_b-T$_1$.

Example 6.12. Let U = {a,b,c} be a universe and U / R = {{a}, {b,c}} be a equivalence relation. Let X = {a,b} ⊆ U, $\tau_R(X)$ = { ϕ, U, {a}, {b,c}}.
N-BO(U) = { ϕ, U, {a}, {b}, {c}, {a,b}, {b,c}, {c,a}}.
Then the operation γ : N-BO(U) → P(U) defined by

$$\gamma(A) = \begin{cases} A \cup \{c\} & \text{if } A = \{a\}\, or\, \{b\} \\ A \cup \{a\} & \text{if } A = \{c\} \\ A & \text{if } A \neq \{a\}, \{b\}, \{c\} \end{cases}$$

for A ∈ N-BO(U). Nγ-BO(U) = {ϕ, U, {a,b}, {b,c}, {c,a}}.
Clearly, (U,τ_R(X)) is Nγ_b-T_1 but not Nγ_b-T_2.

7. Conclusion

In this paper we have introduced Nγ_b-open sets in terms of an operator γ and studied some of its properties. Further it can also be extended in neutrosophic nano topological space and its various properties may be characterised.

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