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CERTAIN PROPERTIES OF JACOBI'S THETA FUNCTIONS

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Abstract: In this paper, certain properties of Jacobi's theta functions have been discussed.

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1. Introduction, Notations and Definitions

For complex variables a and q with $|q| < 1$, the $q-$ shifted factorials are given by

$$(a;q)_0 = 1,\ (a;q)_n = \prod_{r=0}^{n-1}(1 - aq^r),\ (a;q)_\infty = \lim_{n\to\infty}(a;q)_n$$

and for brevity, let

$$(a_1, a_2, ..., a_r; q)_n = (a_1;q)_n(a_2;q)_n...(a_r;q)_n.$$

Jacobi [5] in 1829 defined following four functions which are called Jacobi's theta functions.

$$\theta_1(z,q) = 2\sum_{n=0}^{\infty}(-1)^n q^{\left(n+\frac{1}{2}\right)^2}\sin(2n+1)z, \tag{1}$$

$$\theta_2(z,q) = 2\sum_{n=0}^{\infty} q^{\left(n+\frac{1}{2}\right)^2}\cos(2n+1)z, \tag{2}$$

$$\theta_3(z,q) = 1+2\sum_{n=1}^{\infty} q^{n^2}\cos 2nz, \tag{3}$$

and

$$\theta_4(z,q) = 1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nz. \tag{4}$$

It is saying that no chapter of special functions contains so beautiful results and identities as Jacobi's theta functions. In this paper, our main aim is to discuss the properties of these Jacobi's theta functions. Our attempt will also be to deduce certain identities from the results related to Jacobi's theta functions.
We shall make use of following result in our analysis.
Jacobi's triple product identity [4, App. II (II.28)],

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} z^n = (q^2;q^2)_\infty \left(zq, \frac{q}{z}; q^2\right)_\infty \tag{5}$$

For periodic properties of Jacobi's theta-functions one is referred the chapter 20 of Raniville E. D. book [6] "Special Functions", published in 1960 by Macmillan Press, New York.
By simple manipulation, we can represent these functions as,

$$\theta_1(z,q) = -iq^{\frac{1}{4}}e^{iz}\sum_{n=-\infty}^{\infty} q^{n^2+n}e^{2nzi}. \tag{6}$$

By an appeal of Jacobi's triple product identity (5), (6) yields

$$\theta_1(z,q) = 2q^{\frac{1}{4}}\sin z(q^2;q^2)_\infty \prod_{r=1}^{\infty}\left(1-2q^{2r}\cos 2z+q^{4r}\right). \tag{7}$$

Proceeding as above, we have

$$\theta_2(z,q) = 2q^{\frac{1}{4}}\cos z(q^2;q^2)_\infty \prod_{r=1}^{\infty}\left(1+2q^{2r}\cos 2z+q^{4r}\right), \tag{8}$$

$$\theta_3(z,q) = (q^2;q^2)_\infty \prod_{r=0}^{\infty} \left(1 + 2q^{2r+1}\cos 2z + q^{4r+2}\right), \tag{9}$$

and

$$\theta_4(z,q) = (q^2;q^2)_\infty \prod_{r=0}^{\infty} \left(1 - 2q^{2r+1}\cos 2z + q^{4r+2}\right). \tag{10}$$

Here, we find two different represents of Jacobi's theta functions.
(i) Series representation as given in (1)-(4),
(ii) Product representations as given in (7)-(10).
Following Andrews [1, 2], and Denis [3], partial Jacobi's theta functions are represented as,

$$\theta_{1,N}(z,q) = 2q^{\frac{1}{4}} \sin z(q^2;q^2)_\infty \prod_{r=1}^{N} \left(1 - 2q^{2r}\cos 2z + q^{4r}\right), \tag{11}$$

$$\theta_{2,N}(z,q) = 2q^{\frac{1}{4}} \cos z(q^2;q^2)_\infty \prod_{r=1}^{N} \left(1 + 2q^{2r}\cos 2z + q^{4r}\right), \tag{12}$$

$$\theta_{3,N}(z,q) = (q^2;q^2)_\infty \prod_{r=0}^{N} \left(1 + 2q^{2r+1}\cos 2z + q^{4r+2}\right), \tag{13}$$

and

$$\theta_{4,N}(z,q) = (q^2;q^2)_\infty \prod_{r=0}^{N} \left(1 - 2q^{2r+1}\cos 2z + q^{4r+2}\right). \tag{14}$$

Putting $z = 0$ in (1)-(4) and also in (7)-(10) and equating the corresponding results we get,

$$\theta_1(0,q) = \theta_1(q) = 0$$

$$\theta_2(0,q) = \theta_2(q) = 2q^{\frac{1}{4}} \sum_{n=0}^{\infty} q^{n^2+n} = 2q^{\frac{1}{4}}(q^2;q^2)_\infty(-q^2;q^2)_\infty^2. \tag{15}$$

From (15) we have

$$\sum_{n=0}^{\infty} q^{n^2+n} = (q^2;q^2)_\infty(-q^2;q^2)_\infty^2.$$

Similarly, we have

$$\theta_3(q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = (q^2;q^2)_\infty(-q;q^2)_\infty^2, \tag{16}$$

and

$$\theta_4(q) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} = (q^2;q^2)_\infty(q;q^2)_\infty^2 \tag{17}$$

2. Identities deduced from Jacobi's θ-function

In this section, we establish certain identities by using derivatives of θ-functions. Differentiating $\theta_1(z,q)$ given in (1) we get,

$$\theta_1'(z,q) = 2q^{\frac{1}{4}}\sum_{n=0}^{\infty}(-1)^n q^{n^2+n}(2n+1)\cos(2n+1)z. \tag{18}$$

Differentiating $\theta_1(z,q)$ given in (7) we find,

$$\theta_1'(z,q) = 2q^{\frac{1}{4}}(q^2;q^2)_\infty \prod_{r=1}^{\infty}(1-2q^{2r}\cos 2z + q^{4r})$$

$$\times \left\{\cos z + 4\sin z \sin 2z \sum_{r=1}^{\infty}\frac{q^{2r}}{(1-2q^{2r}\cos 2z + q^{4r})}\right\}. \tag{19}$$

Putting $z=0$ in (18) and (19) and equating these two values, we obtain,

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n^2+n} = (q^2;q^2)_\infty^3, \tag{20}$$

which is well known Jacobi's identity.
Differentiating (18) with respect to z we have

$$\theta_1''(z,q) = -2q^{\frac{1}{4}}\sum_{n=0}^{\infty}(-1)^n(2n+1)^2 q^{n^2+n}\sin(2n+1)z, \tag{21}$$

which for $z=\frac{\pi}{2}$ yields,

$$\theta_1''\left(\frac{\pi}{2},q\right) = -2q^{\frac{1}{4}}\sum_{n=0}^{\infty}(2n+1)^2 q^{n^2+n}. \tag{22}$$

Differentiating (19) with respect to z we get,

$$\theta_1''(z,q) = \left\{ 2q^{\frac{1}{4}}(q^2;q^2)_\infty \prod_{r=1}^{\infty}(1-2q^{2r}\cos 2z + q^{4r}) \right\} \times$$
$$\times \left\{ -\sin z + 2\cos z . \sin 2z \sum_{r=1}^{\infty} \frac{4q^{2r}}{(1-2q^{2r}\cos 2z + q^{4r})} \right.$$
$$\left. + \sin z \sum_{r=1}^{\infty} \frac{8q^{2r}(1+q^{4r})\cos 2z - 16q^{4r}}{(1-2q^{2r}\cos 2z + q^{4r})^2} \right\}. \tag{23}$$

Putting $z = \frac{\pi}{2}$ in (23) we get,

$$\theta_1''\left(\frac{\pi}{2}, q\right) = -2q^{\frac{1}{4}}(q^2;q^2)_\infty(-q^2;q^2)_\infty - 16q^{\frac{1}{4}}(q^2;q^2)_\infty(-q^2;q^2)_\infty^2 \sum_{r=1}^{\infty} \frac{q^{2r}}{(1+q^{2r})^2}. \tag{24}$$

Equating (22) and (24) we have

$$\sum_{n=0}^{\infty}(2n+1)^2 q^{n^2+n} = (q^2;q^2)_\infty(-q^2;q^2)_\infty^2 \left\{ 1 + 8\sum_{r=1}^{\infty} \frac{q^{2r}}{(1+q^{2r})^2} \right\}. \tag{25}$$

Again, let us differentiate (3) with respect to z, thus we have,

$$\theta_3'(z,q) = -4\sum_{n=1}^{\infty} n q^{n^2} \sin 2nz. \tag{26}$$

Differentiating it again, we find

$$\theta_3''(z,q) = -8\sum_{n=1}^{\infty} n^2 q^{n^2} \cos 2nz. \tag{27}$$

Also, differentiating (9) twice we get

$$\theta_3''(z,q) = \left\{ -8(q^2;q^2)_\infty \prod_{r=0}^{\infty}(1+2q^{2r+1}\cos 2z + q^{4r+2}) \right\}$$
$$\times \left\{ \sum_{r=0}^{\infty} \frac{q^{2r+1}\cos 2z}{1+2q^{2r+1}\cos 2z + q^{4r+2}} + \sum_{r=0}^{\infty} \frac{2\sin^2 2z q^{4r+2}}{(1+2q^{2r+1}\cos 2z + q^{4r+2})^2} \right.$$
$$\left. -2\sin^2 2z \left(\sum_{r=0}^{\infty} \frac{q^{2r+1}}{(1+2q^{2r+1}\cos 2z + q^{4r+2})} \right) \right\}. \tag{28}$$

Putting $z=0$ in (27) and (28) and equating the values we have

$$\sum_{n=1}^{\infty} n^2 q^{n^2} = (q^2;q^2)_\infty(-q;q^2)_\infty^2 \sum_{r=0}^{\infty} \frac{q^{2r+1}}{(1+q^{2r+1})^2}. \tag{29}$$

3. Certain Properties of Theta Functions studied by Ramanujan

Another aspect of the $\theta-$ functions studied by Ramanujan is the expansion of theta functions in terms of partial theta functions in the form [2; page 174],

$$\sum_{N=0}^{\infty} T_N(i,j;q;q_1)\theta_{i,N}(z,q) = \theta_j(z,q_1). \tag{30}$$

Now, question is, if such expression exists, what is the form of these coefficients $T_N(i,j;q;q_1)$? Ramanujan's wonderful observation was that in a number of cases these coefficients have very elegant closed form, such as

$$T_n(4,4;q;q^3) = \frac{q^{2n^2}}{(q^2;q^2)_{2n}} \tag{31}$$

and

$$T_n(4,4;q;q^2) = \frac{q^{n^2}(-q;q^2)_\infty}{(q;q^2)_n(q^4;q^4)_n}. \tag{32}$$

From these, Ramanujan derived a large number of $q-$series summations. In order to prove (31) and (32), Andrews [2; (2.4) page 176] established following lemma

$$\sum_{n=0}^{\infty} \frac{(-aq,-\frac{q}{a};q^2)_n(b,c;q^2)_n\left(\frac{q^2}{bc}\right)^n}{(q^2;q^2)_{2n}} = \frac{\left(\frac{q^2}{b},\frac{q^2}{c};q^2\right)_\infty}{\left(q^2,\frac{q^2}{bc};q^2\right)_\infty} \sum_{N=-\infty}^{\infty} \frac{q^{N^2+2N}\left(\frac{a}{bc}\right)^N(b,c;q^2)_N}{\left(\frac{q^2}{b},\frac{q^2}{c};q^2\right)_N} \tag{33}$$

Taking $a=-e^{2iz}$ and $b,c\to\infty$ in (33) we get (31). Again, taking $a=-e^{2iz}$, $b=-q$ and $c\to\infty$ in (33) we get (32).

In order to prove Ramanujan's results stated in the 'Lost' notebook for false theta functions, Andrews [2] established another lemma,

$$\sum_{n=0}^{\infty} \frac{(-a;q)_{n+1}\left(-\frac{q}{a};q\right)_n(b,c;q)_n\left(\frac{q^2}{bc}\right)^n}{(q;q)_{2n+1}}$$
$$= \frac{\left(\frac{q^2}{b},\frac{q^2}{c};q\right)_\infty}{\left(q,\frac{q^2}{bc};q\right)_\infty} \sum_{n=0}^{\infty} \frac{(a^{n+1}+a^{-n})q^{n(n+1)/2}(b,c;q)_n\left(\frac{q^2}{bc}\right)^n}{\left(\frac{q^2}{b},\frac{q^2}{c};q\right)_n} \tag{34}$$

Taking $a = e^{2zi}$, $b, c \to \infty$ in (34) we have

$$(1+e^{2iz})\sum_{n=0}^{\infty}\frac{(-e^{2iz}q,-e^{-2iz}q;q)_n q^{n^2+n}}{(q;q)_{2n+1}} = \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(e^{2iz(n+1)}+e^{-2izn})q^{\frac{3}{2}n(n+1)}. \quad (35)$$

(35) can be expressed as,

$$\sum_{n=0}^{\infty}\theta_{2,n}(z,q^{\frac{1}{2}})\frac{q^{(n+\frac{1}{2})^2}}{(q;q)_{2n+1}} = \theta_2(z,q^{\frac{3}{2}}). \quad (36)$$

Thus we have from (36)

$$T_n(2,2;q^{\frac{1}{2}},q^{\frac{3}{2}}) = \frac{q^{(n+\frac{1}{2})^2}}{(q;q)_{2n+1}}. \quad (37)$$

Taking $a = -e^{2iz}$, $b, c \to \infty$ in (34) and proceeding as above, we get

$$T_n(1,1;q^{\frac{1}{2}},q^{\frac{3}{2}}) = \frac{q^{(n+\frac{1}{2})^2}}{(q;q)_{2n+1}}. \quad (38)$$

4. Lemma

In this section we establish following two lemmas.

Lemma 1.

$$(1+a)\sum_{n=0}^{\infty}\frac{\left(-aq^\mu,-\frac{q^\mu}{a};q^\mu\right)_n (b,c;q^\mu)_n \left(\frac{q^{2\mu}}{bc}\right)^n}{(q^\mu;q^\mu)_{2n+1}}$$
$$= \frac{a\left(\frac{q^{2\mu}}{b},\frac{q^{2\mu}}{c};q^\mu\right)_\infty}{(q^\mu,q^\mu)_\infty\left(\frac{q^{2\mu}}{bc};q^\mu\right)_\infty}\sum_{r=-\infty}^{\infty}\frac{(b,c;q^\mu)_r\left(\frac{aq^{2\mu}}{bc}\right)^r q^{\mu r(r+1)/2}}{\left(\frac{q^{2\mu}}{b},\frac{q^{2\mu}}{c};q^\mu\right)_r}. \quad (39)$$

As, $b, c \to \infty$, (39) yields

$$\sum_{n=0}^{\infty}\frac{\left(-aq^\mu,\frac{-q^\mu}{a};q^\mu\right)_n q^{\mu n(n+1)}}{(q^\mu;q^\mu)_{2n+1}} = \frac{1}{(q^\mu,q^\mu)_\infty}\left(q^{3\mu},-aq^{3\mu},-\frac{q^{3\mu}}{a};q^{3\mu}\right)_\infty. \quad (40)$$

Lemma 2.

$$\sum_{n=0}^{\infty}\frac{\left(-aq^\mu,-\frac{q^\mu}{a};q^{2\mu}\right)_n (b,c;q^{2\mu})_n \left(\frac{q^{2\mu}}{bc}\right)^n}{(q^{2\mu};q^{2\mu})_{2n}}$$
$$= \frac{\left(\frac{q^{2\mu}}{b},\frac{q^{2\mu}}{c};q^{2\mu}\right)_\infty}{\left(q^{2\mu},\frac{q^{2\mu}}{bc};q^{2\mu}\right)_\infty}\sum_{n=-\infty}^{\infty}\frac{(b,c;q^{2\mu})_n\left(\frac{a}{bc}\right)^n q^{n^2+2n}}{\left(\frac{q^{2\mu}}{b},\frac{q^{2\mu}}{c};q^{2\mu}\right)_n}. \quad (41)$$

As, $b, c \to \infty$, (41) yields

$$\sum_{n=0}^{\infty} \frac{\left(-aq^{\mu}, \frac{-q^{\mu}}{a}; q^{2\mu}\right)_n q^{2\mu n^2}}{(q^{2\mu}; q^{2\mu})_{2n}} = \frac{1}{(q^{2\mu}, q^{2\mu})_{\infty}} \left(q^{6\mu}, -aq^{3\mu}, -\frac{q^{3\mu}}{a}; q^{6\mu}\right)_{\infty}. \quad (42)$$

Proof of Lemma 1.

$$(1+a)\sum_{n=0}^{\infty} \frac{\left(-aq^{\mu}, -\frac{q^{\mu}}{a}; q^{\mu}\right)_n (b, c; q^{\mu})_n \left(\frac{q^{2\mu}}{bc}\right)^n}{(q^{\mu}; q^{\mu})_{2n+1}}$$
$$= \sum_{n=0}^{\infty} \frac{(b, c; q^{\mu})_n \left(\frac{aq^{2\mu}}{bc}\right)^n (-aq^{-\mu n}; q^{\mu})_{2n+1} a^{-n} q^{\mu n(n+1)/2}}{(q^{\mu}; q^{\mu})_{2n+1}}.$$

Now, making use of the following result,

$$(z; q)_n = \sum_{r=0}^{n} \frac{(q; q)_n (-1)^n z^r q^{r(r-1)/2}}{(q; q)_r (q; q)_{n-r}}$$

we have

$$\sum_{n=0}^{\infty} \frac{(b, c; q^{\mu})_n \left(\frac{q^{2\mu}}{bc}\right)^n a^{-n} q^{\mu n(n+1)/2}}{(q^{\mu}; q^{\mu})_{2n+1}} \sum_{r=0}^{2n+1} \frac{(q^{\mu}; q^{\mu})_{2n+1} a^r q^{-\mu n r} q^{\mu r(r-1)/2}}{(q^{\mu}, q^{\mu})_r (q^{\mu}; q^{\mu})_{2n+1-r}}$$
$$= \sum_{n=0}^{\infty} \sum_{r=0}^{2n+1} \frac{(b, c; q^{\mu})_n \left(\frac{q^{2\mu}}{bc}\right)^n a^{r-n} q^{\frac{\mu}{2}(r-n)^2 - \frac{\mu}{2}(r-n)}}{(q^{\mu}; q^{\mu})_r (q^{\mu}; q^{\mu})_{2n+1-r}}$$

Putting $r+n+1$ for r we get,

$$= \sum_{n=0}^{\infty} \sum_{r=-n-1}^{n} \frac{(b, c; q^{\mu})_n \left(\frac{q^{2\mu}}{bc}\right)^n a^{r+1} q^{\mu r(r+1)/2}}{(q^{\mu}; q^{\mu})_{r+n+1} (q^{\mu}; q^{\mu})_{n-r}}$$
$$= \sum_{n=0}^{\infty} \sum_{r=-\infty}^{\infty} \frac{(b, c; q^{\mu})_{n+r} \left(\frac{q^{2\mu}}{bc}\right)^{n+r} a^{r+1} q^{\mu r(r+1)/2}}{(q^{\mu}; q^{\mu})_{n+2r+1} (q^{\mu}; q^{\mu})_n}$$
$$= a \sum_{r=-\infty}^{\infty} \frac{(b, c; q^{\mu})_r \left(\frac{q^{2\mu}}{bc}\right)^r a^r q^{\mu r(r+1)/2}}{(q^{\mu}; q^{\mu})_{2r+1}} \; {}_2\Phi_1 \left[\begin{matrix} bq^{\mu r}, cq^{\mu r}; q^{\mu}; \frac{q^{2\mu}}{bc} \\ q^{2\mu+2\mu r} \end{matrix} \right],$$

summing the inner ${}_2\Phi_1$ series by using [4; App.II (II.8)] we get right hand side of (39) after some simplifications.

Proof of Lemma 2.

Left hand side of lemma 2 is

$$\sum_{n=0}^{\infty}\frac{\left(-aq^{\mu},-\frac{q^{\mu}}{a};q^{2\mu}\right)_n(b,c;q^{2\mu})_n\left(\frac{q^{2\mu}}{bc}\right)^n}{(q^{2\mu};q^{2\mu})_{2n}}$$

$$=\sum_{n=0}^{\infty}\frac{(b,c;q^{2\mu})_n\left(\frac{q^{2\mu}}{bc}\right)^n(-aq^{-2\mu n+\mu};q^{2\mu})_{2n}a^{-n}q^{\mu n^2}}{(q^{2\mu};q^{2\mu})_{2n}}.$$

Now, making use of the following result,

$$(z;q)_n=\sum_{r=0}^{n}\frac{(q;q)_n(-z)^rq^{r(r-1)/2}}{(q;q)_r(q;q)_{n-r}}$$

we get

$$\sum_{n=0}^{\infty}\frac{(b,c;q^{2\mu})_n\left(\frac{q^{2\mu}}{bc}\right)^n a^{-n}q^{\mu n^2}}{(q^{2\mu};q^{2\mu})_{2n}}\sum_{r=0}^{2n}\frac{(q^{2\mu};q^{2\mu})_{2n}a^rq^{-2\mu nr}q^{\mu r^2}}{(q^{2\mu},q^{2\mu})_r(q^{2\mu};q^{2\mu})_{2n-r}}$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{2n}\frac{(b,c;q^{2\mu})_n\left(\frac{q^{2\mu}}{bc}\right)^n a^{r-n}q^{\mu(r-n)^2}}{(q^{2\mu};q^{2\mu})_r(q^{2\mu};q^{2\mu})_{2n-r}}$$

Putting $r+n$ for r we have,

$$=\sum_{n=0}^{\infty}\sum_{r=-n}^{n}\frac{(b,c;q^{2\mu})_n\left(\frac{q^{2\mu}}{bc}\right)^n a^rq^{\mu r^2}}{(q^{2\mu};q^{2\mu})_{r+n}(q^{2\mu};q^{2\mu})_{n-r}}=\sum_{n=0}^{\infty}\sum_{r=-\infty}^{\infty}\frac{(b,c;q^{2\mu})_{n+r}\left(\frac{q^{2\mu}}{bc}\right)^{n+r}a^rq^{\mu r^2}}{(q^{2\mu};q^{2\mu})_{n+2r}(q^{2\mu};q^{2\mu})_n}$$

$$=\sum_{r=-\infty}^{\infty}\frac{(b,c;q^{2\mu})_r\left(\frac{q^{2\mu}}{bc}\right)^r a^rq^{\mu r^2}}{(q^{2\mu};q^{2\mu})_{2r}}\ {}_2\Phi_1\left[\begin{array}{l} bq^{2\mu r},cq^{2\mu r};q^{2\mu};\frac{q^{2\mu}}{bc}\\ q^{2\mu+4\mu r}\end{array}\right],$$

summing the inner ${}_2\Phi_1$ series by using [4; App.II (II.8)] we get right hand side of (41) after some simplifications.

For $\mu=1$, (41) reduces into (33) and (39) reduces into (34).

Taking $a=1$ in (40) we get,

$$\sum_{n=0}^{\infty}\frac{(-q^{\mu};q^{\mu})_n^2q^{\mu n(n+1)}}{(q^{\mu};q^{\mu})_{2n+1}}=\frac{(q^{3\mu};q^{3\mu})_{\infty}(-q^{3\mu};q^{3\mu})_{\infty}^2}{(q^{\mu};q^{\mu})_{\infty}}. \tag{43}$$

For $a=-1$, (40) yields

$$\sum_{n=0}^{\infty}\frac{(q^{\mu};q^{\mu})_n^2 q^{\mu n(n+1)}}{(q^{\mu};q^{\mu})_{2n+1}}=\frac{(q^{3\mu};q^{3\mu})_{\infty}^3}{(q^{\mu};q^{\mu})_{\infty}}. \tag{44}$$

Putting $a=1$, in (42) yields

$$\sum_{n=0}^{\infty}\frac{(-q^{\mu};q^{2\mu})_n^2 q^{\mu n^2}}{(q^{2\mu};q^{2\mu})_{2n}}=\frac{(q^{6\mu};q^{6\mu})_{\infty}(-q^{3\mu};q^{6\mu})_{\infty}^2}{(q^{2\mu};q^{2\mu})_{\infty}}. \tag{45}$$

Also, for $a=-1$, (42) yields

$$\sum_{n=0}^{\infty}\frac{(q^{\mu};q^{2\mu})_n^2 q^{2\mu n^2}}{(q^{2\mu};q^{2\mu})_{2n}}=\frac{(q^{6\mu};q^{6\mu})_{\infty}(q^{3\mu};q^{6\mu})_{\infty}^2}{(q^{2\mu};q^{2\mu})_{\infty}}. \tag{46}$$

Putting $a=-e^{2iz}$ in (40) we get,

$$\sum_{n=0}^{\infty}\frac{(e^{2iz}q^{\mu},e^{-2iz}q^{\mu};q^{\mu})_n q^{\mu n(n+1)}}{(q^{\mu};q^{\mu})_{2n+1}}$$

$$=\frac{1}{(q^{\mu};q^{\mu})_{\infty}}(q^{3\mu};q^{3\mu})_{\infty}(e^{2iz}q^{3\mu},e^{-2iz}q^{3\mu};q^{3\mu})_{\infty}. \tag{47}$$

(47) can be expressed as

$$(q^{\mu};q^{\mu})_{\infty}\sum_{n=0}^{\infty}\left\{\prod_{r=1}^{n}(1-2\cos 2z q^{\mu r}+q^{2\mu r})\right\}\frac{q^{\mu n(n+1)}}{(q^{\mu};q^{\mu})_{2n+1}}$$

$$=(q^{3\mu};q^{3\mu})_{\infty}\prod_{r=1}^{\infty}(1-2\cos 2z q^{3\mu r}+q^{6\mu r}). \tag{48}$$

Multiplying both sides of (48) by $2q^{3\mu/8}\sin z$ we have

$$\sum_{n=0}^{\infty}\theta_{1,n}(z,q^{\mu/2})\frac{q^{\mu\left(n+\frac{1}{2}\right)^2}}{(q^{\mu};q^{\mu})_{2n+1}}=\theta_1(z,q^{3\mu/2}). \tag{49}$$

From (49) we have,

$$T_n(1,1;q^{\mu/2},q^{3\mu/2})=\frac{q^{\mu\left(n+\frac{1}{2}\right)^2}}{(q^{\mu};q^{\mu})_{2n+1}}. \tag{50}$$

Putting $a = e^{2iz}$ in (40) we get,

$$(q^\mu; q^\mu)_\infty \sum_{n=0}^{\infty} \left\{ \prod_{r=1}^{n} (1 + 2\cos 2z q^{\mu r} + q^{2\mu r}) \right\} \frac{q^{\mu n(n+1)}}{(q^\mu; q^\mu)_{2n+1}}$$

$$= (q^{3\mu}; q^{3\mu})_\infty \prod_{r=1}^{\infty} (1 + 2\cos 2z q^{3\mu r} + q^{6\mu r}). \tag{51}$$

Multiplying both sides of (51) by $2q^{3\mu/8}\cos z$ we have

$$\sum_{n=0}^{\infty} \theta_{2,n}(z, q^{\mu/2}) \frac{q^{\mu\left(n+\frac{1}{2}\right)^2}}{(q^\mu; q^\mu)_{2n+1}} = \theta_2(z, q^{3\mu/2}). \tag{52}$$

From (52) we have

$$T_n(2, 2; q^{\mu/2}, q^{3\mu/2}) = \frac{q^{\mu\left(n+\frac{1}{2}\right)^2}}{(q^\mu; q^\mu)_{2n+1}}. \tag{53}$$

Differentiating both sides of (48) with respect to z we find

$$(q^\mu; q^\mu)_\infty \sum_{n=0}^{\infty} \left\{ \prod_{r=1}^{n} (1 - 2\cos 2z q^{\mu r} + q^{2\mu r}) \right\}$$

$$\left\{ \sum_{r=1}^{n} \frac{q^{\mu r}}{(1 - 2\cos 2z q^{\mu r} + q^{2\mu r})} \right\} \frac{q^{\mu n(n+1)}}{(q^\mu; q^\mu)_{2n+1}}$$

$$= (q^{3\mu}; q^{3\mu})_\infty \left\{ \sum_{r=1}^{\infty} \frac{q^{3\mu r}}{(1 - 2\cos 2z q^{\mu r} + q^{6\mu r})} \right\} \left\{ \prod_{r=1}^{\infty} (1 - 2\cos 2z q^{3\mu r} + q^{6\mu r}) \right\} \tag{54}$$

Putting $z = 0$ in (54) we have

$$(q^\mu; q^\mu)_\infty \sum_{n=0}^{\infty} (q^\mu; q^\mu)_n^2 \left\{ \sum_{r=1}^{n} \frac{q^{\mu r}}{(1 - q^{\mu r})^2} \right\} \frac{q^{\mu n(n+1)}}{(q^\mu; q^\mu)_{2n+1}}$$

$$= (q^{3\mu}; q^{3\mu})_\infty^3 \sum_{r=1}^{\infty} \frac{q^{3\mu r}}{(1 - q^{3\mu r})^2}. \tag{55}$$

Putting $z = \pi/4$ in (54) we obtain

$$(q^\mu; q^\mu)_\infty \sum_{n=0}^{\infty} \left\{ \sum_{r=1}^{n} \frac{q^{\mu r}}{(1 + q^{2\mu r})} \right\} (-q^{2\mu}; q^{2\mu})_n \frac{q^{\mu n(n+1)}}{(q^\mu; q^\mu)_{2n+1}}$$

$$= (q^{3\mu}; q^{3\mu})_\infty (-q^{6\mu}; q^{6\mu})_\infty \sum_{r=1}^{\infty} \frac{q^{3\mu r}}{(1 + q^{6\mu r})}. \tag{56}$$

Differentiating both sides of (51) with respect to z we have

$$(q^{\mu};q^{\mu})_{\infty}\sum_{n=0}^{\infty}\left\{\prod_{r=1}^{n}(1+2\cos 2zq^{\mu r}+q^{2\mu r})\right\}$$
$$\left\{\sum_{r=1}^{n}\frac{q^{\mu r}}{(1+2\cos 2zq^{\mu r}+q^{2\mu r})}\right\}\frac{q^{\mu n(n+1)}}{(q^{\mu};q^{\mu})_{2n+1}}$$
$$=(q^{3\mu};q^{3\mu})_{\infty}\left\{\sum_{r=1}^{n}\frac{q^{3\mu r}}{(1+2\cos 2zq^{3\mu r}+q^{6\mu r})}\right\}\left\{\prod_{r=1}^{\infty}(1+2\cos 2zq^{3\mu r}+q^{6\mu r})\right\} \quad (57)$$

Putting $z=0$ in (57) we have

$$(q^{\mu};q^{\mu})_{\infty}\sum_{n=0}^{\infty}\sum_{r=1}^{n}(-q^{\mu};q^{\mu})_{n}^{2}\frac{q^{\mu n(n+1)}}{(q^{\mu};q^{\mu})_{2n+1}}\frac{q^{\mu r}}{(1+q^{\mu r})^{2}}$$
$$=(q^{3\mu};q^{3\mu})_{\infty}(-q^{3\mu};q^{3\mu})_{\infty}^{2}\sum_{r=1}^{\infty}\frac{q^{3\mu r}}{(1+q^{3\mu r})^{2}}. \quad (58)$$

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