South East Asian J. of Mathematics and Mathematical Sciences
Vol. 17, No. 3 (2021), pp. 147-172

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

STRONG COUPLED FIXED POINTS OF α-ADMISSIBLE REICH TYPE COUPLED MAPPINGS IN S-METRIC SPACES

G. V. R. Babu, P. Durga Sailaja* and G. Srichandana

Department of Mathematics,
Andhra University, Visakhapatnam - 530003, Andhra Pradesh, INDIA

E-mail : gvr_babu@hotmail.com, sri.chandan3@gmail.com

*Department of Mathematics,
Lendi Institute of Engineering and Technology,
Vizianagaram - 535005, Andhra Pradesh, INDIA

E-mail : sailajadurga@yahoo.com

(**Received:** *Sep. 28, 2020* **Accepted:** *Nov. 02, 2021* **Published:** *Dec. 30, 2021)*

Abstract: In this paper, we introduce α-admissible Reich type cyclic coupled mapping and α-admissible Reich type coupled mapping in S-metric spaces and prove the existence and uniqueness of strong coupled fixed points of such mappings. We give illustrative examples to check the validity of our results.

Keywords and Phrases: S-metric space, cyclic mapping, coupled fixed point, strong coupled fixed point, α-admissible Reich type cyclic coupled mapping, α-admissible Reich type coupled mapping.

2020 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

In 1972, Reich [34] proved that any Reich type contraction on a complete metric space X has a unique fixed point. After this, more works on Reich type contractions appeared ([1], [7], [19], [25], [33]). On the other hand, in 1987, Geo and Lakshmikantham [17] introduced coupled fixed points of nonlinear operators. Later, Bhaskar and Lakshmikantham [16] developed coupled fixed point theory for mixed monotone operators in partially ordered metric spaces. For more works on coupled fixed points, we refer [2], [9], [13], [20], [23], [39]. In 2003, Kirk, Srinivasan and Veeramani [21] introduced cyclic contractions in metric spaces and proved the existence and uniqueness of cyclic contractions in complete metric spaces. Afterwards, various types of cyclic contractions ([4], [15], [22], [27], [29], [30], [31], [32], [35], [40], [41]) were introduced in proving fixed point results. In 2012, Samet, Vetro, Vetro [36] introduced α-admissible maps and studied the existence of fixed points of contraction mappings that are dependent on α-admissible maps in complete metric spaces. Subsequently, existence of coupled fixed points using α-admissible maps were developed ([2], [8], [10], [11], [24], [26], [28]).

In 2012, Sedghi, Shobe and Aliouche [37] introduced S-metric spaces and studied some properties of these spaces. Later, many authors developed coupled fixed point theorems and cyclic contractions on S-metric spaces. Some of them we include [3], [6], [12], [13], [18], [39].

Theorem 1.1. [34] *Let (X, d) be a complete metric space. Let $T : X \to X$ be a self-map on X such that*
$d(Tx, Ty) \leq \alpha d(x, Tx) + \beta d(y, Ty) + \mu d(x, y)$
where $\alpha, \beta, \mu \geq 0$ and $\alpha + \beta + \mu < 1$, then T has a unique fixed point.

Definition 1.2. [16] *Let X be a nonempty set. Let $F : X \times X \to X$ be a mapping. An element $(x, y) \in X \times X$ is said to be a coupled fixed point of F if $F(x, y) = x$ and $F(y, x) = y$.*

Definition 1.3. [9] *Let X be a nonempty set. Let $F : X \times X \to X$ be a mapping. An element $(x, x) \in X \times X$ is said to be a strong coupled fixed point of F if $F(x, x) = x$.*

Choudhury, Maity and Konar [9] extended the notion of cyclic mapping to the case of mappings defined on $X \times X$ as follows.

Definition 1.4. [9] *Let A and B be two nonempty subsets of X. A mapping $F : X \times X \to X$ is said to be cyclic with respect to A and B if $F(A, B) \subseteq B$ and $F(B, A) \subseteq A$.*

Definition 1.5. [9] *Let A and B be two nonempty subsets of a metric space (X, d). A mapping $F : X \times X \to X$ is called a cyclic coupled Kannan type contraction with respect to A and B if F is cyclic with respect to A and B and satisfy the following: for some $k \in (0, \frac{1}{2})$,*

$$d(F(x, y), F(u, v)) \leq k(d(x, F(x, y)) + d(u, F(u, v))), \tag{1.1}$$

where $x, v \in A$ and $y, u \in B$.

Theorem 1.6. [9] *Let A and B be two nonempty closed subsets of a complete met-*

ric space (X, d). Let $F : X \times X \to X$ be a cyclic coupled Kannan type contraction with respect to A and B and $A \cap B \neq \emptyset$. Then F has a strong coupled fixed point in $A \cap B$.

Definition 1.7. ([2], [36]) *Let (X, d) be a metric space. Let $F : X \times X \to X$ and $\alpha : X^2 \times X^2 \to [0, +\infty)$. We say that F is α-admissible if*
$\alpha((x, y), (u, v)) \geq 1$ *implies* $\alpha((F(x, y), F(y, x)), (F(u, v), F(v, u))) \geq 1$
for all $(x, y), (u, v) \in X \times X$.

Definition 1.8. [37] *Let X be a nonempty set. An S-metric on X is a function $S : X^3 \to [0, +\infty)$ that satisfies the following conditions: for each $x, y, z, a \in X$*

(S1) $S(x, y, z) \geq 0$,

(S2) $S(x, y, z) = 0$ *if and only if* $x = y = z$ *and*

(S3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S-metric space.

Throughout this paper, we denote the set of all reals by $\mathbb{R}$, the set of all natural numbers by $\mathbb{N}$, and $\mathbb{R}^+ = [0, +\infty)$.

Example 1.9. [37] Let (X, d) be a metric space. We define $S : X^3 \to \mathbb{R}^+$ by $S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$ for all $x, y, z \in X$. Then S is an S-metric on X and S is called the S-metric induced by the metric d.

Example 1.10. [14] Let $X = \mathbb{R}$ and let $S(x, y, z) = |y + z - 2x| + |y - z|$ for all $x, y, z \in X$. Then (X, S) is an S-metric space.

Example 1.11. [38] Let $\mathbb{R}$ be the real line. Then $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$ is an S-metric on $\mathbb{R}$. This S-metric is called the usual S-metric on $\mathbb{R}$.

Example 1.12. [3] Let $X = \mathbb{R}$, and let $S(x, y, z) = \max\{|x - z|, |y - z|\}$ for all $x, y, z \in \mathbb{R}$. Then (X, S) is an S-metric space.

Example 1.13. Let $X = \mathbb{R}^+$ and

$$S(x, y, z) = \begin{cases} 0 & \text{if } \; x = y = z \\ \max\{x, y, z\} & \text{otherwise} \end{cases}$$

for all $x, y, z \in \mathbb{R}^+$. Then S is an S-metric on X.

Lemma 1.14. [37] *In an S-metric space, we have $S(x, x, y) = S(y, y, x)$.*

Lemma 1.15. [14] *Let (X, S) be an S-metric space. Then*

$S(x,x,z) \leq 2S(x,x,y) + S(y,y,z)$.

Definition 1.16. [37] *Let (X,S) be an S-metric space.*

(i) A sequence $\{x_n\} \subseteq X$ is said to converge to a point $x \in X$ if $S(x_n,x_n,x) \to 0$ as $n \to +\infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n,x_n,x) < \epsilon$ and we denote it by $\lim\limits_{n\to+\infty} x_n = x$.

(ii) A sequence $\{x_n\} \subseteq X$ is called Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n,x_n,x_m) < \epsilon$ for all $n,m \geq n_0$.

(iii) An S-metric space (X,S) is said to be complete if each Cauchy sequence in X is convergent.

Lemma 1.17. [37] *Let (X,S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x, then x is unique.*

Lemma 1.18. [37] *Let (X,S) be an S-metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim\limits_{n\to+\infty} x_n = x$ and $\lim\limits_{n\to+\infty} y_n = y$, then $\lim\limits_{n\to+\infty} S(x_n,x_n,y_n) = S(x,x,y)$.*

Lemma 1.19. [5] *Let (X,S) be an S-metric space. Let $\{x_n\}$, $\{y_n\}$ be two sequences in X and $\{x_n\}$ converges to x in X. Then $\lim\limits_{n\to+\infty} S(x_n,x_n,y_n) = \lim\limits_{n\to+\infty} S(x,x,y_n)$.*

Motivated by Theorem 1.6, the work of Choudhury, Maity and Konar [9], we introduce α-admissible Reich type cyclic coupled mapping, α-admissible Reich type coupled mapping in S-metric spaces and prove the existence and uniqueness of strong coupled fixed points of such mappings in complete S-metric spaces. In Section 3, we draw some corollaries and give examples to check the validity of our results.

2. α-admissible Reich type cyclic coupled mapping

In the following we define α-admissible mappings to the case of S-metric spaces.

Definition 2.1. [6] *Let (X,S) be an S-metric space. Let $T : X \to X$ and $\alpha : X \times X \times X \to \mathbb{R}^+$. We say that T is α-admissible, if $x,y,z \in X$, $\alpha(x,y,z) \geq 1$ implies $\alpha(Tx,Ty,Tz) \geq 1$.*

Definition 2.2. *Let (X,S) be an S-metric space. An α-admissible map T on X is said to be triangular α-admissible if*
$\alpha(x,x,z) \geq 1$ and $\alpha(z,z,y) \geq 1$ implies $\alpha(x,x,y) \geq 1$.

We now extend Definition 1.7 to the case of S-metric spaces.

Definition 2.3. *Let (X,S) be an S-metric space. Let $F : X \times X \to X$ and*

$\alpha : X^2 \times X^2 \times X^2 \to \mathbb{R}^+$. We say that F is α-admissible if
$\alpha((x,y),(u,v),(w,z)) \geq 1$ implies
$\alpha((F(x,y),F(y,x)),(F(u,v),F(v,u)),(F(w,z),F(z,w))) \geq 1$
for all $(x,y),(u,v),(w,z) \in X \times X$.

Example 2.4. Let $X = \mathbb{R}^+$ and (X,S) be an S-metric space defined as in Example 1.12. We define $F : X \times X \to X$ by $F(x,y) = 2x + y$ and $\alpha : X^2 \times X^2 \times X^2 \to \mathbb{R}^+$ by $\alpha((x,y),(u,v),(w,z)) = e^{(x+u+w)-(y+v+z)}$.
Let $\alpha((x,y),(u,v),(w,z)) \geq 1$. That is $e^{(x+u+w)-(y+v+z)} \geq 1$.
We now consider

$$\begin{aligned}
&\alpha((F(x,y),F(y,x)),(F(u,v),F(v,u)),(F(w,z),F(z,w))) \\
&\qquad = \alpha((2x+y,2y+x),(2u+v,2v+u),(2w+z,2z+w)) \\
&\qquad = e^{(2x+y+2u+v+2w+z)-(2y+x+2v+u+2z+w)} \\
&\qquad = e^{(x+u+w)-(y+v+z)} \geq 1.
\end{aligned}$$

Therefore F is α-admissible.

Definition 2.5. *Let (X,S) be an S-metric space. A mapping $F : X \times X \to X$ is said to be triangular α-admissible if*

(i) F is α-admissible and

(ii) $\alpha((x,y),(x,y),(u,v)) \geq 1$ and $\alpha((u,v),(u,v),(w,z)) \geq 1$ implies $\alpha((x,y),(x,y),(w,z)) \geq 1$ for all $(x,y),(u,v),(w,z) \in X \times X$.

Example 2.6. Let $X = \mathbb{R}^+$ and (X,S) be an S-metric space defined as in Example 1.12. We define $F : X \times X \to X$ by $F(x,y) = x + y$ and $\alpha : X^2 \times X^2 \times X^2 \to \mathbb{R}^+$ by $\alpha((x,y),(u,v),(w,z)) = e^{(2x+2y)-(u+v+w+z)}$.
Let $\alpha((x,y),(u,v),(w,z)) \geq 1$.
That is $e^{(2x+2y)-(u+v+w+z)} \geq 1$ if and only if $2x + 2y \geq u + v + w + z$. Hence

$$\begin{aligned}
&\alpha((F(x,y),F(y,x)),(F(u,v),F(v,u)),(F(w,z),F(z,w))) \\
&\qquad = \alpha((x+y,y+x),(u+v,v+u),(w+z,z+w)) \\
&\qquad = e^{4(x+y)-2(u+v+w+z)} \geq 1.
\end{aligned}$$

Therefore F is α-admissible.
Now, suppose that $\alpha((x,y),(x,y),(u,v)) \geq 1$ and $\alpha((u,v),(u,v),(w,z)) \geq 1$
i.e., $e^{(2x+2y)-(x+y+u+v)} \geq 1$ and $e^{(2u+2v)-(u+v+w+z)} \geq 1$
i.e., $e^{(x+y)-(u+v)} \geq 1$ and $e^{(u+v)-(w+z)} \geq 1$
if and only if $x + y \geq u + v$ and $u + v \geq w + z$ which implies that $x + y \geq w + z$.
Then we have $e^{(x+y)-(w+z)} \geq 1$ or $e^{(2x+2y)-(x+y+w+z)} \geq 1$.
Thus $\alpha((x,y),(x,y),(w,z)) \geq 1$. Therefore F is triangular α-admissible.

In the following, we define α-admissible Reich type cyclic coupled mapping.

Definition 2.7. *Let (X, S) be an S-metric space. Let A and B be two nonempty subsets of X. Suppose that $\alpha : X^2 \times X^2 \times X^2 \to \mathbb{R}^+$, $F : X \times X \to X$ and F is α-admissible. We say that F is an α-admissible Reich type cyclic coupled mapping on X if F is cyclic with respect to A and B satisfying the following inequality: there exist a, b, c, $d \geq 0$ with $a + b + c + d < 1$ such that*

$$\begin{aligned} \alpha((x,y),(u,v),(w,z))&S(F(x,y),F(u,v),F(w,z)) \\ &\leq \frac{a}{2}[S(x,u,w)+S(y,v,z)] \\ &+\frac{b}{2}[S(x,x,F(x,y))+S(y,y,F(y,x))] \\ &+\frac{c}{2}[S(u,u,F(u,v))+S(v,v,F(v,u))] \\ &+\frac{d}{2}[S(w,w,F(w,z))+S(z,z,F(z,w))] \end{aligned} \tag{2.1}$$

where $x, u, z \in A$ and $y, v, w \in B$.

Remark 2.8. *If $A = B = X$ in Definition 2.7 then we say that F is an α-admissible Reich type coupled mapping on X.*

Proposition 2.9. *Let (X, S) be an S-metric space. Let A and B be two nonempty subsets of X. Let $F : X \times X \to X$ be triangular α-admissible mapping on X. Suppose that there exist $x_0, y_0 \in X$ such that*

(a) $\alpha((x_0, y_0), (x_0, y_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1$

(b) $\alpha((y_0, x_0), (y_0, x_0), (F(y_0, x_0), F(x_0, y_0))) \geq 1$

(c) $\alpha((F(y_0, x_0), F(x_0, y_0)), (F(y_0, x_0), F(x_0, y_0)), (x_0, y_0)) \geq 1$

(d) $\alpha((F(x_0, y_0), F(y_0, x_0)), (F(x_0, y_0), F(y_0, x_0)), (y_0, x_0)) \geq 1.$

Then there exist sequences $\{x_n\}$ in A and $\{y_n\}$ in B such that

(i) $\alpha((x_n, y_n), (x_n, y_n), (y_n, x_n)) \geq 1$ *for* $n = 0, 1, 2, ...$;

(ii) $\alpha((x_{n+1}, y_{n+1}), (x_{n+1}, y_{n+1}), (y_n, x_n)) \geq 1$ *for* $n = 0, 1, 2, ...$;

(iii) $\alpha((x_n, y_n), (x_n, y_n), (y_{n+1}, x_{n+1})) \geq 1$ *for* $n = 0, 1, 2, 3, ...$.

Proof. Let $x_0 \in A$ and $y_0 \in B$ be arbitrary. We define sequences $\{x_n\}$ and $\{y_n\}$ by

$$x_{n+1} = F(y_n, x_n), \quad y_{n+1} = F(x_n, y_n), \quad n = 0, 1, 2, ... \ . \tag{2.2}$$

Since F is cyclic, we have $x_n \in A$ and $y_n \in B$ for $n = 0, 1, 2, ...$

From (a), (b), (c) and (d), we have

$$\alpha((x_0, y_0), (x_0, y_0), (y_1, x_1)) \geq 1; \tag{2.3}$$

$$\alpha((y_0, x_0), (y_0, x_0), (x_1, y_1)) \geq 1; \tag{2.4}$$

$$\alpha((x_1, y_1), (x_1, y_1), (x_0, y_0)) \geq 1; \tag{2.5}$$

$$\alpha((y_1, x_1), (y_1, x_1), (y_0, x_0)) \geq 1. \tag{2.6}$$

Since F is α-admissible, from (2.3), (2.4), (2.5) and (2.6), we have the following:
$\alpha((F(x_0, y_0), F(y_0, x_0)), (F(x_0, y_0), F(y_0, x_0)), (F(y_1, x_1), F(x_1, y_1))) \geq 1$
that is

$$\alpha((y_1, x_1), (y_1, x_1), (x_2, y_2)) \geq 1; \tag{2.7}$$

$\alpha((F(y_0, x_0), F(x_0, y_0)), (F(y_0, x_0), F(x_0, y_0)), (F(x_1, y_1), F(y_1, x_1))) \geq 1$
that is

$$\alpha((x_1, y_1), (x_1, y_1), (y_2, x_2)) \geq 1; \tag{2.8}$$

$\alpha((F(x_1, y_1), F(y_1, x_1)), (F(x_1, y_1), F(y_1, x_1)), (F(x_0, y_0), F(y_0, x_0)) \geq 1$
that is

$$\alpha((y_2, x_2), (y_2, x_2), (y_1, x_1)) \geq 1; \tag{2.9}$$

and $\alpha((F(y_1, x_1), F(x_1, y_1)), (F(y_1, x_1), F(x_1, y_1)), (F(y_0, x_0), F(x_0, y_0)) \geq 1$
that is

$$\alpha((x_2, y_2), (x_2, y_2), (x_1, y_1)) \geq 1. \tag{2.10}$$

By using the inequalities (2.5) and (2.3), and since F is triangular α-admissible, we get
$\alpha((x_1, y_1), (x_1, y_1), (y_1, x_1)) \geq 1.$
Since F is α-admissible and proceeding as above, we get
$\alpha((y_2, x_2), (y_2, x_2), (x_2, y_2)) \geq 1.$
Again, since F is α-admissible, we have $\alpha((x_3, y_3), (x_3, y_3), (y_3, x_3)) \geq 1.$
On continuing this process, we get

$$\alpha((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), (y_{2n+1}, x_{2n+1})) \geq 1 \tag{2.11}$$

for $n = 0, 1, 2, ...$. Now, from the inequalities (2.3) and (2.6), and since F is triangular α-admissible, we get

$\alpha((x_0,y_0),(x_0,y_0),(y_0,x_0)) \geq 1$. Since F is α-admissible, we get
$\alpha((y_1,x_1),(y_1,x_1),(x_1,y_1)) \geq 1$ which gives that
$\alpha((x_2,y_2),(x_2,y_2),(y_2,x_2)) \geq 1$ and proceeding as above, we get

$$\alpha((x_{2n},y_{2n}),(x_{2n},y_{2n}),(y_{2n},x_{2n})) \geq 1 \quad \text{for} \quad n=0,1,2,... \quad . \tag{2.12}$$

From the inequalities (2.11) and (2.12), we get
$\alpha((x_n,y_n),(x_n,y_n),(y_n,x_n)) \geq 1$ for $n=0,1,2,...$.

Therefore (i) holds.

From (i), we have $\alpha((x_1,y_1),(x_1,y_1),(y_1,x_1)) \geq 1$ and from the inequality (2.6), we have $\alpha((y_1,x_1),(y_1,x_1),(y_0,x_0)) \geq 1$, and since F is triangular α-admissible, we get $\alpha((x_1,y_1),(x_1,y_1),(y_0,x_0)) \geq 1$. Since F is α-admissible,
$\alpha((y_2,x_2),(y_2,x_2),(x_1,y_1)) \geq 1$. Again, since F is α-admissible,
$\alpha((x_3,y_3),(x_3,y_3),(y_2,x_2)) \geq 1$. On continuing this process, we get

$$\alpha((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),(y_{2n},x_{2n})) \geq 1 \quad \text{for} \quad n=0,1,2,... \; . \tag{2.13}$$

From the inequality (2.6), $\alpha((y_1,x_1),(y_1,x_1),(y_0,x_0)) \geq 1$.
Since F is α-admissible, we get $\alpha((x_2,y_2),(x_2,y_2),(x_1,y_1)) \geq 1$ and from (i) we have $\alpha((x_1,y_1),(x_1,y_1),(y_1,x_1)) \geq 1$. Since F is triangular α-admissible, we have $\alpha((x_2,y_2),(x_2,y_2),(y_1,x_1)) \geq 1$ which gives that
$\alpha((y_3,x_3),(y_3,x_3),(x_2,y_2)) \geq 1$. Again, since F is α-admissible,
$\alpha((x_4,y_4),(x_4,y_4),(y_3,x_3)) \geq 1$. On continuing this process, we get

$$\alpha((x_{2n},y_{2n}),(x_{2n},y_{2n}),(y_{2n-1},x_{2n-1})) \geq 1 \quad \text{for} \quad n=1,2,... \; . \tag{2.14}$$

From the inequalities (2.13) and (2.14), we get
$\alpha((x_{n+1},y_{n+1}),(x_{n+1},y_{n+1}),(y_n,x_n)) \geq 1$ for $n=0,1,2,...$. Thus (ii) holds.
Now, from the inequality (2.3), we have $\alpha((x_0,y_0),(x_0,y_0),(y_1,x_1)) \geq 1$. Since F is α-admissible, we get $\alpha((y_1,x_1),(y_1,x_1),(x_2,y_2)) \geq 1$. Again, since F is α-admissible, we have $\alpha((x_2,y_2),(x_2,y_2),(y_3,x_3)) \geq 1$ and proceeding as above, we get

$$\alpha((x_{2n},y_{2n}),(x_{2n},y_{2n}),(y_{2n+1},x_{2n+1})) \geq 1 \quad \text{for} \quad n=0,1,2,... \; . \tag{2.15}$$

Now, from the inequality (2.4), we have $\alpha((y_0,x_0),(y_0,x_0),(x_1,y_1)) \geq 1$; Since F is α-admissible, we get that $\alpha((x_1,y_1),(x_1,y_1),(y_2,x_2)) \geq 1$
which gives $\alpha((y_2,x_2),(y_2,x_2),(x_3,y_3)) \geq 1$. Again, since F is α-admissible,
$\alpha((x_3,y_3),(x_3,y_3),(y_4,x_4)) \geq 1$ and proceeding as above, we get

$$\alpha((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),(y_{2n+2},x_{2n+2})) \geq 1 \tag{2.16}$$

for $n = 0, 1, 2, ...$.
From the inequalities (2.15) and (2.16), we have
$\alpha((x_n, y_n), (x_n, y_n), (y_{n+1}, x_{n+1})) \geq 1$ for $n = 0, 1, 2, ...$. Therefore (iii) holds.

Theorem 2.10. *Let (X, S) be a complete S-metric space. Let A and B be two nonempty subsets of X. Let $F : X \times X \to X$ be an α-admissible Reich type cyclic coupled mapping on X and F is a triangular α-admissible mapping. Suppose that there exist $x_0, y_0 \in X$ such that (a), (b), (c), (d) of Proposition 2.9 hold. Suppose that either*

(i) F is continuous, or

(ii) when ever $\{x_n\}$ is a sequence in A and $\{y_n\}$ is a sequence in B such that $x_n \to x$ in A and $y_n \to y$ in B, we have $\alpha((x_n, y_n), (x_n, y_n), (x, y)) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Then $A \cap B \neq \emptyset$ and F has a strong coupled fixed point in $A \cap B$.

Proof. Let $x_0 \in A$ and $y_0 \in B$ be arbitrary. We define sequences $\{x_n\}$ and $\{y_n\}$ as in (2.2). Since F is cyclic, we have $x_n \in A$ and $y_n \in B$ for $n = 0, 1, 2, ...$.
We now prove that

$$S(x_n, x_n, y_{n+1}) + S(y_n, y_n, x_{n+1}) \leq t^n [S(x_0, x_0, y_1) + S(y_0, y_0, x_1)] \tag{2.17}$$

for $n \in \mathbb{N}$ by using induction on n, where $t = \frac{a+\frac{b}{2}+\frac{c}{2}+\frac{d}{2}}{1-[\frac{b}{2}+\frac{c}{2}+\frac{d}{2}]}$.
From (ii) of Proposition 2.9 we have $\alpha((x_1, y_1), (x_1, y_1), (y_0, x_0)) \geq 1$.
By using (2.1) we get that

$$\begin{aligned}
S(x_1, x_1, y_2) &= S(y_2, y_2, x_1) \\
&= S(F(x_1, y_1), F(x_1, y_1), F(y_0, x_0)) \\
&\leq \alpha((x_1, y_1), (x_1, y_1), (y_0, x_0)) S(F(x_1, y_1), F(x_1, y_1), F(y_0, x_0)) \\
&\leq \tfrac{a}{2}[S(x_1, x_1, y_0) + S(y_1, y_1, x_0)] \\
&\quad + \tfrac{b}{2}[S(x_1, x_1, F(x_1, y_1)) + S(y_1, y_1, F(y_1, x_1))] \\
&\quad + \tfrac{c}{2}[S(x_1, x_1, F(x_1, y_1)) + S(y_1, y_1, F(y_1, x_1))] \\
&\quad + \tfrac{d}{2}[S(y_0, y_0, F(y_0, x_0)) + S(x_0, x_0, F(x_0, y_0))] \\
&= [\tfrac{a}{2} + \tfrac{d}{2}][S(x_1, x_1, y_0) + S(y_1, y_1, x_0)] \\
&\qquad + [\tfrac{b}{2} + \tfrac{c}{2}][S(x_1, x_1, y_2) + S(y_1, y_1, x_2)].
\end{aligned}$$

That is

$$\begin{aligned}
S(x_1, x_1, y_2) \leq [\frac{a}{2} + \frac{d}{2}][S(x_1, x_1, y_0) + S(y_1, y_1, x_0)] \\
+[\frac{b}{2} + \frac{c}{2}][S(x_1, x_1, y_2) + S(y_1, y_1, x_2)].
\end{aligned} \tag{2.18}$$

From (iii) of Proposition 2.9 we have $\alpha((x_0,y_0),(x_0,y_0),(y_1,x_1)) \geq 1$ and by using (2.1) we get that

$$\begin{aligned} S(y_1,y_1,x_2) &= S(F(x_0,y_0),F(x_0,y_0),F(y_1,x_1)) \\ &\leq \alpha((x_0,y_0),(x_0,y_0),(y_1,x_1))S(F(x_0,y_0),F(x_0,y_0),F(y_1,x_1)) \\ &\leq \tfrac{a}{2}[S(x_0,x_0,y_1)+S(y_0,y_0,x_1)] \\ &\quad + \tfrac{b}{2}[S(x_0,x_0,F(x_0,y_0))+S(y_0,y_0,F(y_0,x_0))] \\ &\quad + \tfrac{c}{2}[S(x_0,x_0,F(x_0,y_0))+S(y_0,y_0,F(y_0,x_0))] \\ &\quad + \tfrac{d}{2}[S(y_1,y_1,F(y_1,x_1))+S(x_1,x_1,F(x_1,y_1))] \\ &= [\tfrac{a}{2}+\tfrac{b}{2}+\tfrac{c}{2}][S(x_0,x_0,y_1)+S(y_0,y_0,x_1)] \\ &\qquad + \tfrac{d}{2}[S(y_1,y_1,x_2)+S(x_1,x_1,y_2)]. \end{aligned}$$

That is

$$\begin{aligned} S(y_1,y_1,x_2) \leq [\frac{a}{2}+\frac{b}{2}+\frac{c}{2}][S(x_0,x_0,y_1)+S(y_0,y_0,x_1)] \\ +\frac{d}{2}[S(y_1,y_1,x_2)+S(x_1,x_1,y_2)]. \end{aligned} \tag{2.19}$$

Now, from the inequalities (2.18) and (2.19), we have

$$S(x_1,x_1,y_2)+S(y_1,y_1,x_2) \leq t[S(x_0,x_0,y_1)+S(y_0,y_0,x_1)] \tag{2.20}$$

where

$$t = \frac{a+\frac{b}{2}+\frac{c}{2}+\frac{d}{2}}{1-[\frac{b}{2}+\frac{c}{2}+\frac{d}{2}]}. \tag{2.21}$$

Then $0 \leq t < 1$. From (ii) of Proposition 2.9, we have $\alpha((x_2,y_2),(x_2,y_2),(y_1,x_1)) \geq 1$. By using (2.1), We have

$$\begin{aligned} S(x_2,x_2,y_3) &= S(y_3,y_3,x_2) \\ &= S(F(x_2,y_2),F(x_2,y_2),F(y_1,x_1)) \\ &\leq \alpha((x_2,y_2),(x_2,y_2),(y_1,x_1))S(F(x_2,y_2),F(x_2,y_2),F(y_1,x_1)) \end{aligned}$$

which gives that

$$\begin{aligned} S(x_2,x_2,y_3) \leq [\frac{a}{2}+\frac{d}{2}][S(y_1,y_1,x_2)+S(x_1,x_1,y_2)]+[\frac{b}{2}+\frac{c}{2}][S(x_2,x_2,y_3) \\ +S(y_2,y_2,x_3)]. \end{aligned} \tag{2.22}$$

From (iii) of Proposition 2.9, we have $\alpha((x_1,y_1),(x_1,y_1),(y_2,x_2)) \geq 1$.
We now consider

$$\begin{aligned} S(y_2,y_2,x_3) &= S(F(x_1,y_1),F(x_1,y_1),F(y_2,x_2)) \\ &\leq \alpha((x_1,y_1),(x_1,y_1),(y_2,x_2))S(F(x_1,y_1),F(x_1,y_1),F(y_2,x_2)) \end{aligned}$$

which on using (2.1) gives that

$$\begin{aligned} S(y_2, y_2, x_3) \leq [\frac{a}{2} + \frac{b}{2} + \frac{c}{2}][S(x_1, x_1, y_2) + S(y_1, y_1, x_2)] \\ + \frac{d}{2}[S(x_2, x_2, y_3) + S(y_2, y_2, x_3)]. \end{aligned} \tag{2.23}$$

By using the inequalities (2.22) and (2.23), we have

$$\begin{aligned} S(x_2, x_2, y_3) + S(y_2, y_2, x_3) &\leq t[S(x_1, x_1, y_2) + S(y_1, y_1, x_2)] \\ &\leq t^2[S(x_0, x_0, y_1) + S(y_0, y_0, x_1)], \text{ (by using (2.20)).} \end{aligned}$$

We now assume that (2.17) is true for n. We now prove that it holds for $n+1$. From (ii) of Proposition 2.9 we have $\alpha((x_{n+1}, y_{n+1}), (x_{n+1}, y_{n+1}), (y_n, x_n)) \geq 1$ and from (iii) of Proposition 2.9 we have $\alpha((x_n, y_n), (x_n, y_n), (y_{n+1}, x_{n+1})) \geq 1$. We now consider

$$\begin{aligned} &S(x_{n+1}, x_{n+1}, y_{n+2}) + S(y_{n+1}, y_{n+1}, x_{n+2}) \\ &\quad = S(y_{n+2}, y_{n+2}, x_{n+1}) + S(y_{n+1}, y_{n+1}, x_{n+2}) \\ &\quad = S(F(x_{n+1}, y_{n+1}), F(x_{n+1}, y_{n+1}), F(y_n, x_n)) \\ &\qquad + S(F(x_n, y_n), F(x_n, y_n), F(y_{n+1}, x_{n+1})) \\ &\quad \leq \alpha((x_{n+1}, y_{n+1}), (x_{n+1}, y_{n+1}), (y_n, x_n)) \\ &\qquad\qquad S(F(x_{n+1}, y_{n+1}), F(x_{n+1}, y_{n+1}), F(y_n, x_n)) \\ &\qquad + \alpha((x_n, y_n), (x_n, y_n), (y_{n+1}, x_{n+1})) \\ &\qquad\qquad S(F(x_n, y_n), F(x_n, y_n), F(y_{n+1}, x_{n+1})) \\ &\quad \leq [a + \tfrac{b}{2} + \tfrac{c}{2} + \tfrac{d}{2}][S(x_n, x_n, y_{n+1}) + S(y_n, y_n, x_{n+1})] \\ &\qquad + [\tfrac{b}{2} + \tfrac{c}{2} + \tfrac{d}{2}][S(x_{n+1}, x_{n+1}, y_{n+2}) + S(y_{n+1}, y_{n+1}, x_{n+2})] \end{aligned}$$

which implies that

$$\begin{aligned} S(x_{n+1}, x_{n+1}, y_{n+2}) + S(y_{n+1}, y_{n+1}, x_{n+2}) &\leq \frac{a + \frac{b}{2} + \frac{c}{2} + \frac{d}{2}}{1 - [\frac{b}{2} + \frac{c}{2} + \frac{d}{2}]}[S(x_n, x_n, y_{n+1}) \\ &\qquad + S(y_n, y_n, x_{n+1})] \\ &= t[S(x_n, x_n, y_{n+1}) + S(y_n, y_n, x_{n+1})] \\ &\leq t^{n+1}[S(x_0, x_0, y_1) + S(y_0, y_0, x_1)] \\ &\qquad \text{(by our assumption on (2.17)).} \end{aligned}$$

Therefore by mathematical induction, the inequality (2.17) is true for all $n \in \mathbb{N}$. From (i) of Proposition 2.9, we have $\alpha((x_n, y_n), (x_n, y_n), (y_n, x_n)) \geq 1$. We consider

$$\begin{aligned} S(x_{n+1}, x_{n+1}, y_{n+1}) &= S(y_{n+1}, y_{n+1}, x_{n+1}) \\ &= S(F(x_n, y_n), F(x_n, y_n), F(y_n, x_n)) \\ &\leq \alpha((x_n, y_n), (x_n, y_n), (y_n, x_n)) \\ &\qquad S(F(x_n, y_n), F(x_n, y_n), F(y_n, x_n)) \\ &\leq aS(x_n, x_n, y_n) \\ &\qquad + [\tfrac{b}{2} + \tfrac{c}{2} + \tfrac{d}{2}][S(x_n, x_n, F(x_n, y_n)) + S(y_n, y_n, F(y_n, x_n))] \end{aligned}$$

$$\begin{aligned}
&\leq a[2S(x_n, x_n, y_{n+1}) + 2S(y_n, y_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, y_{n+1})] \\
&\quad + [\tfrac{b}{2} + \tfrac{c}{2} + \tfrac{d}{2}][S(x_n, x_n, y_{n+1}) + S(y_n, y_n, x_{n+1})] \\
&= [2a + \tfrac{b}{2} + \tfrac{c}{2} + \tfrac{d}{2}][S(x_n, x_n, y_{n+1}) + S(y_n, y_n, x_{n+1})] \\
&\quad + aS(x_{n+1}, x_{n+1}, y_{n+1})
\end{aligned}$$

which on using the inequality (2.17) gives that

$$S(x_{n+1}, x_{n+1}, y_{n+1}) \leq pt^n[S(x_0, x_0, y_1) + S(y_0, y_0, x_1)] \tag{2.24}$$

where $p = \frac{2a+\frac{b}{2}+\frac{c}{2}+\frac{d}{2}}{1-a}$. We now consider

$$\begin{aligned}
S(x_n, x_n, x_{n+1}) + S(y_n, y_n, y_{n+1}) &\leq 2S(x_n, x_n, y_n) + S(y_n, y_n, x_{n+1}) \\
&\quad + 2S(y_n, y_n, x_n) + S(x_n, x_n, y_{n+1}) \\
&= 4S(x_n, x_n, y_n) + S(y_n, y_n, x_{n+1}) \\
&\quad + S(x_n, x_n, y_{n+1}) \\
&\leq 4t^{n-1}p[S(x_0, x_0, y_1) + S(y_0, y_0, x_1)] \\
&\quad + t^n[S(x_0, x_0, y_1) + S(y_0, y_0, x_1)] \\
&\quad \text{(by (2.24) and (2.17))} \\
&= t^n[S(x_0, x_0, y_1) + S(y_0, y_0, x_1)][\tfrac{4p}{t} + 1] \\
&= t^n K
\end{aligned}$$

where $K = [S(x_0, x_0, y_1) + S(y_0, y_0, x_1)][\frac{4p}{t} + 1]$.
Let m, n be positive integers with $m > n$. We consider

$$\begin{aligned}
S(x_n, x_n, x_m) + S(y_n, y_n, y_m) &\leq 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_m) \\
&\quad + 2S(y_n, y_n, y_{n+1}) + S(y_{n+1}, y_{n+1}, y_m) \\
&\leq 2S(x_n, x_n, x_{n+1}) + 2S(x_{n+1}, x_{n+1}, x_{n+2}) + ... \\
&\quad + S(x_{m-1}, x_{m-1}, x_m) \\
&\quad + 2S(y_n, y_n, y_{n+1}) + 2S(y_{n+1}, y_{n+1}, y_{n+2}) + \\
&\quad ... + S(y_{m-1}, y_{m-1}, y_m) \\
&\leq 2t^n K + 2t^{n+1}K + ... + t^{m-1}K \\
&\leq 2t^n K + 2t^{n+1}K + ... + 2t^{m-1}K \\
&= 2K[t^n + t^{n+1} + ... + t^{m-1}] \\
&\leq 2K\tfrac{t^n}{1-t}.
\end{aligned}$$

On taking limits as $n \to +\infty$, since $0 < t < 1$, we have $\lim_{n,m\to+\infty}(S(x_n, x_n, x_m) + S(y_n, y_n, y_m)) = 0$. Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences and hence convergent. Since A and B are closed subsets of X and $\{x_n\} \subset A$, $\{y_n\} \subset B$, there exist $x \in A$ and $y \in B$ such that

$$x_n \to x, \qquad y_n \to y \quad \text{as} \quad n \to +\infty. \tag{2.25}$$

We now prove $F(x, x) = x$.
By using the inequality (2.24), we get $\lim_{n\to+\infty} S(x_n, x_n, y_n) = 0$. Now, by Lemma 1.18, we have $S(x, x, y) = 0$ and hence $x = y$ so that $A \cap B \neq \emptyset$ and $x \in A \cap B$.

Now suppose that (i) holds. That is F is continuous. Then we have
$x = \lim_{n\to+\infty} x_{n+1} = \lim_{n\to+\infty} F(y_n, x_n) = F(y, x) = F(x, x)$.
Now suppose that (ii) holds. Since $x_n \to x$ in A and $y_n \to y$ in B, by our assumption we have $\alpha((x_n, y_n), (x_n, y_n), (x, y)) \geq 1$. Now, by (2.1) and (2.2), we have

$$\begin{aligned}
S(x, x, F(x, x)) &\leq 2S(x, x, y_{n+1}) + S(y_{n+1}, y_{n+1}, F(x, x)) \\
&= 2S(x, x, y_{n+1}) + S(F(x_n, y_n), F(x_n, y_n), F(x, x)) \\
&\leq 2S(x, x, y_{n+1}) \\
&\quad + \alpha((x_n, y_n), (x_n, y_n), (x, x))S(F(x_n, y_n), F(x_n, y_n), F(x, x)) \\
&\leq 2S(x, x, y_{n+1}) + \tfrac{a}{2}[S(x_n, x_n, x) + S(y_n, y_n, x)] \\
&\qquad + \tfrac{b}{2}[S(x_n, x_n, F(x_n, y_n)) + S(y_n, y_n, F(y_n, x_n))] \\
&\qquad + \tfrac{c}{2}[S(x_n, x_n, F(x_n, y_n)) + S(y_n, y_n, F(y_n, x_n))] \\
&\qquad + \tfrac{d}{2}[S(x, x, F(x, x)) + S(x, x, F(x, x))] \\
&= 2S(x, x, y_{n+1}) + \tfrac{a}{2}[S(x_n, x_n, x) + S(y_n, y_n, x)] \\
&\qquad + [\tfrac{b}{2} + \tfrac{c}{2}][S(x_n, x_n, y_{n+1}) + S(y_n, y_n, x_{n+1})] \\
&\qquad + \tfrac{d}{2}[S(x, x, F(x, x)) + S(x, x, F(x, x))].
\end{aligned}$$

On taking limits as $n \to +\infty$, we get

$$\begin{aligned}
S(x, x, F(x, x)) &\leq 2S(x, x, y) + \tfrac{a}{2}[S(x, x, x) + S(y, y, x)] \\
&\quad + [\tfrac{b+c}{2}][S(x, x, y) + S(y, y, x)] + dS(x, x, F(x, x))
\end{aligned}$$

which implies that $(1 - d)S(x, x, F(x, x)) \leq 0$.
Thus $S(x, x, F(x, x)) = 0$ so that $x = F(x, x)$.
Therefore (x, x) is a strong coupled fixed point of F.

In order to get the uniqueness of coupled fixed point in Theorem 2.10, we consider the following hypothesis:
Condition (U): For any $(x, y), (u, v) \in A \times B$, there exists $(z, t) \in A \times B$ such that the following hold:

(u_1) $\alpha((z, t), (z, t), (x, y)) \geq 1$; (u_2) $\alpha((t, z), (t, z), (y, x)) \geq 1$;

(u_3) $\alpha((x, y), (x, y), (z, t)) \geq 1$; (u_4) $\alpha((y, x), (y, x), (t, z)) \geq 1$;

(u_5) $\alpha((z, t), (z, t), (u, v)) \geq 1$; (u_6) $\alpha((t, z), (t, z), (v, u)) \geq 1$;

(u_7) $\alpha((u, v), (u, v), (z, t)) \geq 1$; (u_8) $\alpha((v, u), (v, u), (t, z)) \geq 1$.

Theorem 2.11. *Under the hypotheses of Theorem 2.10, and $2a+3b+3c+3d < 2$, and Condition (U), F has a unique strong coupled fixed point.*

Proof. Suppose (x, x) and (y, y) are two strong coupled fixed points of F. We define the sequences $\{z_n\} \subseteq A$ and $\{t_n\} \subseteq B$ as $z_{n+1} = F(t_n, z_n)$ and $t_{n+1} = F(z_n, t_n)$. From (u_4), we have $\alpha((x, x), (x, x), (t_0, z_0)) \geq 1$. Since F is α-admissible, we get that $\alpha((x, x), (x, x), (z_1, t_1)) \geq 1$.

Again, since F is α-admissible, we get that $\alpha((x,x),(x,x),(t_2,z_2)) \geq 1$.
On continuing this process, we get $\alpha((x,x),(x,x),(t_{2n},z_{2n})) \geq 1$ for $n = 0,1,2,\dots$.
From (u_3), we have $\alpha((x,x),(x,x),(z_0,t_0)) \geq 1$. Since F is α-admissible, we get that $\alpha((x,x),(x,x),(t_1,z_1)) \geq 1$.
Again, since F is α-admissible, we get that $\alpha((x,x),(x,x),(z_2,t_2)) \geq 1$.
On continuing this process, we get $\alpha((x,x),(x,x),(t_{2n+1},z_{2n+1})) \geq 1$ for $n = 0,1,2,\dots$.
Therefore we have $\alpha((x,x),(x,x),(t_n,z_n)) \geq 1$ for $n = 0,1,2,\dots$.
Now, by using (u_1) and (u_2) and on proceeding as the similar lines as above, we obtain that $\alpha((z_n,t_n),(z_n,t_n),(x,x)) \geq 1$ for $n = 0,1,2,\dots$.
Since $\alpha((x,x),(x,x),(t_n,z_n)) \geq 1$ and by using (2.1), We have

$$\begin{aligned}
S(z_{n+1},z_{n+1},x) &= S(x,x,z_{n+1}) = S(F(x,x),F(x,x),F(t_n,z_n)) \\
&\leq \alpha((x,x),(x,x),(t_n,z_n))S(F(x,x),F(x,x),F(t_n,z_n)) \\
&\leq \tfrac{a}{2}[S(x,x,t_n)+S(x,x,z_n)] + [b+c][S(x,x,F(x,x))] \\
&\quad + \tfrac{d}{2}[S(t_n,t_n,(t_n,z_n)) + S(z_n,z_n,F(z_n,t_n))] \\
&= \tfrac{a}{2}[S(x,x,t_n)+S(x,x,z_n)] + \\
&\quad + \tfrac{d}{2}[S(t_n,t_n,z_{n+1}) + S(z_n,z_n,t_{n+1})].
\end{aligned}$$

Now, since $\alpha((z_n,t_n),(z_n,t_n),(x,x)) \geq 1$ and by using (2.1), we have

$$\begin{aligned}
S(t_{n+1},t_{n+1},x) &= S(F(z_n,t_n),F(z_n,t_n),F(x,x)) \\
&\leq \alpha(F(z_n,t_n),F(z_n,t_n),F(x,x))S(F(z_n,t_n),F(z_n,t_n),F(x,x)) \\
&\leq \tfrac{a}{2}[S(z_n,z_n,x)+S(t_n,t_n,x)] \\
&\quad + \tfrac{[b+c]}{2}[S(z_n,z_n,F(z_n,t_n)) + S(t_n,t_n,(t_n,z_n))] \\
&= \tfrac{a}{2}[S(x,x,z_n)+S(x,x,t_n)] \\
&\quad + \tfrac{[b+c]}{2}[S(z_n,z_n,t_{n+1}) + S(t_n,t_n,z_{n+1})].
\end{aligned}$$

We consider

$$\begin{aligned}
S(z_{n+1},z_{n+1},x) + S(t_{n+1},z_{n+1},x) &\leq a[S(x,x,t_n)+S(x,x,z_n)] \\
&\quad + [\tfrac{b}{2}+\tfrac{c}{2}+\tfrac{d}{2}][S(z_n,z_n,t_{n+1}) + S(t_n,t_n,z_{n+1})] \\
&\leq a[S(x,x,t_n)+S(x,x,z_n)] \\
&\quad + [\tfrac{b}{2}+\tfrac{c}{2}+\tfrac{d}{2}][2S(z_n,z_n,x) + S(x,x,t_{n+1}) \\
&\qquad + 2S(t_n,t_n,x) + S(x,x,z_{n+1})] \\
&= [a+b+c+d](S(x,x,t_n)+S(x,x,z_n)] \\
&\quad + [\tfrac{b}{2}+\tfrac{c}{2}+\tfrac{d}{2}][S(x,x,t_{n+1}) + S(x,x,z_{n+1})]
\end{aligned}$$

which implies that

$$\begin{aligned}
S(z_{n+1},z_{n+1},x) + S(t_{n+1},t_{n+1},x) &\leq \tfrac{a+b+c+d}{1-[\frac{b}{2}+\frac{c}{2}+\frac{d}{2}]}[S(x,x,t_n)+S(x,x,z_n)] \\
&\leq p_1[S(x,x,t_n)+S(x,x,z_n)] \\
&\leq p_1^n[S(x,x,t_0)+S(x,x,z_0)]
\end{aligned}$$

where $0 \leq p_1 = \frac{a+b+c+d}{1-[\frac{b}{2}+\frac{c}{2}+\frac{d}{2}]} < 1$.

On taking the limits as $n \to +\infty$, we have
$\lim_{n\to+\infty}[S(z_{n+1}, z_{n+1}, x) + S(t_{n+1}, t_{n+1}, x)] \le 0$
which implies that
$\lim_{n\to+\infty} S(z_{n+1}, z_{n+1}, x) = 0$ and $\lim_{n\to+\infty} S(t_{n+1}, t_{n+1}, x) = 0$ so that
$\lim_{n\to+\infty} z_{n+1} = x = \lim_{n\to+\infty} t_{n+1}$.
Similarly, we can prove that
$\lim_{n\to+\infty}[S(z_{n+1}, z_{n+1}, y) + S(t_{n+1}, t_{n+1}, y)] \le 0$ so that
$\lim_{n\to+\infty} z_{n+1} = y = \lim_{n\to+\infty} t_{n+1}$.

Therefore $x = y$. Thus (x, x) is a unique strong coupled fixed point of F.

Proposition 2.12. *Let (X, S) be an S-metric space. Let $F : X \times X \to X$ be α-admissible mapping on X. Suppose that there exist $x_0, y_0 \in X$ such that*

(a) $\alpha((x_0, y_0), (x_0, y_0), (F(x_0, y_0), F(y_0, x_0))) \ge 1$

(b) $\alpha((y_0, x_0), (y_0, x_0), (F(y_0, x_0), F(x_0, y_0))) \ge 1$

(c) $\alpha((x_0, y_0), (x_0, y_0), (y_0, x_0)) \ge 1$.

Then there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

(i) $\alpha((x_n, y_n), (x_n, y_n), (y_n, x_n)) \ge 1$ *for* $n = 0, 1, 2, \ldots$;

(ii) $\alpha((x_n, y_n), (x_n, y_n), (x_{n+1}, y_{n+1})) \ge 1$ *for* $n = 0, 1, 2, 3, \ldots$;

(iii) $\alpha((y_n, x_n), (y_n, x_n), (y_{n+1}, x_{n+1})) \ge 1$ *for* $n = 0, 1, 2, \ldots$.

Proof. Let $x_0, y_0 \in X$ be arbitrary. We define sequences $\{x_n\}$ and $\{y_n\}$ by

$$x_{n+1} = F(x_n, y_n), \quad y_{n+1} = F(y_n, x_n), \quad n = 0, 1, 2, \ldots \quad . \tag{2.26}$$

From (a), we have $\alpha((x_0, y_0), (x_0, y_0), (x_1, y_1)) \ge 1$. Since F is α-admissible, we have
$\alpha((F(x_0, y_0), F(y_0, x_0)), (F(x_0, y_0), F(y_0, x_0)), (F(x_1, y_1), F(y_1, x_1))) \ge 1$.
That is $\alpha((x_1, y_1), (x_1, y_1), (x_2, y_2)) \ge 1$. Again, since F is α-admissible, we have $\alpha((x_2, y_2), (x_2, y_2), (x_3, y_3)) \ge 1$. On continuing this process, we get
$\alpha((x_n, y_n), (x_n, y_n), (x_{n+1}, y_{n+1})) \ge 1$ for $n = 0, 1, 2, \ldots$.

Therefore (ii) holds.

Similarly, from (b) we prove (iii) and from (c) we prove (i).

Theorem 2.13. *Let (X, S) be a complete S-metric space. Let $F : X \times X \to X$ be an α-admissible Reich type coupled mapping on X. Suppose that there exist $x_0, y_0 \in X$ such that (a), (b), (c) of Proposition 2.12 hold. Suppose that either* *(i) F is continuous, or*

(ii) when ever $x_n \to x$ and $y_n \to y$ for all $x, y \in X$, we have $\alpha((x_n, y_n), (x_n, y_n), (x, y)) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Then F has a strong coupled fixed point in X.

Proof. Let $x_0, y_0 \in X$ be arbitrary. By Proposition 2.12 there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $x_{n+1} = F(x_n, y_n), \quad y_{n+1} = F(y_n, x_n)$, $n = 0, 1, 2, ...$ and (i), (ii), (iii) of Proposition 2.12 hold. Now proceeding on the similar lines as in the proof of Theorem 2.10, we obtain the conclusion of the theorem.

For the uniqueness of strong coupled fixed point in Theorem 2.13, we consider the following hypothesis:

Condition (U_1): For each $(x, y), (u, v) \in X \times X$ there exists a point $(z, t) \in X \times X$ such that the following hold:

(u_1) $\alpha((x, y), (x, y), (z, t)) \geq 1$

(u_2) $\alpha((y, x), (y, x), (t, z)) \geq 1$

(u_3) $\alpha((u, v), (u, v), (z, t)) \geq 1$

(u_4) $\alpha((v, u), (v, u), (t, z)) \geq 1$.

Theorem 2.14. *Under the hypotheses of Theorem 2.13, if $a+3d < 1$ and Condition (U_1) holds then F has a unique strong coupled fixed point.*

Proof. Suppose (x, x) and (y, y) be two strong coupled fixed points of F. Hence $x, y \in X$. Let $z_0 = z$ and $t_0 = t$ be arbitrary in X. We define sequences $\{z_n\}$, $\{t_n\}$ as $z_{n+1} = F(z_n, t_n)$ and $t_{n+1} = F(t_n . z_n)$ then $z_n, t_n \in X$.

From (u_1), we have $\alpha((x, x), (x, x), (z_0, t_0)) \geq 1$. Since F is α-admissible, we get that $\alpha((x, x), (x, x), (z_n, t_n)) \geq 1$. We consider

$$\begin{aligned}
S(x, x, z_{n+1}) &= S(F(x, x), F(x, x), F(z_n, t_n)) \\
&\leq \alpha((x, x), (x, x), (z_n, t_n)) S(F(x, x), F(x, x), F(z_n, t_n)) \\
&\leq \tfrac{a}{2}[S(x, x, z_n) + S(x, x, t_n) + [b + c][S(x, x, F(x, x))] \\
&\quad + \tfrac{d}{2}[S(z_n, z_n, z_{n+1}) + S(t_n, t_n, t_{n+1})]
\end{aligned}$$

From (u_2), we have $\alpha((x, x), (x, x), (t_0, z_0)) \geq 1$. Since F is α-admissible, we get that $\alpha((x, x), (x, x), (t_n, z_n)) \geq 1$. We consider

$$\begin{aligned}
S(x, x, t_{n+1}) &= S(F(x, x), F(x, x), F(t_n, z_n)) \\
&\leq \alpha((x, x), (x, x), (t_n, z_n)) S(F(x, x), F(x, x), F(t_n, z_n)) \\
&\leq \tfrac{a}{2}[S(x, x, t_n) + S(x, x, z_n)] + [b + c][S(x, x, F(x, x))] \\
&\quad + \tfrac{d}{2}[S(t_n, t_n, t_{n+1}) + S(z_n, z_n, z_{n+1})].
\end{aligned}$$

Now, we have

$$\begin{aligned} S(x,x,z_{n+1})+S(x,x,t_{n+1}) &\le a[S(x,x,z_n)+S(x,x,t_n)] \\ &\quad + d[S(z_n,z_n,z_{n+1})+S(t_n,t_n,t_{n+1})] \\ &\le a[S(x,x,z_n)+S(x,x,t_n)] \\ &\quad + d[2S(z_n,z_n,x)+S(x,x,z_{n+1}) \\ &\qquad + 2S(t_n,t_n,x)+S(x,x,t_{n+1})] \end{aligned}$$

which implies that

$$\begin{aligned} S(x,x,z_{n+1})+S(x,x,t_{n+1}) &\le \tfrac{a+2d}{1-d}[S(x,x,z_n)+S(x,x,t_n)] \\ &\le P^{n+1}[S(x,x,z_0)+S(x,x,t_0)] \\ &\quad \text{(similar as in the proof of (2.3))} \end{aligned}$$

where $0 \le P = \frac{a+2d}{1-d} < 1$.
On taking the limits as $n \to +\infty$, we have
$\lim\limits_{n\to+\infty}[S(x,x,z_{n+1})+S(x,x,t_{n+1})] \le 0$
which implies that $\lim\limits_{n\to+\infty} S(x,x,z_{n+1}) = 0$ and $\lim\limits_{n\to+\infty} S(x,x,t_{n+1}) = 0$ so that $\lim\limits_{n\to+\infty} z_{n+1} = x = \lim\limits_{n\to+\infty} t_{n+1}$. Similarly, we prove that
$\lim\limits_{n\to+\infty}[S(y,y,z_{n+1})+S(y,y,t_{n+1})] \le 0$ so that $\lim\limits_{n\to+\infty} z_{n+1} = y = \lim\limits_{n\to+\infty} t_{n+1}$.
Therefore $x = y$. Thus (x,x) is a unique strong coupled fixed point of F.

3. Corollaries and Examples

If we choose $\alpha \equiv 1$, $b = c = d = 0$, in the inequality (2.1), then we have the following corollary from Theorem 2.11.

Corollary 3.1. *Let (X,S) be a complete S-metric space. Let A and B be two nonempty closed subsets of X. Let $F : X \times X \to X$ be a cyclic mapping with respect to A and B satisfying the following inequality: there exists $0 \le a < 1$ such that*

$$S(F(x,y),F(u,v),F(w,z)) \le \frac{a}{2}(S(x,u,w)+S(y,v,z)),$$

where $x,u,z \in A$ and $y,v,w \in B$. Then F has a unique strong coupled fixed point in $A \cap B$.

By choosing $\alpha \equiv 1$ and $A = B = X$ in Theorem 2.11, we have the following corollary from Theorem 2.11.

Corollary 3.2. *Let (X,S) be a complete S-metric space. Let $F : X \times X \to X$ be a mapping satisfying the following inequality:*
there exist a, b, c, $d \ge 0$ with $a+b+c+d < 1$ such that

$$\begin{aligned} S(F(x,y),F(u,v),F(w,z)) &\le \tfrac{a}{2}(S(x,u,w)+S(y,v,z)) \\ &\quad + \tfrac{b}{2}(S(x,x,F(x,y))+S(y,y,F(y,x))) \end{aligned}$$

$$+\frac{c}{2}(S(u,u,F(u,v))+S(v,v,F(v,u)))$$
$$+\frac{d}{2}(S(w,w,F(w,z))+S(z,z,F(z,w))),$$

where $x,u,z,y,v,w \in X$. Then F has a unique strong coupled fixed point in X.

In the following, we give examples in support of our results.

Example 3.3. Let $X=[0,1]$. We define $S: X^3 \to [0,+\infty)$ by

$$S(x,y,z)=\begin{cases} 0 & \text{if } x=y=z \\ \max\{x,y,z\} & \text{otherwise.}\end{cases}$$

Then (X,S) is a complete S-metric space.
Let $A=B=X$. We define $F: X\times X \to X$ by $F(x,y)=\frac{x+y}{20}$ so that F is cyclic with respect to A and B.
We define $\alpha: X^2\times X^2\times X^2 \to [0,+\infty)$ by

$$\alpha((x,y),(u,v),(w,z))=\begin{cases} 1 & \text{if } (x,y)\neq(u,v) \quad \text{or} \quad (u,v)\neq(w,z) \\ & \quad \text{or} \quad (x,y)\neq(w,z) \\ 0 & \text{otherwise.}\end{cases}$$

Here F is α-admissible. For, assume that $\alpha((x,y),(u,v),(w,z)) \geq 1$. Then we have $(x,y)\neq(u,v)$ or $(u,v)\neq(w,z)$ or $(x,y)\neq(w,z)$.
If $(x,y)\neq(u,v)$ then $\frac{x+y}{20}\neq\frac{u+v}{20}$ and $\frac{y+x}{20}\neq\frac{v+u}{20}$.
If $(u,v)\neq(w,z)$ then $\frac{u+v}{20}\neq\frac{w+z}{20}$ and $\frac{v+u}{20}\neq\frac{z+w}{20}$.
If $(x,y)\neq(w,z)$ then $\frac{x+y}{20}\neq\frac{w+z}{20}$ and $\frac{y+x}{20}\neq\frac{z+w}{20}$.
Then we have $\alpha((\frac{x+y}{20},\frac{y+x}{20}),(\frac{u+v}{20},\frac{v+u}{20}),(\frac{w+z}{20},\frac{z+w}{20}))\geq 1$.
That is $\alpha((F(x,y),F(y,x)),(F(u,v),F(v,u)),(F(w,z),F(z,w)))\geq 1$.
Clearly, F is triangular α-admissible. For, assume that
$\alpha((x,y),(x,y),(u,v))\geq 1$ and $\alpha((u,v),(u,v),(w,z))\geq 1$.
Then $(x,y)\neq(u,v)$ and $(u,v)\neq(w,z)$ so that $(x,y)\neq(w,z)$.

Therefore $\alpha((x,y),(x,y),(w,z))\geq 1$. Also, conditions (a), (b), (c), (d) of Proposition 2.9 are satisfied for $x_0=0$ and $y_0=1$. Here $F(0,1)=\frac{1}{20}$ and $F(1,0)=\frac{1}{20}$.
(a) $\alpha((x_0,y_0),(x_0,y_0),(F(x_0,y_0),F(y_0,x_0)))=\alpha((0,1),(0,1),(\frac{1}{20},\frac{1}{20}))=1$
(b) $\alpha((y_0,x_0),(y_0,x_0),(F(y_0,x_0),F(x_0,y_0)))=\alpha((1,0),(1,0),(\frac{1}{20},\frac{1}{20}))=1$
(c) $\alpha((F(y_0,x_0),F(x_0,y_0)),(F(y_0,x_0),F(x_0,y_0)),(x_0,y_0))$
$$=\alpha((\tfrac{1}{20},\tfrac{1}{20}),(\tfrac{1}{20},\tfrac{1}{20}),(0,1))=1$$
(d) $\alpha((F(x_0,y_0),F(y_0,x_0)),(F(x_0,y_0),F(y_0,x_0)),(y_0,x_0))$
$$=\alpha((\tfrac{1}{20},\tfrac{1}{20}),(\tfrac{1}{20},\tfrac{1}{20}),(0,1))=1.$$
We now verify the inequality (2.1). Let $x,u,z,y,v,w\in[0,1]$.
If $(x,y)\neq(u,v)$ or $(u,v)\neq(w,z)$ or $(x,y)\neq(w,z)$ then we have

$$\begin{aligned}
\alpha((x,y),(u,v),(w,z))&S(F(x,y),F(u,v),F(w,z))\\
&= S(F(x,y),F(u,v),F(w,z))\\
&= \max\{\tfrac{x+y}{20},\tfrac{u+v}{20},\tfrac{w+z}{20}\}\\
&\leq \tfrac{1}{20}[x+y+u+v+w+z]\\
&\leq \tfrac{3}{20}[\max\{x,u,w\}+\max\{y,v,z\}]\\
&\leq \tfrac{a}{2}[S(x,u,w)+S(y,v,z)]+\tfrac{b}{2}[S(x,x,F(x,y))+S(y,y,F(y,x))]\\
&\quad+\tfrac{c}{2}[S(u,u,F(u,v))+S(v,v,F(v,u))]\\
&\quad+\tfrac{d}{2}[S(w,w,F(w,z))+S(z,z,F(z,w))]
\end{aligned}$$

for $a=\frac{3}{10}$ and $b,c,d\geq 0$ with $a+b+c+d<1$.
In all the remaining cases, the inequality (2.1) holds trivially.
Now for any (x,y), $(u,v)\in A\times B$, we choose $(z,t)\in A\times B$ such that $(z,t)\neq(x,y)$ and $(z,t)\neq(u,v)$ so that condition (U) holds, by the definition of α.

Therefore F satisfies all the hypotheses of Theorem 2.11 and $(0,0)$ is a unique strong coupled fixed point of F.

The following is an example in support of Theorem 2.10 for which (ii) of Theorem 2.10 holds and this example illustrates the significance of α-admissible function in Theorem 2.10.

Example 3.4. Let $X=[0,2]$. We define $S:X^3\to[0,+\infty)$ by

$$S(x,y,z)=\begin{cases}0 & \text{if } x=y=z\\ \max\{x,y,z\} & \text{otherwise.}\end{cases}$$

Then (X,S) is a complete S-metric space.
Let $A=[0,1]$ and $B=[1,2]$. We define $F:X\times X\to X$ by

$$F(x,y)=\begin{cases}\frac{2+y-x}{2} & \text{if } x\in A \text{ and } y\in B\\ 0 & \text{otherwise.}\end{cases}$$

Then $F(A,B)=[1,2]\subset B$ and $F(B,A)=\{0\}$ so that is cyclic with respect to A and B. We define $\alpha:X^2\times X^2\times X^2\to[0,+\infty)$ by

$$\alpha((x,y),(u,v),(w,z))=\begin{cases}1 & \text{if } x\leq y,\ u\leq v \text{ and } w\leq z\\ & \quad\text{or } x\geq y,\ u\geq v \text{ and } w\geq z\\ \frac{1}{12} & \text{otherwise.}\end{cases}$$

Here F is α-admissible. For, assume that $\alpha((x,y),(u,v),(w,z))\geq 1$. Then we have either $x\leq y, u\leq v$ and $w\leq z$ or $x\geq y, u\geq v$ and $w\geq z$.
If $x\leq y, u\leq v$, and $w\leq z$ then $\frac{2+y-x}{2}\geq\frac{2+x-y}{2}$, $\frac{2+v-u}{2}\geq\frac{2+u-v}{2}$ and $\frac{2+z-w}{2}\geq\frac{2+w-z}{2}$.
If $x\geq y, u\geq v$, and $w\geq z$ then $\frac{2+y-x}{2}\leq\frac{2+x-y}{2}$, $\frac{2+v-u}{2}\leq\frac{2+u-v}{2}$ and $\frac{2+z-w}{2}\leq\frac{2+w-z}{2}$.

Then we have $\alpha((\frac{2+y-x}{2}, \frac{2+x-y}{2}), (\frac{2+v-u}{2}, \frac{2+u-v}{2}), (\frac{2+z-w}{2}, \frac{2+w-z}{2})) = 1$.
That is $\alpha((F(x,y), F(y,x)), (F(u,v), F(v,u)), (F(w,z), F(z,w))) = 1$.
Clearly, F is triangular α-admissible. For, assume that
$\alpha((x,y),(x,y),(u,v)) \geq 1$ and $\alpha((u,v),(u,v),(w,z)) \geq 1$.
Then $x \leq y$, $u \leq v$ or $x > y$, $u > v$; and $u \leq v$, $w \leq z$ or $u > v$, $w > z$.
If $x \leq y; u \leq v$ and $u \leq v; w \leq z$ then we have $x \leq y; w \leq z$
so that $\alpha((x,y),(x,y),(w,z)) = 1$.
If $x \leq y$; $u \leq v$ and $u > v$; $w > z$. This case will not arise.
If $x > y$, $u > v$ and $u \leq v$, $w \leq z$. Also, this case will not arise.
If $x > y$, $u > v$ and $u > v$, $w > z$ then we have $x > y$, $w > z$
so that $\alpha((x,y),(x,y),(w,z)) = 1$.
Also, conditions (a), (b), (c), (d) Proposition 2.9 are satisfied for $x_0 = 1$ and $y_0 = 1$.
We now verify (ii) of Theorem 2.10.

Suppose that $\{x_n\} \subset A$ and $\{y_n\} \subset B$ such that $x_n \to x \in A$ and $y_n \to y \in B$.
We now prove $\alpha((x_n,y_n),(x_n,y_n),(x,y)) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.
Here we have $x_n \leq y_n$ for all n and on taking the limits as $n \to +\infty$, we get $x \leq y$ which implies that $\alpha((x_n,y_n),(x_n,y_n),(x,y)) \geq 1$.

We now verify the inequality (2.1). Let $x,u,z \in A$ and $y,v,w \in B$ so that $x \leq y$, $u \leq v$, and $z \leq w$. Then we have

$$\begin{aligned}
&\alpha((x,y),(u,v),(w,z))S(F(x,y),F(u,v),F(w,z)) \\
&\quad = \tfrac{1}{12}\max\{\tfrac{2+y-x}{2}, \tfrac{2+u-v}{2}, \tfrac{2+z-w}{2}\} \\
&\quad \leq \tfrac{1}{12}[S(x,x,F(x,y)) + S(u,u,F(u,v)) + S(w,w,F(w,z))] \\
&\quad \leq \tfrac{1}{12}[S(x,x,F(x,y)) + S(y,y,F(y,x))] \\
&\qquad + \tfrac{1}{12}[S(u,u,F(u,v)) + S(v,v,F(v,u))] \\
&\qquad + \tfrac{1}{12}[S(w,w,F(w,z)) + S(z,z,F(z,w))] \\
&\quad = \tfrac{a}{2}[S(x,u,w) + S(y,v,z)] \\
&\qquad + \tfrac{b}{2}[S(x,x,F(x,y)) + S(y,y,F(y,x))] \\
&\qquad + \tfrac{c}{2}[S(u,u,F(u,v)) + S(v,v,F(v,u))] \\
&\qquad + \tfrac{d}{2}[S(w,w,F(w,z)) + S(z,z,F(z,w))],
\end{aligned}$$

where $a = 0$, $b = \frac{1}{6}$, $c = \frac{1}{6}$ and $d = \frac{1}{6}$ with $a+b+c+d < 1$.
Therefore the inequality (2.1) holds.
If $x = y = u = v = w = z = 1$ then we have $S(F(x,y),F(u,v),F(w,z)) = 0$. In this case also, the inequality (2.1) holds trivially.
Further, it is easy to see that Condition (U) holds.

Therefore F satisfies all the hypotheses of Theorem 2.11 and $(1,1)$ is a unique strong coupled fixed point of F.

Here we observe that if we relax α from the inequality 2.1 then F may not satisfy the inequality 2.1.

For, by choosing $x = 0, y = 2, u = 0, v = 2, w = 0, z = 2$, we have

$$\begin{aligned}
S(F(x,y), F(u,v), F(w,z)) = 2 & \\
&\nleq 2(a+b+c+d) \\
&= \tfrac{a}{2}[S(0,0,2) + S(2,2,0)] \\
&\quad + \tfrac{b}{2}[S(0,0,2) + S(2,2,0)] \\
&\quad + \tfrac{c}{2}[S(0,0,2) + S(2,2,0)] \\
&\quad + \tfrac{d}{2}[S(2,2,0) + S(0,0,2)] \\
&= \tfrac{a}{2}[S(x,u,w) + S(y,v,z)] \\
&\quad + \tfrac{b}{2}[S(x,x,F(x,y)) + S(y,y,F(y,x))] \\
&\quad + \tfrac{c}{2}[S(u,u,F(u,v)) + S(v,v,F(v,u))] \\
&\quad + \tfrac{d}{2}[S(w,w,F(w,z)) + S(z,z,F(z,w))]
\end{aligned}$$

for any $a, b, c, d \geq 0$ with $a + b + c + d < 1$.
Hence the inequality 2.1 fails to hold when $\alpha \equiv 1$.

The following is an example in support of Theorem 2.13.

Example 3.5. Let $X = [0, 1]$. We define $S : X^3 \to [0, \infty)$ by

$$S(x,y,z) = \begin{cases} 0 & \text{if } x = y = z \\ \max\{x,y,z\} & \text{otherwise.} \end{cases}$$

Then (X, S) is a complete S-metric space. We define $F : X \times X \to X$ by $F(x, y) = xy$. We define $\alpha : X^2 \times X^2 \times X^2 \to [0, \infty)$ by

$$\alpha((x,y),(u,v),(w,z)) = \begin{cases} 1 & \text{if } x = y = u = v = w = z \\ \frac{1}{8} & \text{otherwise.} \end{cases}$$

Here F is α-admissible. For, assume that $\alpha((x, y), (u, v), (w, z)) \geq 1$. Then we have $x = y = u = v = w = z$. We have

$$\begin{aligned}
&\alpha((F(x,y), F(y,x)), (F(u,v), F(v,u)), (F(w,z), F(z,w))) \\
&\qquad = \alpha((xy, yx), (uv, vu), (wz, zw) \\
&\qquad = 1(\text{since } xy = yx = uv = vu = wz = zw).
\end{aligned}$$

We now verify conditions (a), (b), (c) of Proposition 2.12 for $x_0 = 1$ and $y_0 = 1$.
Here $F(1, 1) = 1$. We have

(a) $\alpha((x_0, y_0), (x_0, y_0), (F(x_0, y_0), F(y_0, x_0))) = \alpha((1, 1), (1, 1), (1, 1)) = 1$
(b) $\alpha((y_0, x_0), (y_0, x_0), (F(y_0, x_0), F(x_0, y_0))) = \alpha((1, 1), (1, 1), (1, 1)) = 1$
(c) $\alpha((x_0, y_0), (x_0, y_0), (y_0, x_0)) = \alpha((1, 1), (1, 1), (1, 1)) = 1$.

Thus conditions (a), (b), (c) of Proposition 2.12 are satisfied.
We now verify the inequality (2.1). Let $x, u, z, y, v, w \in X$.
Case (i): If $x = y = u = v = w = z$ then we have $xy = uv = wz$. We consider
$\alpha((x, y), (u, v), (w, z))S(F(x, y), F(u, v), F(w, z)) = 1(S(xy, uv, wz)) = 0$.

Therefore the inequality (2.1) holds trivially.

<u>Case</u> (ii): If atleast two of x, y, u, v, w, z are not equal, in this case we consider

$$\begin{aligned}
&\alpha((x,y),(u,v),(w,z))S(F(x,y),F(u,v),F(w,z))\\
&\quad = \tfrac{1}{8}S(F(x,y),F(u,v),F(w,z)) = \tfrac{1}{8}S(xy,uv,wz) = \tfrac{1}{8}[\max\{xy,uv,wz\}]\\
&\quad \leq \tfrac{1}{8}[xy+uv+wz] \leq \tfrac{1}{8}x + \tfrac{1}{8}u + \tfrac{1}{8}w\\
&\quad \leq \tfrac{1}{8}[S(x,u,w)] + \tfrac{1}{8}[S(u,u,F(u,v))] + \tfrac{1}{8}[S(w,w,F(w,z))]\\
&\quad \leq \tfrac{a}{2}[S(x,u,w)+S(y,v,z)] + +\tfrac{b}{2}[S(x,x,F(x,y))+S(y,y,F(y,x))]\\
&\qquad +\tfrac{c}{2}[S(u,u,F(u,v))+S(v,v,F(v,u))]+\tfrac{d}{2}[S(w,w,F(w,z))+S(z,z,F(z,w))]
\end{aligned}$$

with $a = \frac{1}{4}$, $b = 0$, $c = \frac{1}{4}$, $d = \frac{1}{4}$.

Therefore the inequality (2.1) holds for $a = \frac{1}{4}$, $b = 0$, $c = \frac{1}{4}$, $d = \frac{1}{4}$.

Here we observe that Condition (U_1) fails to hold for any $(x,y),(u,v) \in X \times X$ with $x \neq u$ and $y \neq v$. Also, we observe that $a + 3d \nless 1$ and F has two strong coupled fixed points $(0,0)$ and $(1,1)$.

The following is an example in support of Theorem 2.14.

Example 3.6. Let $X = [0, \frac{1}{2}]$. We define $S : X^3 \to [0,+\infty)$ by

$$S(x,y,z) = \begin{cases} 0 & \text{if } x = y = z \\ \max\{x,y,z\} & \text{otherwise.} \end{cases}$$

Then (X,S) is a complete S-metric space. We define $F : X \times X \to X$ by $F(x,y) = \frac{x}{6(1+y)}$. We define $\alpha : X^2 \times X^2 \times X^2 \to [0,+\infty)$ by

$$\alpha((x,y),(u,v),(w,z)) = \begin{cases} 1 & \text{if } x = y = u = v \quad \text{and} \quad 0 \leq w,z \leq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Here F is α-admissible. For, assume that $\alpha((x,y),(u,v),(w,z)) \geq 1$. Then we have $x = y = u = v$ and $0 \leq w,z \leq \frac{1}{2}$. Now we have

$$\begin{aligned}
&\alpha((F(x,y),F(y,x)),(F(u,v),F(v,u)),(F(w,z),F(z,w)))\\
&\quad = \alpha((\tfrac{x}{6(1+y)},\tfrac{y}{6(1+x)}),(\tfrac{u}{6(1+v)},\tfrac{v}{6(1+u)}),(\tfrac{w}{6(1+z)},\tfrac{z}{6(1+w)}))\\
&\quad = 1(\text{since } \tfrac{x}{6(1+y)} = \tfrac{y}{6(1+x)} = \tfrac{u}{6(1+v)} = \tfrac{v}{6(1+u)} \text{ and } 0 \leq \tfrac{w}{6(1+z)},\tfrac{z}{6(1+w)} \leq \tfrac{1}{2}).
\end{aligned}$$

We now verify conditions (a), (b), (c) of Proposition 2.12 for $x_0 = 0$ and $y_0 = 0$. Here $F(0,0) = 0$. We have

(a) $\alpha((x_0,y_0),(x_0,y_0),(F(x_0,y_0),F(y_0,x_0))) = \alpha((0,0),(0,0),(0,0)) = 1$

(b) $\alpha((y_0,x_0),(y_0,x_0),(F(y_0,x_0),F(x_0,y_0))) = \alpha((0,0),(0,0),(0,0)) = 1$

(c) $\alpha((x_0,y_0),(x_0,y_0),(y_0,x_0)) = \alpha((0,0),(0,0),(0,0)) = 1$.

Thus conditions (a), (b), (c) of Proposition 2.12 are satisfied. We now verify the inequality (2.1). Let $x,u,z,y,v,w \in X$. If $x = y = u = v$ and $0 \leq w,z \leq \frac{1}{2}$ then we have

$S(F(x,y),F(u,v),F(w,z)) = \max\{\frac{x}{6(1+y)}, \frac{u}{6(1+v)}, \frac{w}{6(1+z)}\} = \max\{\frac{x}{6(1+y)}, \frac{w}{6(1+z)}\}$

Case (i): Suppose that $S(F(x,y),F(u,v),F(w,z)) = \frac{w}{6(1+z)}$. In this, case we consider

$$\alpha((x,y),(u,v),(w,z))S(F(x,y),F(u,v),F(w,z)) = \frac{w}{6(1+z)} \leq \frac{a}{2}[w+z] \leq \frac{a}{2}[S(x,u,w)+S(y,v,z)] \text{ with } a = \frac{1}{3}.$$

Case (ii): Suppose that $S(F(x,y),F(u,v),F(w,z)) = \frac{x}{6(1+y)}$. Then we have

$$\begin{aligned}\alpha((x,y),(u,v),(w,z))&S(F(x,y),F(u,v),F(w,z))\\ &= \frac{x}{6(1+y)} \leq \frac{1}{6}\max\{x,u,w\}\\ &\leq \frac{a}{2}[S(x,u,w)+S(y,v,z)] \text{ with } a = \frac{1}{3}.\end{aligned}$$

Therefore the inequality (2.1) holds with $a = \frac{1}{3}$, $b = c = d = 0$.
Now, for $(x,y),(u,v) \in X \times X$, $(u_1),(u_2),(u_3)$ and (u_4) of Condition (U_1) hold for any $(z,t) \in [0,\frac{1}{2}]$. Also, $a+3d<1$ and F has a unique strong coupled fixed point $(0,0)$.

Acknowledgement

The authors are very thankful to the referees for their valuable comments and suggestions.

References

[1] Ali Zafer A., and Ali L., Generalization of Reich's fixed point theorem, Int. Journal of Math. Analysis, 3 (25) (2009), 1239-1243.

[2] Ansari A. H., Işik H., Radenović S., Coupled fixed point theorems for contractive mappings involving new function classes and applications, Filomat, 31:7 (2017), 1893-1907.

[3] Babu G. V. R., Kumssa Leta Bekere, Fixed points of (α,ψ,φ)- generalized weakly contractive maps and property (P) in S-metric spaces, Filomat, 31:14 (2017), 4469-4481.

[4] Babu G. V. R., Sarma K. K. M., Kumari V. A. and Kumar P. Sudheer, Fixed point results of various cyclic contractions in metric spaces, Int. Jour. of Adv. in Math., 2018 (5) (2018), 1-13.

[5] Babu G. V. R., Sailaja P. D., Srichandana G., Common fixed points via C_k-class functions in S-metric spaces, J. Fixed Point Theory, 2020 (1) (2020), 22 pages.

[6] Babu G. V. R., Sailaja P. D., Srichandana G., Fixed points of almost generalized weakly contractive maps with rational expressions in S-metric spaces, Malaya Journal of Matematik, 8 (2) (2020), 593-601.

[7] Balasubramanian S., An extended Reich fixed point theorem, Math. FA, 9 (2014), 9 pages.

[8] Budhia L., Aydi H., Ansari A. H., Dhananjay Gopal, Some new fixed point results in rectangular metric spaces with an application to fractional-order functional differential equations, Nonlinear Analysis: Modelling and Control, 25 (4) (2020), 580-597.

[9] Choudhury B. S. and Maity P., Konar P., Fixed point results for coupling on metric spaces, U. P. B. Sci. Bull., series A, 79 (1) (2017), 77-88.

[10] Dhananjay G., Abbas M., Vetro C., Some new fixed point theorems in Menger PM-spaces with application to Volterra type integral equation, Applied Mathematics and Computation, 232 (2014), 955-967.

[11] Dhananjay G., Fixed points of alpha-type F-contractive mappings with an application to non- linear fractional differential equation, Acta Mathematica Scientia, 36 (3) (2016), 957-970.

[12] Došenović T., Radenović S., Sedghi S., Generalized metric spaces: Survey, TWMS. J. Pure Appl. Math., 9(1) (2018), 3-17.

[13] Dung N. V., On coupled common fixed points for mixed weakly monotone maps in partially ordered S-metric spaces, Fixed Point Theory and Appl., 2013 (48) (2013), 17 pages.

[14] Dung N. V., Hieu N. T. and Radojević S., Fixed point theorems for g-monotone maps on partially ordered S-metric spaces, Filomat, 28:9 (2014), 1885-1898.

[15] Dung N. V. and Radenović S., Remarks on theorems for cyclic quasi - contractions in uniformly convex Banach spaces, Krag. J. Math., 40 (2) (2016), 272-279.

[16] Gnana Bhaskar T. and Lakshmikantham V., Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal., 65 (2006), 1379-1393.

[17] Guo D., Lakshmikantham V., Coupled fixed points of nonlinear operators with applications, Nonlinear Anal., 11 (1987), 623-632.

[18] Gupta A., Cyclic contraction on S-metric space, Int. J. Anal. Appl., 3 (2) (2013), 119-130.

[19] Hardy G. E. and Rogers T. D., A generalization of a fixed point theorem of Reich, Canad. Math. Bull., 16 (2) (1973), 201-206.

[20] Huang H., Deng G., Došenović T., Hussain N., Note on recent common coupled fixed point results in multiplicative metric spaces, Applied Mathematics and Nonlinear Sciences, 3 (2) (2018), 659-668.

[21] Kirk W. A., Srinivasan P. S. and Veeramani P., Fixed points for mapping satisfying cyclical contractive conditions, Fixed Point Theory, 4(1) (2003), 79-89.

[22] Kumari P. S. and Panthi D., Connecting various types of cyclic contractions and contractive self-mapping with Hardy-Rogers self-mappings, Fixed Point Theory and Appl., 15 (2016), 19 pages.

[23] Lakshmikantham V. and Ćirić L., Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal., 70 (12) (2009), 4341-4349.

[24] Lakzian H., Dhananjay G. and Sintunavarat W., New fixed point results for mappings of contractive type with an application to nonlinear fractional differential equations, J. Fixed Point Theory and Appl., 18 (2) (2016), 251-266.

[25] Lazăr T., Moţ G., Petrušel G., and Szentesi S., The theory of Reich's fixed point theorem for multivalued operators, Fixed Point Theory and Appl., 2010 (2010), 10 pages.

[26] Nantadilok J., Coupled best proximity point theorems for $\alpha - \psi$-proximal contractive multimaps, Fixed Point Theory and Appl., 2015 (30) (2015), 14 pages.

[27] Pacurar M., and Rus I. A., Fixed point theory for cyclic φ-contractions, Nonlinear Anal., 72 (2010), 1181-1187.

[28] Parvaneh V., Hussain N., Hosseinzadeh H., and Salimi P., Coupled fixed point results for α-admissible Mizoguchi-Takahashi contractions in b-metric spaces with applications, Sahand Communications in Mathematical Analysis (SCMA), 7 (1) (2017), 85-104.

[29] Radenović S., A note on fixed point theory for cyclic weaker Meir-Keeler function in complete metric spaces, Int. J. Anal. Appl., 7 (1) (2015), 16-21.

[30] Radenović S., Chandok S., Shatanawi W., Some cyclic fixed point results for contractive mappings, University Though, Publication in Nature Sciences, 6 (2) (2016), pp.doi:10.5937/univtho6-11813.

[31] Radenović S., Some remarks on mappings satisfying cyclical contractive conditions, Afrika Matematika, 27 (1-2) (2016), 291-295.

[32] Radenović S., Classical fixed point results in 0-complete partial metric spaces via cyclic-type extension, The Allahabad Mathematical Society, 31 (1) (2016), 39-55.

[33] Rangamma M. and Mallikarjun Reddy P., A fixed point theorem for Reich contractions in generalized Cone b-metric spaces, International Journal of Mathematical Analysis, 10 (6) (2016), 271-278.

[34] Reich S., Fixed points of contractive functions, Bull. Univ. Mat. Italiana,

[35] Rus A., Cyclic representations and fixed points, Ann. T. Popoviciu Seminar Funct. Eq. Approx. Convexity, 3 (2005), 171-178.

[36] Samet B., Vetro C., and Vetro P., Fixed point theorem for (α, ψ)-contractive type mappings, Nonlinear Anal. Theory, Methods and Applications, 75 (4) (2012), 2154-2165.

[37] Sedghi S., Shobe N. and Alouche A., A generalization of fixed point theorem in S-metric spaces, Mat. Vesnik, 64 (3) (2012), 258-266.

[38] Sedghi S. and Dung N. V., Fixed point theorems on S-metric spaces, Mat. Vesnik, 66 (1) (2014), 113-124.

[39] Zhou M. and Liu X. L., On coupled common fixed point theorems for Geraghty-type contraction mappings using mixed weakly monotone property in partially ordered S-metric spaces, J. Funct. Spaces, 2016 (2016), 9 pages.

[40] Zoran K., Radenović S., A note on Pata-type cyclic contractions, Sarajevo Journal of Mathematics, 11 (24) (2) (2015), 1-11.

[41] Zoran K., Nastasi A., Radenović S., Vetro P., Remarks on the paper Fixed point theorems for cyclic contractions in C*-algebra-valued b-metric spaces, Adv. Oper. Theory, 1 (1) (2016), 93-104.