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Reducibility of ordinary double hypergeometric functions

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Abstract: In this paper, ordinary double hypergeometric series have been reduced into single series.
Keywords and Phrases: Ordinary hypergeometric series, summation/product formula/ double ordinary hypergeometric series.

1. Introduction, Notation and Definition In this paper, we have established certain results involving double hypergeometric series and single hypergeometric series. Certain product formulae have been deduced from the results. Results established in this paper are quite interesting and useful. An explicit representation of the generalized hypergeometric series is given by

$$ {}_rF_s\left[\begin{array}{l} a_1, a_2, a_3, ..., a_r; z \\ b_1, b_2, b_3, ..., b_s \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n ... (a_r)_n z^n}{(b_1)_n (b_2)_n ... (b_s)_n (1)_n}, \tag{1.1} $$

valid for $|z| < 1$, provided no zeros appear in the denominator. Here $a_1, a_2, a_3, ..., a_r$ and $b_1, b_2, b_3, ..., b_s$ and z are assumed to be complex numbers.
The shifted factorial is denoted by

$$ (a_n) = \left\{\begin{array}{ll} 1, & , n = 0 \\ a(a+1)...(a+n-1), & n \geq 1 \end{array}\right\} \tag{1.2} $$

The generalized ordinary double hypergeometric series is given as,

$$ F^{p:q;t}_{l:m;s}\left[\begin{array}{l} (a_p) : (b_q); (c_t); x, y \\ (\alpha_l) : (\beta_m); (\gamma_s) \end{array}\right] = F\left[\begin{array}{l} (a_p) : (b_q); (c_t); x, y \\ (\alpha_l) : (\beta_m); (\gamma_s) \end{array}\right] $$

$$ = \sum_{n,r=0}^{\infty} \frac{[(a_p)]_{n+r} [(b_q)]_n [(c_t)]_r x^n y^r}{[\alpha_l]_{n+r} [\beta_m]_s [\gamma_s]_r n! r!} \tag{1.3} $$

[4;(1.3.28) p.27]

We shall use the following sums due to Verma and Jain [2] in order to establish our main results.

$$ {}_3F_2\left[\begin{array}{l} -n, x, y; 1 \\ -n-x, -n-y \end{array}\right] = \frac{(1)_n(1+x+y)_n(1+x)_m(1+y)_m}{(1+x)_n(1+y)_n(1)_m(1+x+y)_m}, \tag{1.4} $$

where m is greatest integer $\leq \dfrac{n}{2}$. [2;(2.6) p.1024]

$$ {}_3F_2\left[\begin{array}{l} -n, -n-x, y; 1 \\ 1+x, -n-y \end{array}\right] = \frac{(1)_n(1+x-y)_m(1+y)_m}{(1+y)_n(1)_m(1+x)_m}, \tag{1.5} $$

where m is greatest integer $\leq \dfrac{n}{2}$. [2;(2.7) p.1024]

$$ {}_3F_2\left[\begin{array}{l} -n, -n-2x, y; 1 \\ -n-x, 2y+1 \end{array}\right] = \frac{(1)_n(1+x+y)_n(1+x)_m(1+y)_m}{(1+x)_n(1)_m(1+x+y)_m(1+2y)_n}, \tag{1.6} $$

where m is greatest integer $\leq \dfrac{n}{2}$. [2;(2.13) p.1026]

$$ {}_3F_2\left[\begin{array}{l} -n, -n-2x, 1+y; 1 \\ 1-n-x, 2y+1 \end{array}\right] = \frac{(-1)^n(1)_n(1+x+y)_n(1+y)_m(1+x)_m}{(x)_n(1+2y)_n(1)_m(1+x+y)_m}, \tag{1.7} $$

where m is greatest integer $\leq \dfrac{n}{2}$. [2;(2.17) p.1026]

We shall also use the following identity,

$$ \sum_{n=0}^{\infty}\sum_{r=0}^{n} A(r,n) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} A(r,n+r) \tag{1.8} $$

[3;(2.1.1) p.100]

2. Main Results: Our main results are

$$ F\left[\begin{array}{l} -; 1+x, 1+y; x, y; z, -z \\ 1+x+y; --; -; \end{array}\right] = (1+z)\ {}_2F_1\left[\begin{array}{l} 1+x, 1+y; z^2 \\ 1+x+y \end{array}\right] \tag{2.1} $$

$$ F\left[\begin{array}{l} 1+x; 1+y; y; z, -z \\ -; 1+x; 1+x; \end{array}\right] = (1+z)\ {}_2F_1\left[\begin{array}{l} 1+x-y, 1+y; z^2 \\ 1+x \end{array}\right] \tag{2.2} $$

$$ F\left[\begin{array}{l} 1+2x, 1+2y; 1+x, y; z, -z \\ 1+x+y; 1+2x; 1+2y; \end{array}\right] = (1+z)\ {}_2F_1\left[\begin{array}{l} 1+x, 1+y; z^2 \\ 1+x+y \end{array}\right] \tag{2.3} $$

$$F\left[\begin{array}{l} 1+2x;1+x;y;z,-z \\ -;1+2x;1+2y \end{array}\right] = {}_3F_2\left[\begin{array}{l} \frac{1}{2}+\frac{x}{2}+\frac{y}{2},1+\frac{x}{2}+\frac{y}{2},1+x;z^2 \\ \\ \frac{1}{2}+y,1+x+y \end{array}\right]$$

$$+\frac{(1+x+y)}{(1+2y)}z\ {}_3F_2\left[\begin{array}{l} 1+\frac{x}{2}+\frac{y}{2},\frac{3}{2}+\frac{x}{2}+\frac{y}{2},1+x;z^2 \\ \\ \frac{3}{2}+y,1+x+y \end{array}\right]. \quad (2.4)$$

$$F\left[\begin{array}{l} 1+2x,1+2y;x,1+y;z,-z \\ 1+x+y;1+2x;1+2x; \end{array}\right] = (1-z)\ {}_2F_1\left[\begin{array}{l} 1+x,1+y;z^2 \\ 1+x+y \end{array}\right] \quad (2.5)$$

$$F\left[\begin{array}{l} 1+2x;x;1+y;z,-z \\ -;1+2x;1+2y \end{array}\right] = {}_3F_2\left[\begin{array}{l} \frac{1}{2}+\frac{x}{2}+\frac{y}{2},1+\frac{x}{2}+\frac{y}{2},1+x;z^2 \\ \\ \frac{1}{2}+y,1+x+y \end{array}\right]$$

$$-\frac{(1+x+y)}{(1+2y)}z\ {}_3F_2\left[\begin{array}{l} 1+\frac{x}{2}+\frac{y}{2},\frac{3}{2}+\frac{x}{2}+\frac{y}{2},1+x;z^2 \\ \\ \frac{3}{2}+y,1+x+y \end{array}\right]. \quad (2.6)$$

3. Proof of the Results:

(a) The proof of (2.1) is multiply by Ω_n on both side of (1.4) and summing with respect to n from zero to infinity, we get

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n}\frac{(-n)_r(x)_r(y)_r}{(-n-x)_r(-n-y)_r(1)_r}\Omega_n = \sum_{n=0}^{\infty}\frac{(1)_n(1+x+y)_n(1+x)_m(1+y)_m}{(1+x)_n(1+y)_n(1+x+y)_m(1)_m}\Omega_n. \quad (3.1)$$

By using the identity (1.8) on L.H.S. of (3.1), we get

$$\sum_{n=0,r=0}^{\infty}\frac{(x)_r(y)_r(-1)^r(1+x)_n(1+y)_n(1)_{n+r}}{(1)_n(1+x)_{n+r}(1)_r(1+y)_{n+r}}\Omega_{n+r} = \sum_{n=0}^{\infty}\frac{(1)_n(1+x+y)_n(1+x)_m(1+y)_m}{(1+x)_n(1+y)_n(1+x+y)_m(1)_n}. \quad (3.2)$$

On putting $\Omega_n = \frac{(1+x)_n(1+y)_n}{(1+x+y)_n(1)_n}z^n$ in (3.2) we get

$$\sum_{n,r=0}^{\infty}\frac{1}{(1+x+y)_{n+r}}\frac{(1+x)_n(1+y)_nz^n}{(1)_n}\frac{(x)_r(y)_rz^r(-1)^r}{(1)_r} = \sum_{n=0}^{\infty}\frac{(1+x)_m(1+y)_m}{(1+x+y)_m(1)_m}z^n. \quad (3.3)$$

It can be written as

$$\sum_{n,r=0}^{\infty}\frac{1}{(1+x+y)_{n+r}}\frac{(1+x)_n(1+y)_nz^n}{(1)_n}\frac{(x)_r(y)_rz^r(-1)^r}{(1)_r}$$

$$= \sum_{n=0}^{\infty} \frac{(1+x)_n(1+y)_n z^{2n}}{(1+x+y)_n(1)_n} + \sum_{n=0}^{\infty} \frac{(1+x)_n(1+y)_n z^{2n+1}}{(1+x+y)_n(1)_n}. \tag{3.4}$$

On simplification of (3.4) we get the result (2.1).

(b) The proof (2.2) is multiply by Ω_n both sides of (1.5) and summing with respect to n from zero to infinity, we get

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{(-n)_r(-n-x)_r(y)_r}{(1+x)_r(-n-y)_r(1)_r}\Omega_n = \sum_{n=0}^{\infty} \frac{(1)_n(1+x-y)_m(1+y)_m}{(1+y)_n(1+x)_m(1)_m}\Omega_n. \tag{3.5}$$

By using the identity (1.8) on L.H.S. of (3.5), we get

$$\sum_{n,r=0}^{\infty} \frac{(1)_{n+r}(1+x)_{n+r}}{(1+y)_{n+r}}\Omega_{n+r} \frac{(1+y)_n(-1)^r(y)_r}{(1+x)_n(1)_n(1+x)_r(1)_r} = \sum_{n=0}^{\infty} \frac{(1)_n(1+x-y)_m(1+y)_m}{(1+y)_n(1+x)_m(1)_m}\Omega_n. \tag{3.6}$$

Now put $\Omega_n = \frac{(1+y)_n}{(1)_n}z^n$ and simplify (3.6) we get the result (2.2).

(c) The proof (2.3) and (2.4), multiply by Ω_n both sides of (1.6) and summing with respect to n from zero to infinity, we get

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{(-n)_r(-n-2x)_r(y)_r}{(-n-x)_r(2y+1)_r(1)_r}\Omega_n = \sum_{n=0}^{\infty} \frac{(1)_n(1+x+y)_n(1+x)_m(1+y)_m}{(1+x)_n(1+2y)_n(1+x+y)_m(1)_m}\Omega_n. \tag{3.7}$$

By using the identity (1.8) on L.H.S. of (3.7), we get

$$\sum_{n,r=0}^{\infty} \frac{(1)_{n+r}(1+2x)_{n+r}}{(1+x)_{n+r}}\Omega_{n+r} \frac{(1+x)_n(-1)^r(y)_r}{(1+2x)_n(1)_n(1+2y)_r(1)_r}$$

$$= \sum_{n=0}^{\infty} \frac{(1)_n(1+x+y)_n(1+x)_m(1+y)_m}{(1+x)_n(1+2y)_n(1+x+y)_m(1)_m} \tag{3.8}$$

Now put $\Omega_n = \frac{(1+x)_n(1+2y)_n}{(1+x+y)_n(1)_n}z^n$ in (3.8) and simplify, we get the result (2.3) and putting $\Omega_n = \frac{(1+x)_n}{(1)_n}z^n$ in (3.8) simplify it we get the result (2.4). **(d)** The proof (2.5) and (2.6), multiply by Ω_n both sides of (1.7) and summing with respect to n from zero to infinity, we get

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{(-n)_r(-n-2x)_r(1+y)_r}{(1-n-x)_r(2y+1)_r}\Omega_n = \sum_{n=0}^{\infty} \frac{(-1)^n(1)_n(1+x+y)_n(1+x)_m(1+y)_m}{(x)_n(1)_m(1+2y)_n(1+x+y)_m}\Omega_n. \tag{3.9}$$

By using the identity (1.8) on L.H.S. of (3.9), we get

$$\sum_{n,r=0}^{\infty}\frac{(1)_{n+r}(1+2x)_{n+r}}{(x)_{n+r}}\Omega_{n+r}\frac{(x)_n(-1)^r(1+y)_r}{(1+2x)_n(1)_n(1+2y)_r(1)_r}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^r(1)_n(1+x+y)_n(1+x)_m(1+y)_m}{(x)_n(1+2y)_n(1+x+y)_m(1)_m} \tag{3.10}$$

Now put $\Omega_n = \dfrac{(x)_n(1+2y)_n}{(1+x+y)_n(1)_n}z^n$ in (3.10) and simplify, we get the result (2.5) and putting $\Omega_n = \dfrac{(x)_n}{(1)_n}z^n$ in (3.10) simplify it we get the result (2.6).

4. Product Formulae In this section, by setting the value of Ω_n we get the product formulas. By putting $\Omega_n = \dfrac{(1+x)_n(1+y)_n}{(1)_n}z^n$ in (3.2) we get,

$$ {}_2F_0\left[\begin{array}{l} x, y; -z \\ 0 \end{array}\right] {}_2F_0\left[\begin{array}{l} 1+x, 1+y; z \\ - \end{array}\right] = {}_4F_1\left[\begin{array}{l} \frac{x+y+1}{2}, \frac{2+x+y}{2}, 1+x, 1+y; 4z^2 \\ 1+x+y \end{array}\right]$$

$$+(1+x+y)z\ {}_4F_1\left[\begin{array}{l} \frac{(2+x+y)}{2}, \frac{(3+x+y)}{2}, 1+x, 1+y; 4z^2 \\ 1+x+y \end{array}\right]. \tag{4.1}$$

provided any one numerator parameter is a negative integer.

By setting $\Omega_n = \dfrac{(1+y)_n}{(1+x)_n(1)_n}z^n$ in (3.6) we get

$$ {}_1F_1\left[\begin{array}{l} 1+y; -z \\ 1+x \end{array}\right] {}_1F_1\left[\begin{array}{l} y; z \\ 1+x \end{array}\right]$$

$$= {}_2F_3\left[\begin{array}{l} 1+x-y, 1+y; \frac{z^2}{4} \\ \frac{(1+x)}{2}, \frac{(2+x)}{2}, 1+x \end{array}\right] + \frac{z}{(1+x)}\ {}_2F_3\left[\begin{array}{l} 1+x-y, 1+y; \frac{z^2}{4} \\ \frac{(2+x)}{2}, \frac{(3+x)}{2}, 1+x \end{array}\right] \tag{4.2}$$

provided any one numerator parameter is a negative integer

Now setting $\Omega_n = \dfrac{(1+x)_n}{(1+2x)_n(1)_n}z^n$ in (3.8) we get

$$ {}_1F_1\left[\begin{array}{l} y; -z \\ 1+2y \end{array}\right] {}_1F_1\left[\begin{array}{l} 1+x; z \\ 1+2x \end{array}\right]$$

$$= {}_2F_3\left[\begin{array}{l} \frac{1+x+y}{2}, \frac{2+x+y}{2}; \frac{z^2}{4} \\ \frac{1}{2}+y, \frac{1}{2}+x, 1+x+y \end{array}\right] + \frac{1+x+y}{(1+2x)(1+2y)}z\ {}_2F_3\left[\begin{array}{l} \frac{2+x+y}{2}, \frac{3+x+y}{2}; \frac{z^2}{4} \\ \frac{3}{2}+y, \frac{3}{2}+x, 1+x+y \end{array}\right]. \tag{4.3}$$

provided any one numerator parameter is a negative integer
and setting $\Omega_n = \frac{(x)_n}{(1+2x)_n(1)_n} z^n$ in (3.10) we get

$$ {}_1F_1\left[\begin{matrix} 1+y; -z \\ 1+2y \end{matrix}\right] {}_1F_1\left[\begin{matrix} x; z \\ 1+2x \end{matrix}\right] $$

$$ = {}_2F_3\left[\begin{matrix} \frac{1+x+y}{2}, \frac{2+x+y}{2}; \frac{z^2}{4} \\ \frac{1+2x}{2}, \frac{1+2y}{2}, 1+x+y \end{matrix}\right] - \frac{1+x+y}{(1+2x)(1+2y)} z\, {}_2F_3\left[\begin{matrix} 1+\frac{x+y}{2}, \frac{3+x+y}{2}; \frac{z^2}{4} \\ \frac{3}{2}+y, \frac{3}{2}+x, 1+x+y \end{matrix}\right]. \tag{4.4} $$

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