South East Asian J. of Mathematics and Mathematical Sciences
Vol. 17, No. 3 (2021), pp. 209-214

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

A COMMON FIXED POINT THEOREM FOR COMPATIBLE MAPS IN COMPLEX VALUED METRIC SPACES

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(**Received:** *Oct. 20, 2020* **Accepted:** *Aug. 23, 2021* **Published:** *Dec. 30, 2021)*

Abstract: In this article, we prove a common fixed point theorem for compatible map in complex valued metric spaces without using the notion of continuity for two pairs of map. Our result generalizes and extends the results of Karapinar [5] and Noorwali [6].

Keywords and Phrases: Contractive map, interpolation, common fixed point, compatible mapping, complex valued metric space, partial ordered set.

2020 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

In 2011, Azam, Fisher and Khan [1] introduced the notion of complex valued metric space, which is a generalization of the classical metric space and established sufficient conditions for the existence of a common fixed points of a pair of mapping satisfying a contractive condition. The study of existence of common fixed point developed from commutativety to compatibility and similarly weakly commutativety to weakly compatibility. Also we can say non-commutativety of mapping grown from non-compatibility by some property.

A complex number $z \in \mathbb{C}$ is an ordered pair of real numbers, whose first co-ordinate is called $Re(z)$ and second co-ordinate is called $Im(z)$.

In 1968, Kannan [3, 4] introduced a contraction mapping which is non-continuous and gave a fixed point theorem: If X is a complete metric space and $T : X \to X$ is a mapping satisfying

$$d(Tx, Ty) \leq \alpha \left[d(x, Tx) + d(y, Ty)\right], \quad \forall x, y \in X \text{ and } \alpha \in [0, 1).$$

Then T has a unique fixed point.
Many authors generalized the contractive condition and properties of Kannan in different types of metric space. Recently in 2018, one of the generalization done by Karapinar [5] and Noorwali [6]. They introduced a Kannan-type contraction called interpolative Kannan types contraction.
In our main result we have generalized the above contraction using compatible mapping in complex valued metric space.

2. Preliminaries

First we recall some notations and definitions that will be utilized in our subsequent discussion.

Definition 2.1. *Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows:*
$z_1 \preceq z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$.
Consequently, we can infer that $z_1 \preceq z_2$ if one of the following conditions is satisfied:
(i) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,
(ii) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(iii) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$,
(iv) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,
In particular, we write $z_1 \preceq z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iv) is satisfied and we write $z_1 \prec z_2$ if only (iii) is satisfied.
We Note that $0 \preceq z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$, and $z_1 \preceq z_2, z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

The following definition is recently introduced by Azam et al. [1].

Definition 2.2. *Let X be a nonempty set whereas $\mathbb{C}$ is the set of complex numbers. Suppose that the mapping $d : X \times X \to \mathbb{C}$, satisfies the following conditions:*
(d1) $0 \preceq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(d3) $d(x, y) \preceq d(x, z) + d(z, y)$, for all $x, y, z \in X$.
Then d is called a complex valued metric on X, and (X, d) is called a complex valued metric space.

Example 2.3. Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by $d(z_1, z_2) = e^{ik}|z_1 - z_2|, k \in (0, 1)$. Then (X, d) is a complex valued metric space.

Lemma 2.4. *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 2.5. *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$,* *where m in X.*

Definition 2.6. *Let (X, d) be a complex valued metric space. Then X is said to be complete if every Cauchy sequence in X is convergent in X.*

Definition 2.7. *A point $x \in X$ is said to be a fixed point of T if $Tx = x$.*

Definition 2.8. *A point $x \in X$ is said to be a common fixed point of T and S if $Tx = Sx = x$.*

Definition 2.9. *Let f and g be self-map of a set X (i.e., $f, g : X \to X$). If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.*

Definition 2.10. [2] *Two self-map A and S of a complex valued metric space (X, d) are called compatible if $\lim_{n\to\infty} d(ASx_n, SAx_n, z) = 0$, $z \in X$, where $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.*

Definition 2.11. *Let T and S be self- mappings of a nonempty set X. The mapping T and S are weakly compatible if $TSx = STx$ whenever $Tx = Sx$ for some x that is if T and S have a coincidence point at x.*

Definition 2.12. *Two self- maps T and S of a metric space (X, d) are said to be weakly commutating if $d(TSx, STx) \leq d(Sx, Tx)$ for all x in X.*

In a similar mode, we introduce the notion of weak commutatively in complex valued metric space as follows:

Definition 2.13. *Two self- map T and S of a metric space (X, d) are said to be weakly commutating if $d(TSx, STx) \preceq d(Tx, Sx)$, for all x in X.*

Definition 2.14. *Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by*
$d(x, y) = i|x - y|$, $\forall x, y \in X$.
Then (X, d) is a complex valued metric space.
We define $Sx = x$ and $Tx = x$, then clearly $d(TSx, STx) \preceq d(Tx, Sx)$.
Thus, S and T are weakly commutating.

Definition 2.14. *Two self-map S and T of a complex valued metric space (X, d) are called non-compatible if there exists at least one sequence $\{x_n\}$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$. But $\lim_{n\to\infty} d(STx_n, TSx_n)$ is either non zero or not exist.*

3. Main Results

Theorem 3.1. *Let A, B, S and T be four self-map of a complete complex valued metric space (X, d) such that*

1. $A(X) \subset S(X);\ B(X) \subset T(X)$.
2. For all $x, y \in X, x \neq y$,

$$d(Ax, By) \preceq \lambda[d(Tx, Ax)]^{\alpha}[d(By, Sy)]^{1-\alpha}, \tag{3.1}$$

for some $\lambda \in [0, 1)$ *and* $\alpha \in (0, 1)$.
3. The pairs $\{A, T\}$ *and* $\{B, S\}$ *are compatible.*
4. A *and* T *have a coincidence point.*
5. B *and* S *have a coincidence point.*
Then, also A, B, S *and* T *have a unique common fixed point in* X.

Proof. Suppose x_0 be an arbitrary point in X. We define a sequence $\{y_{2n}\}$ in X such that

$$y_{2n} = Ax_{2n} = Sx_{2n+1},$$

$$y_{2n+1} = Bx_{2n+1} = Tx_{2n+2},$$

Now using (3.1), we have

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(Ax_{2n}, Bx_{2n+1}) \\ &\preceq \lambda[d(Tx_{2n}, Ax_{2n})]^{\alpha}[d(Bx_{2n+1}, Sx_{2n+1})]^{1-\alpha} \\ &= \lambda[d(y_{2n-1}, y_{2n})]^{\alpha}[d(y_{2n+1}, y_{2n})]^{1-\alpha} \\ \text{Or,} \quad [d(y_{2n}, y_{2n+1})]^{\alpha} &\preceq \lambda[d(y_{2n-1}, y_{2n})]^{\alpha} \\ \text{Or,} \quad d(y_{2n}, y_{2n+1}) &\preceq \lambda^{\frac{1}{\alpha}}[d(y_{2n-1}, y_{2n})] \\ \text{Or,} \quad |d(y_{2n}, y_{2n+1})| &\leq \lambda|d(y_{2n-1}, y_{2n})| \end{aligned}$$

Hence,

$$|d(y_{2n}, y_{2n+1})| \leq \lambda|d(y_{2n-1}, y_{2n})| \leq \lambda^2|d(y_{2n-2}, y_{2n-1})| \leq ... + \lambda^{2n}|d(y_0, y_1)|$$

Or,

$$|d(y_{2n+1}, y_{2n+2})| \leq \lambda^{2n}|d(y_0, y_1)| \tag{3.2}$$

Similarly,

$$\begin{aligned} d(y_{2n+1}, y_{2n+2}) &= d(Bx_{2n+1}, Ax_{2n+2}) \\ &= d(Ax_{2n+2}, Bx_{2n+1}) \\ &\preceq \lambda[d(Tx_{2n+2}, Ax_{2n+2})]^{\alpha}[d(Sx_{2n+1}, Bx_{2n+1})]^{1-\alpha} \\ &= \lambda[d(y_{2n+1}, y_{2n+2})]^{\alpha}[d(y_{2n}, y_{2n+1})]^{1-\alpha} \\ \text{Or,} \quad |d(y_{2n+1}, y_{2n+2})| &\leq \lambda|d(y_{2n}, y_{2n+1})| \end{aligned} \tag{3.3}$$

From (3.2) and (3.3) we get

$$|d(y_{2n}, y_{2n+1})| \leq \lambda^{2n}|d(y_0, y_1)|$$

For any $m > n$,

$$\begin{aligned} |d(y_{2n}, y_{2m})| &\leq |d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) + ... + d(y_{2m-1}, y_{2m})| \\ &\leq [\lambda^{2n} + \lambda^{2n+1} + \lambda^{2m-1}]|d(y_0, y_1)| \\ &\leq \left[\frac{\lambda^{2n}}{1-\lambda}\right] |d(y_0, y_1)| \to 0 \text{ as } n, m \to \infty. \end{aligned}$$

Hence $\{y_{2n}\}$ is a Cauchy sequence and since X is complete, $\{y_{2n}\}$ converges to point z (say) in X and its subsequences $Ax_{2n}, Bx_{2n+1}, Sx_{2n+1}$ and Tx_{2n+2} of sequences $\{y_{2n}\}$ also converges to the point z.
Since $A(X) \subset S(X)$, there exists a point $u \in X$ s.t. $z = Su$.
Now,

$$\begin{aligned} d(Bu, z) &\preceq d(Bu, Ax_{2n}) + d(Ax_{2n}, z) \\ &\preceq d(Ax_{2n}, z) + \lambda[d(Tx_{2n}, Ax_{2n})]^{\alpha}[d(Bu, Su)]^{1-\alpha} \end{aligned}$$

When $n \to \infty$, $Tx_{2n} \to z$, $Ax_{2n} \to z$ and putting $Su = z$ we get $d(Bu, z) \preceq \lambda[d(Bu, z)]^{1-\alpha} < \lambda[d(Bu, z)]$, which is only possible when $d(Bu, z) = 0$, so we have $Bu = z$.
That is, $Bu = Su = Z$, showing that u is a coincidence point of S and B. Since, the pair of map B and S are compatible, $BSu = SBu$, i.e., $Bz = Sz$. Again $B(X) \subset T(X)$, there exists a point $v \in X$ s.t. $z = Tv$.
Now,

$$\begin{aligned} d(Av, z) &\preceq d(Av, Bx_{2n+1}) + d(Bx_{2n+1}, z) \\ &\preceq d(Bx_{2n+1}, z) + \lambda[d(Tv, Av)]^{\alpha}[d(Sx_{2n+1}, Bx_{2n+1})]^{1-\alpha} \end{aligned}$$

When $n \to \infty$, $Bx_{2n+1} \to z$, $Sx_{2n+1} \to z$ and putting $Tv = z$ we get

$$d(Az, u) \preceq \lambda[d(Tv, Av)]^{\alpha} \quad \text{i.e.} \quad |d(Az, u)| \leq \lambda|[d(Tv, Av)]^{\alpha}|,$$

which is only possible when $d(Av, z) = 0$, so we have $Az = z$, i.e., $Av = Tv = z$, showing that u is a coincidence point of A and T. Since the pair of map T and A are compatible so $TAv = ATv$ (because every compatible mapping is weakly compatible) Now we shall show that z is a fixed point of B.

$$\begin{aligned} d(z, Bz) &= d(Av, Bz) \\ &\preceq \lambda[d(Tv, Av)]^{\alpha}[d(Sz, Bz)]^{1-\alpha}. \end{aligned}$$

Therefore, $|d(z, Bz)| \leq 0$, i.e. $|d(z, Bz)| = 0$. This gives $Bz = z$, i.e., $Bz = Sz = z$. Now we shall show that z is a fixed point of A.

$$\begin{aligned} d(z, Az) &= d(Az, Bu) \\ &\preceq \lambda[d(Tz, Az)]^{\alpha}[d(Su, Bu)]^{1-\alpha}. \end{aligned}$$

Therefore, $|d(z, Az)| \leq 0$, i.e. $|d(z, Az)| = 0$. Which gives $Az = z$, i.e., $Az = Tz = z$.

Hence, $Az = Sz = Bz = Tz = z$, showing that Z is the common fixed point of A, B, S and T.

Uniqueness: If possible let w be the another common fixed point of A, B, S and T. then we have $Aw = Sw = Bw = Tw = w$.

$$\begin{aligned} d(z, w) &= d(Az, Bw) \\ &\preceq \lambda[d(Tz, Az)]^{\alpha}[d(Sw, Bw)]^{1-\alpha}. \end{aligned}$$

i.e. $d(z, w) = 0$ implies that $z = w$.

Therefore z is the unique common fixed point of A, B, S and T.

4. Conclusion

This article investigates common fixed point theorems for four self-mappings using the concept of compatible map in complex valued metric spaces without using notion of continuity.

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