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SOME NEW RESULTS ON F-CONTRACTIONS IN RECTANGULAR b-METRIC SPACES WITH AN APPLICATION

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Abstract: In the present paper, some new fixed point theorems have been provided for F-contraction on rectangular b-metric spaces in which maps need not be continuous. Especially, we derive common fixed point theorem for two pairs of weakly compatible mappings for new type of F-contraction on rectangular b-metric spaces (not necessarily continuous). Our results not only generalize many known results in the literature, but also improve some of the results therein. In addition, the results are justified by appropriate examples and deployed to examine the existence and uniqueness of solution for a system of Volterra integral equation.

Keywords and Phrases: Fixed point, F-contractions, rectangular b-metric space.

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1. Introduction and Preliminaries

In 1922, the celebrated result of Banach, popularly, known as Banach Contraction principle was proved. Many authors generalized this fixed point theorem in different metric spaces like quasi metric space, partial metric space ([6], [24]), b-metric space, quasi partial b-metric space ([11], [12]), rectangular metric space and cone rectangular metric space. Some authors proved the existence and uniqueness of common fixed points for two or more than two maps [27]. In 1968, Kannan [18] extended Banach fixed point theorem without assuming the continuity, which was the turning point in fixed point theory. In the recent years Santosh Kumar [20], Luambano Sholastica et. al. [29] and Lucas Wangwe et. al. [32] obtained fixed point results on F-contractions. Some fixed point theorems for multivalued F-contractions in partial metric space were given by Santosh Kumar [21]. In 2020, Vujakovic et. al. [30] derived Wardowski type F-contractions fixed point results for the setting of four continuous maps and Nicola et. al. [10] obtained fixed point theorems on W-contractions of Junck-Ciric-Wardowski type in metric space. Also, Vujakovic et. al. [31] established fixed point results of F-contractions for triangular $\alpha-$ admissible and triangular weak $\alpha-$ admissible mappings in metric-like spaces. Recently, Stojan et. al. [26] proved fixed point theorems on F-contractions with only one condition. i.e. F only be strictly increasing.

In 1989, Bakhtin [3] introduced b-metric space by multiplying right hand side of the triangle inequality with some real number. Then many fixed point theorems for different contractions on b-metric spaces have been proved ([1], [14], [15]).

Definition 1.1. [3] *Let X be a non-empty set with the coefficient $s \geqslant 1$, and the mapping $d : X \times X \to [0, \infty)$ satisfies the following:*

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, y) \leqslant s[d(x, z) + d(z, y)]$; $\forall x, y, z \in X$.

Then d is called a b-metric on X and (X, d) is called a b-metric space with coefficient s.

In 2000, Branciari [4] introduced rectangular(generalized) metric space(RMS) by replacing triangular inequality by rectangular one in the context of fixed point theorem.

Definition 1.2. [4] *Let X be a nonempty set and the mapping $d : X \times X \to [0, \infty)$ satisfies the following:*

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(3) $d(x, y) \leqslant d(x, w) + d(w, z) + d(z, y)$ *for all* $x, y \in X$ *and for all distinct points* $w, z \in X - \{x, y\}$ *[rectangular property].*

Then (X, d) *is called a rectangular(generalized) metric space(in short RMS).*

In the year 2002, Das [7] obtained fixed point theorem for a class of generalized metric spaces. In the sequel, Azam and Arshad [2] proved the well known Kannan fixed point theorem for rectangular metric space. In 2015, George [13] introduced the concept of rectangular b-metric space, which was not necessarily hausdorff and which generalized the concept of metric space, RMS and b-metric space. He also proved Banach and Kannan fixed point theorems for rectangular b-metric space.

Definition 1.3. [13] *Let* X *be a nonempty set with the coefficient* $s \geqslant 1$, *and the mapping* $d : X \times X \to [0, \infty)$ *satisfies the following:*

(1) $d(x, y) = 0$ *if and only if* $x = y$;

(2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(3) $d(x, y) \leqslant s[d(x, w) + d(w, z) + d(z, y)]$ *for all* $x, y \in X$ *and for all distinct points* $w, z \in X - \{x, y\}$.

Then (X, d) *is called a rectangular b- metric space(in short RbMS).*

Remark 1.1. *Every metric space is a RMS and every RMS is a RbMS(with coefficient* $s = 1$*), but converse not necessarily true.*

The concepts of convergence, Cauchy sequence and completeness in a RbMS are defined same as metric space.

Remark 1.2.

(1) Limit of a sequence in a RbMS is not necessarily unique and also every convergent sequence in RbMS is not necessarily a Cauchy sequence. Further, RbMS is not a continuous map.

(2) The open balls in RbMs are not necessarily open and (X, d) *is not Hausdorff.*

We modify the Example 1.7 [13] as below, justifying the above statements.

Example 1.1. Let $X = A \cup B$,where $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ and B is the set of all positive integers. Define $d : X \times X \to [0, \infty)$ such that $d(x, y) = d(y, x)$ for all

$x, y \in X$ and

$$d(x,y) = \begin{cases} 0, & ; x = y \\ 2\alpha & ; x, y \in A \\ \frac{\alpha}{2ny} & ; x \in A \quad \text{and} \quad y \in \{2,3\} \\ \alpha & ; \text{otherwise} \end{cases} \tag{1.1}$$

where $\alpha > 0$ is a constant. Then (X, d) is a RbMS with coefficient $s = 2 > 1$ but not a RMS , b-metric space and metric space.

As well known, a sequence in a RbMS may have two limits. However, there is a special situation where this is not possible, and this will be used in some proofs.

Theorem 1.1. [28] *Let (X, d) be a RbMS with $s \geqslant 1$ and let $\{x_n\}$ be a b-rectangular-Cauchy sequence in X such that $x_n \neq x_m$ whenever $n \neq m$. Then $\{x_n\}$ can converge to at most one point.*

In 2012, Wardowski [33] introduced a new type of contraction called F-contraction and obtained a fixed point for complete metric space. Subsequently, several papers have dealt with the F-contraction mappings and their extensions (see [1], [15], [19], [26]).

Definition 1.4. [33] *Let (X, d) be a metric space, then a mapping $T : X \to X$ is said to be a Wardowski F-contraction if there exists $\tau > 0$ such that $d(Tx, Ty) > 0$ implies*

$$\tau + F(d(Tx, Ty)) \leqslant F(d(x, y)); \qquad \textit{for all} \quad x, y \in X. \tag{1.2}$$

where, $F : (0, \infty) \to R$ is a mapping satisfying the following conditions:

(F_1) *F is strictly increasing.*

(F_2) *For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$.*

(F_3) *There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.*

From now on, let F be the set of all continuous functions $F : (0, \infty) \to R$ satisfying F_1 to F_3.

Example 1.2. Such examples of F are as follows:

1. $F(x) = ln(x); x > 0$.

2. $F(x) = x + ln(x); x > 0$.

3. $F(x) = ln(x^2 + x); x > 0.$

4. $F(x) = \frac{-1}{x^p}; p > 0.$

5. $F(x) = x - \frac{1}{x}; x > 0.$

6. $F(x) = \frac{1}{1-e^x}; x > 0.$

Since F is an increasing function, it is easily seen that every Wardoski's F-contraction mapping is a contraction mapping and hence continuous [33]. However, the mappings which have been found here need not be continuous. Also, in our results, we are dealing with the discontinuity of the metric space.

In the present paper, we extend Wardoski's theorem for two maps in complete rectangular b-metric space(RbMS). Also, we derive some common fixed point theorems for two pairs of weakly compatible mappings for new type of F-contraction on RbMS. Some examples are provided to illustrate our results.
Let us recall the following definitions.

Definition 1.5. *Let X be a non-empty set and $T_1, T_2 : X \to X$. If $w = T_1x = T_2x$ for some $x \in X$, then x is called a coincidence point of T_1 and T_2, and w is called a point of coincidence of T_1 and T_2.*

Definition 1.6. [17] *Let X be a non-empty set and $T_1, T_2 : X \to X$. The pair $\{T_1, T_2\}$ is said to be weakly compatible if $T_1T_2t = T_2T_1t$, whenever $T_1t = T_2t$ for some t in X.*

2. Main Results

The following result, generalizes Wardoski's theorem for the setting of four maps in RbMS.

Theorem 2.1. *Suppose that A,B,S and T are self-maps of a complete RbMS with coefficient $s > 1$ such that $AX \subset TX, BX \subset SX$ and if their exists $\tau > 0$ such that $d(Ax, By) > 0$ implies*

$$\tau + F(d(Ax, By)) \leqslant F(\alpha(x, y)); \forall x, y \in X. \tag{2.1}$$

where,

$$\alpha(x, y) = max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty)\}$$

If one of the ranges AX, BX, TX and SX is a closed subset of (X, d), then

(i) A and S have a coincidence point.

(ii) B and T have a coincidence point.

Moreover, if the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible, then A, B, T and S have a unique common fixed point.

Proof. Let $x_0 \in X$. Since $AX \subset TX$, there exists $x_1 \in X$ such that $Tx_1 = Ax_0$, and $BX \subset SX$, there exists $x_2 \in X$ such that $Sx_2 = Bx_1$. Continuing this process, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X defined by

$$y_{2n} = Tx_{2n+1} = Ax_{2n}, \qquad y_{2n+1} = Sx_{2n+2} = Bx_{2n+1} \qquad \forall n \in \mathbb{N}$$

First of all it is shown that $\{y_n\}$ is a Cauchy sequence in the RbMS.
From the contraction condition (2.1) with $x = x_{2k}$ and $y = x_{2k+1}$, one obtains

$$\tau + F(d(Ax_{2k}, Bx_{2k+1})) \leqslant F(\alpha(x_{2k}, x_{2k+1}))$$

where,

$$\alpha(x_{2k}, x_{2k+1}) = max\{d(y_{2k-1}, y_{2k}), d(y_{2k+1}, y_{2k})\}$$

If $\alpha(x_{2k}, x_{2k+1}) = d(y_{2k+1}, y_{2k})$, then $\tau \leqslant 0$, which contradicts with $\tau > 0$.
This means $\alpha(x_{2k}, x_{2k+1}) = d(y_{2k-1}, y_{2k})$. Therefore, one finds that

$$\tau + F(d(y_{2k}, y_{2k+1})) \leqslant F(d(y_{2k-1}, y_{2k})) \tag{2.2}$$

Similarly, one obtains

$$\tau + F(d(y_{2k+1}, y_{2k+2})) \leqslant F(d(y_{2k}, y_{2k+1})) \tag{2.3}$$

Therefore, from (2.2)and (2.3),

$$F(d(y_n, y_{n+1})) \leqslant F(d(y_n, y_{n-1})) - \tau \qquad ;\forall \;\; n \geqslant 1. \tag{2.4}$$

Likewise,

$$F(d(y_n, y_{n+1})) \leqslant F(d(y_{n-1}, y_{n-2})) - 2\tau$$

Continuing this process, one arrives at

$$F(d(y_n, y_{n+1})) \leqslant F(d(y_0, y_1)) - n\tau \tag{2.5}$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} F(d(y_n, y_{n+1})) = -\infty$$

which together with (1.4 F2) gives

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \tag{2.6}$$

By using (1.4 F3) , there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty}(d(y_n, y_{n+1}))^k F(d(y_n, y_{n+1})) = 0. \tag{2.7}$$

From (2.5), one infers the following for all $n \in \mathbb{N}$.

$$(d(y_n, y_{n+1}))^k (F(d(y_n, y_{n+1})) - F(d(y_0, y_1))) \leqslant -(d(y_n, y_{n+1}))^k n\tau \leqslant 0. \tag{2.8}$$

From (2.6), (2.7) and letting $n \to \infty$ in (2.8), one gets

$$\lim_{n\to\infty}(n(d(y_n, y_{n+1}))^k) = 0. \tag{2.9}$$

This implies that, there exists $n_0 \in N$, such that $n(d(y_n, y_{n+1}))^k \leqslant 1$ for all $n \geqslant n_0$

$$d(y_n, y_{n+1}) \leqslant \frac{1}{n^{1/k}} \qquad \text{for all} \qquad n \geqslant n_0. \tag{2.10}$$

Since (X, d) is a RbMS, one gets

$$d(y_n, y_{n+p}) \leqslant s[d(y_{n+p}, y_{n+p-1}) + d(y_{n+p-1}, y_{n-1}) + d(y_{n-1}, y_n)]; p > 0$$

Again using the same property of RbMS and from (2.10), one arrives at

$$d(y_{n+p}, y_n) \leqslant \frac{s}{1-s}\{\frac{1}{(n+p-1)^{1/k}} + \frac{1}{(n-1)^{1/k}}\}$$

Thus, $\{y_n\}$ is a b-rectangular Cauchy sequence. Since X is complete, there exists $w \in X$ such that

$$\lim_{n\to\infty} y_n = w. \tag{2.11}$$

which yields

$$d(w, w) = \lim_{n\to\infty} d(y_n, w) = 0.$$

Thus, one finds that

$$\lim_{n\to\infty} d(Ax_{2n}, w) = \lim_{n\to\infty} d(Tx_{2n+1}, w) = 0.$$

and

$$\lim_{n\to\infty} d(Bx_{2n-1}, w) = \lim_{n\to\infty} d(Sx_{2n}, w) = 0. \tag{2.12}$$

Now without loss of generality, one can suppose that SX is a closed subset of the RbMS (X, d). From(2.12), there exists $z \in X$ such that $w = Sz$.
Claim: $d(Az, w) = 0$.

Suppose to the contrary, that $d(Az, w) > 0$. In this case, there exists an $n_1 \in \mathbb{N}$ such that $d(Az, y_{2n}) > 0$ for all $n \geqslant n_1$. (otherwise, there exists $n_2 \in \mathbb{N}$ such that $y_n = Az$ for all $n \geqslant n_2$, which implies that $y_n \to Az$. This is a contradiction, since $w \neq Az$).
Since $d(Az, y_{2n}) > 0$, from contractive condition (2.1), one gets

$$\tau + F(d(Az, y_{2n})) \leqslant F(\alpha(z, x_{2n})) \tag{2.13}$$

where,

$$\alpha(z, x_{2n}) = max\{d(Sz, y_{2n-1}), d(Az, Sz), d(y_{2n}, y_{2n-1}), d(Az, y_{2n-1})\}$$

Taking $n \to \infty$ in (2.13), one concludes that

$$\tau + F(\lim_{n\to\infty} d(Az, y_{2n})) \leqslant F(\lim_{n\to\infty} d(Az, y_{2n-1}))$$

which is a contradict with $\tau > 0$. So,

$$Az = w = Sz \tag{2.14}$$

Hence, A and S have coincidence point z. Since, $AX \subset TX$ and (2.14), we have $w \in TX$. So, there exists $v \in X$ such that $w = Tv$. With the use of similar procedure, one can deduce that $Bv = w = Tv$. Hence, B and T have coincidence point v.
Since, the pair $\{A, S\}$ is weakly compatible, from (2.14) one comes across

$$Aw = ASz = SAz = Sw$$

Next, one claims that $d(Aw, w) = 0$. Suppose, $d(Aw, w) > 0$.
From contractive condition(2.1), one can derive

$$\tau + F(d(Aw, y_{2n})) \leqslant F(\alpha(w, x_{2n})) \tag{2.15}$$

where,

$$\alpha(w, x_{2n}) = max\{d(Sw, y_{2n-1}), d(Aw, Sw), d(y_{2n}, y_{2n-1}), d(Aw, y_{2n-1})\}$$

Taking $n \to \infty$, from (2.15), one has

$$\tau + F(\lim_{n\to\infty} d(Aw, y_{2n}) \leqslant F(\lim_{n\to\infty} d(Aw, y_{2n-1}))$$

Again it contradicts with the fact that $\tau > 0$.
So, $Aw = w = Sw$. Therefore, w is the common fixed point of A and S. Similarly, $Bw = w = Tw$.
Hence, w is the common fixed point of A, B, S and T. It is easy to check that w is the unique common fixed point.

If one puts $A = B$ and $S = T$, the contractive condition (2.1) leads to be the following result:

Corollary 2.1. *Suppose that A and T are self-maps of a complete RbMS with $s > 1$ such that $AX \subset TX$ and if their exists $\tau > 0$ such that $d(Ax, Ay) > 0$ implies*

$$\tau + F(d(Ax, Ay)) \leqslant F(\alpha(x, y)) \quad ; \forall x, y \in X. \tag{2.16}$$

where,

$$\alpha(x, y) = max\{d(Tx, Ty), d(Ax, Tx), d(Ay, Ty), d(Ax, Ty)\}$$

If one of the ranges AX and TX is a closed subset of (X, d), then A and T have a coincidence point. Moreover, if the pair $\{A, T\}$ is weakly compatible, then A and T have a unique common fixed point.

Example 2.1. Let $X = U \cup V$,where $U = \{1, \frac{1}{2}, \frac{1}{3}\}$ and V={2,3}.
Define $d : X \times X \to [0, \infty)$ such that $d(x, y) = d(y, x)$ for all $x, y \in X$ and

$$d(x, y) = 0, \text{if} \quad x = y,$$

$$d(1, \frac{1}{2}) = d(1, \frac{1}{3}) = d(\frac{1}{2}, \frac{1}{3}) = 1; d(1, 2) = d(\frac{1}{2}, 2) = d(\frac{1}{3}, 2) = \frac{1}{8},$$
$$d(1, 3) = d(\frac{1}{2}, 3) = d(\frac{1}{3}, 3) = \frac{1}{12}, d(2, 3) = \frac{1}{2}.$$

Note that (X, d) is a RbMS with coefficient $s = 2$, but not a RMS and metric space. Define the mappings $T, A : X \to X$ by $T(x) = \frac{1}{x}$ and $A(x) = \begin{cases} 1, & ; x \in U \\ x - 1 & ; x \in V \end{cases}$
We have $AX \subset TX = X$.
For the case, when $x \in U$ and $y \in V$ (vice versa)
For $y = 2$, we have trivial case. So take $y = 3$, one gets

$$d(Ax, Ay) = d(1, 2) = \frac{1}{8},$$

$$d(Tx, Ty) = d(\frac{1}{x}, \frac{1}{y}) = d(\frac{1}{x}, \frac{1}{3}) = 1 \quad \text{or} \quad \frac{1}{8} \quad \text{or} \quad \frac{1}{12},$$

$$d(Ax, Tx) = d(1, \frac{1}{x}) = 0 \quad \text{or} \quad \frac{1}{8} \quad \text{or} \quad \frac{1}{12},$$
$$d(Ay, Ty) = d(y-1, \frac{1}{y}) = \frac{1}{8},$$

and

$$d(Ax, Ty) = d(1, \frac{1}{y}) = 1.$$

Hence,

$$\alpha(x, y) = 1$$

From (2.16)

$$\tau + ln\frac{1}{8} \leqslant ln1$$

Implies that

$$\tau \leqslant ln8$$

Now, for the case, when $x \in V$ and $y \in V$.
It is trivial when $x = y$, take $x \neq y$.
This implies, we can take $x = 2$ and $y = 3$ (vice versa).
One obtains

$$d(Ax, Ay) = d(1, 2) = \frac{1}{8}$$
$$d(Tx, Ty) = d(\frac{1}{x}, \frac{1}{y}) = d(\frac{1}{2}, \frac{1}{3}) = 1$$
$$d(Ax, Tx) = d(1, \frac{1}{2}) = 1,$$
$$d(Ay, Ty) = d(y-1, \frac{1}{y}) = d(2, \frac{1}{3}) = \frac{1}{8},$$

and

$$d(Ax, Ty) = d(1, \frac{1}{y}) = d(1, \frac{1}{3}) = 1.$$

From (2.16)

$$\tau + ln\frac{1}{8} \leqslant ln1$$

That is

$$\tau \leqslant ln8$$

Also, the case is trivial, when $x \in U$ and $y \in U$.
At last, for all cases, let $\tau = ln8$ and $F(x) = lnx$. The equation (2.16) is satisfied.
Hence, 1 is the unique common fixed point of A and T.

If we put $A = B$ and $S = T = I$ (the identity map on X) in (2.1), we obtain following:

Corollary 2.2. *Let A be a self-map of a complete RbMS with $s > 1$ and if there exists $\tau > 0$ such that $d(Ax, Ay) > 0$ implies*

$$\tau + F(d(Ax, Ay)) \leqslant F(\alpha(x, y)) \tag{2.17}$$

where,

$$\alpha(x, y) = max\{d(x, y), d(x, Ax), d(y, Ay)\}$$

Then A has a unique fixed point in X.

Example 2.2. We have seen that the function A given in Example 2.1 with the same metric space satisfies corollary 2.2 for $\tau = ln2$ and $F(x) = lnx$.

Next, in the sequel the following is proved.

Theorem 2.2. *Suppose that T and S are self-maps of a complete RbMS with $s > 1$ and if their exists $\tau > 0$ such that $d(Tx, Ty) > 0$ implies*

$$\tau + F(d(Tx, Ty)) \leqslant F(d(Sx, Sy)) \tag{2.18}$$

If one of the ranges TX and SX is a closed subset of (X, d), then T and S have a coincidence point. Moreover, if the pair $\{S, T\}$ is weakly compatible, then T and S have a unique common fixed point.

Proof. Consider the sequence $\{x_n\}$, where $y_n = Tx_n = Sx_{n+1}$. Adopting a similar process as in previous theorem, it is easy to prove that S and T have unique common fixed point.

Now, If we take $S = I$(the identity map on X), we have Wardoski's F-contraction.

Corollary 2.3. *Let (X,d) be a RbMS with $s > 1$ and $T : X \to X$ satisfying the following with $\tau > 0$*

$$\tau + F(d(Tx, Ty)) \leqslant F(d(x, y)) \tag{2.19}$$

Then T have a unique fixed point in X.

Example 2.3. Let $X = \{1, 2, 3, 4\}$. Define $d : X \times X \to [0, \infty)$ such that $d(x, y) = d(y, x)$ for all $x, y \in X$ and

$$d(1, 2) = 10; d(1, 3) = d(2, 3) = 1; d(1, 4) = d(2, 4) = d(3, 4) = 2.$$

Then, (X, d) is a RbMS with coefficient $s = 2(> 1)$, but not a RMS.

Define $T : X \to X$ by $T(x) = \begin{cases} 1, & ; x = 1, 2, 3 \\ 3, & ; x = 4 \end{cases}$

T satisfies equation (2.19) for $\tau = ln2$ and $F(\alpha) = ln\alpha$. So, 1 is the unique fixed point of T.

3. Application

In this section, we find existence and uniqueness of solution of the following system of integral equation of Volterra type:

$$\begin{aligned} u(t) &= f(t) + \int_0^t K_1(t, s, u(s))\, ds \\ u(t) &= f(t) + \int_0^t K_2(t, s, u(s))\, ds \\ u(t) &= f(t) + \int_0^t K_3(t, s, u(s))\, ds \\ u(t) &= f(t) + \int_0^t K_4(t, s, u(s))\, ds \end{aligned} \tag{3.1}$$

where $t \in [0, a], a > 0$ and $K_i : [0, a] \times [0, a] \times R \to R$ $(i \in 1, 2, 3, 4)$ and $f : R \to R$ are continuous functions. For $u \in C([0, a], R) = X(say)$, define supremum norm as:

$$\|u\|_\tau = \sup_{t \in [0,a]} |u(t)|^2 e^{-\tau t}$$

where $\tau > 0$ is taken arbitrary. Let $C([0, a], R)$ be endowed with the metric

$$d_\tau(u, v) = \sup_{t \in [0,a]} \{|u(t) - v(t)|^2 e^{-\tau t}\} \qquad ; \forall u, v \in C([0, a], R)$$

Here (X, d_τ) is a complete RbMS with $s = 3/2$. Notice that it is not a metric space and RMS.
Let $I = [0, a]$ and defined $T_i : C(I, R) \to C(I, R)$ defined by

$$T_i u(t) = f(t) + \int_0^t K_i(t, s, u(s))\, ds \tag{3.2}$$

$;\forall\ u \in C(I, R),\ t \in I,\ i \in \{1, 2, 3, 4\}$. Clearly, u^* is a solution of (3.1) if and only if it is a common fixed point of T_i for $i\ \in \{1, 2, 3, 4\}$.
We are equipped the following condition to prove our result.

Theorem 3.1. *Suppose that the following hypothesis hold:*

1. *For all* $t \in I, u \in C(I, R)$,
   $T_1T_4u(t) = T_4T_1u(t),$ *whenever* $T_1u(t) = T_4u(t)$,
   $T_2T_3u(t) = T_3T_2u(t),$ *whenever* $T_2u(t) = T_3u(t)$,

2. *Assume that there exist $\tau > 1$, such that*

$$|K_1(t,s,u) - K_2(t,s,v)|^2 \leqslant \tau e^{-\tau}|\alpha(u,v)|^2$$

$\forall t, s \in [0,a]$ *and* $u, v \in X$, *where*

$$\alpha(u,v) = max\{|Su - Tv|, |Au - Su|, |Bv - Tv|, |Au - Tv|\}$$

Then (3.1) has a unique solution u^*(say).

Proof. By above assumption, we have

$$\begin{aligned}
|T_1u(t) - T_2v(t)|^2 &\leqslant \int_0^t |K_1(t,s,u(s) - K_2(t,s,v(s)|^2 ds \\
&\leqslant \int_0^t \tau e^{-\tau}(|\alpha(u,v)|^2 e^{-\tau s})e^{\tau s} ds \\
&\leqslant \tau e^{-\tau}\|\alpha(u,v)\|_\tau \frac{1}{\tau}e^{\tau t} \\
\tau + ln\|T_1u(t) - T_2v(t)\|_\tau &\leqslant ln\|\alpha(u,v)\|_\tau
\end{aligned}$$

This implies $\tau + F(d(T_1u, T_2v)) \leqslant F(\alpha(u,v))$ where, $F(x) = lnx$
Putting $A = T_1, B = T_2, T = T_3$ and $S = T_4$, then all the conditions given in Theorem (2.1)are satisfied. Therefore A, B, S and T have a unique common fixed point $u^* \in C(I,R)$; i.e, u^* is a unique solution of system (3.1).

4. Conclusion

Throughout the paper, we have generalized Wardoski F-contraction fixed point theorems in Rectangular b-metric space. An example is also provided for the justification of our results. Finally, we successfully apply our result to examine the existence and uniqueness of system of Volterra integral. System of Volterra integral equations appear in scientific applications in engineering, physics, chemistry and populations growth models (one may refer [16], [22], [23], [25], [34], [35]).

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