South East Asian J. of Mathematics and Mathematical Sciences
Vol. 17, No. 3 (2021), pp. 101-108

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

ON RADIUS PROBLEMS FOR SOME SUBCLASSES OF ANALYTIC UNIVALENT FUNCTIONS

U. Vadivelan, S. Sunil Varma* and Thomas Rosy*

Department of Mathematics,
Nazareth College of Arts and Science,
Chennai – 600062, Tamil Nadu, INDIA

E-mail : vadivelanu@gmail.com

*Department of Mathematics,
Madras Christian College, Tambaram,
Chennai - 600059, Tamil Nadu, INDIA

E-mail : sunu_79@yahoo.com, thomas.rosy@gmail.com

(**Received:** *Jan. 09, 2021* **Accepted:** *Oct. 31, 2021* **Published:** *Dec. 30, 2021)*

Abstract: In this article we compute the radii of the largest disks for which the functions in the class $\mathcal{S}$ of normalized, analytic and univalent functions belong to certain subclasses of it.

Keywords and Phrases: Analytic function, univalent function, radius problem, polynomial equations.

2020 Mathematics Subject Classification: 30C45.

1. Introduction

Let $\mathcal{A}$ be the class of normalised analytic functions f defined on the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ with Taylor's series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

and $\mathcal{S}$ denote the subclass of it containing univalent functions [3].

The radius problem for subclasses of $\mathcal{A}$ or $\mathcal{S}$ is defined as follows: if $\mathcal{M}$ is a class of functions and $\mathcal{P}$ is a property which the functions in $\mathcal{M}$ may or may not possess in the disk $|z| < r$, the radius for the property $\mathcal{P}$ in the set $\mathcal{M}$ is the largest R such that every function in the set $\mathcal{M}$ has the property $\mathcal{P}$ in each disk $\{z \in \mathbb{C} : |z| < r\}$ for every $r < R$. In other words, if $\mathcal{F}$ and $\mathcal{G}$ are two given subsets of $\mathcal{A}$ then the $\mathcal{G}$ radius of f in $\mathcal{F}$ is the largest R such that for each $f \in \mathcal{F}, r^{-1}f(rz) \in \mathcal{G}$ for each $r \leq R$ [4].

Several authors had considered this problem for various pairs of subclasses of $\mathcal{S}$. Gavrilov [5] in 1970 showed that the radius of univalence of functions in the class $\mathcal{A}$ satisfying the inequality $|a_n| \leq n$ is the real root of the equation $2(1-r)^3-(1+r) = 0$ and the same for those functions satisfying $|a_n| \leq M$ is $1-\sqrt{M/(1+M)}$. In 1982, Yamashitha [13] showed that the radius of univalence obtained by Gavrilov is also the radii of starlikeness for the corresponding functions.The radius of starlikeness and convexity of functions in the class $\mathcal{S}$ were found to be $tanh(\pi/4)$ and $2-\sqrt{3}$ respectively [4]. The radius of univalence of certain combination of two analytic functions was obtained by [2, 7, 8].

In this paper we obtain few radii results for functions in the class $\mathcal{S}$ of analytic, normalized univalent functions to be in certain standard subclasses of it defined and studied by various authors.

2. Preliminaries

We recall certain standard subclasses of $\mathcal{S}$ and the respective sufficient conditions for a function $f \in \mathcal{S}$ to be in these subclasses.

Definition 2.1. [12] *A function $f \in \mathcal{S}$ is said to be in the class $\mathcal{D}$ if*

$$\Re(f'(z)) > |zf''(z)|, \ for\ all\ z \in \Delta.$$

Theorem 2.1. [12] *A function $f \in \mathcal{D}$ if it satisfies the condition*

$$\Sigma_{n=2}^{\infty} n^2|a_n| \leq 1.$$

Definition 2.2. [10] *A function $f \in \mathcal{S}$ is said to be in the class $\mathcal{UCD}$ if*

$$\Re(f'(z)) > 2|zf''(z)|, \ for\ all\ z \in \Delta.$$

Theorem 2.2. [10] *Let $f \in \mathcal{S}$. If f satisfies*

$$\Sigma_{n=2}^{\infty} n(2n-1)|a_n| \leq 1$$

then $f \in \mathcal{UCD}$.

Definition 2.3. [6, 9] *A function $f \in \mathcal{S}$ is said to be in the class $\mathcal{UCD}(\alpha)$ if*

$$\Re(f'(z)) > \alpha|zf''(z)|, \; for\ all\ z \in \Delta, \alpha \geq 0.$$

Theorem 2.3. [10, 11] *Let $f \in \mathcal{S}$. If f satisfies*

$$\Sigma_{n=2}^{\infty} n[n\alpha + (1-\alpha)]|a_n| \leq 1$$

then $f \in \mathcal{UCD}(\alpha)$.

Definition 2.4. [10] *A function $f \in \mathcal{S}$ is said to be in the class $\mathcal{SD}$ if*

$$\Re\left(\frac{f(z)}{z}\right) > \left|f'(z) - \frac{f(z)}{z}\right|, \; for\ all\ z \in \Delta.$$

Theorem 2.4. [10] *Let $f \in \mathcal{S}$. If f satisfies*

$$\Sigma_{n=2}^{\infty} n|a_n| \leq 1$$

then $f \in \mathcal{SD}$.

Definition 2.5. [10] *A function $f \in \mathcal{S}$ is said to be in the class $\mathcal{SD}(\alpha)$ if*

$$\Re\left(\frac{f(z)}{z}\right) > \alpha\left|f'(z) - \frac{f(z)}{z}\right|, \; for\ all\ z \in \Delta, \alpha \geq 0.$$

Theorem 2.5. [10] *Let $f \in \mathcal{S}$. If f satisfies*

$$\Sigma_{n=2}^{\infty} [1 + \alpha(n-1)|a_n| \leq 1$$

then $f \in \mathcal{SD}(\alpha)$.

Definition 2.6. [10] *A function $f \in \mathcal{S}$ is said to be in the class $\mathcal{SD}(\alpha, \beta)$ if*

$$\Re\left(\frac{f(z)}{z}\right) > \beta\left|f'(z) - \frac{f(z)}{z}\right| + \alpha, \; for\ all\ z \in \Delta, \; 0 \leq \alpha < 1, \; 0 \leq \beta \leq 1.$$

Theorem 2.6. [10] *Let $f \in \mathcal{S}$. If f satisfies*

$$\Sigma_{n=2}^{\infty} [\beta n + (1-\beta)]|a_n| \leq 1 - \alpha$$

then $f \in \mathcal{SD}(\alpha, \beta)$.

Definition 2.7. [1] *A function $f \in \mathcal{S}$ is said to be in the class $\mathcal{SR}(\alpha, \beta)$ if*

$$\Re(f'(z)) > \beta|f'(z) - 1| + \alpha, \; for\ all\ z \in \Delta, \; 0 \leq \alpha < 1, \; 0 \leq \beta \leq 1.$$

Theorem 2.7. [1] *Let $f \in \mathcal{S}$. If f satisfies*

$$\Sigma_{n=2}^{\infty} n|a_n| \leq \frac{1-\alpha}{1+\beta}$$

then $f \in \mathcal{SD}(\alpha, \beta)$.

3. Main Results

We now obtain the radii results for functions in the class $\mathcal{S}$ to be in the above subclasses.

Theorem 3.1. *The $\mathcal{D}-$ radius of $f \in \mathcal{S}$ is the real root of the equation $r^4 + 8r^3 - 11r^2 + 12r - 1 = 0$ lying in $(0, 1)$.*

Proof. Let $f \in \mathcal{S}$ be given by (1). For $0 < r_0 < 1$,

$$\frac{1}{r_0} f(r_0 z) = z + \sum_{n=2}^{\infty} a_n r_0^{n-1} z^n.$$

Since $|a_n| \leq n,\ n \geq 2$,

$$\sum_{n=2}^{\infty} n^2 |a_n| r_0^{n-1} \leq \sum_{n=2}^{\infty} n^3 r_0^{n-1} = \frac{1 + 4r_0 + r_0^2}{(1-r_0)^4} - 1.$$

If $\frac{1}{r_0} f(r_0 z) \in \mathcal{D}$ then we must have

$$\frac{1 + 4r_0 + r_0^2}{(1-r_0)^4} - 1 = 1$$
$$\implies 2r_0^4 + 8r_0^3 - 11r_0^2 + 12r_0 - 1 = 0.$$

Thus r_0 is the root of the equation $2r^4 + 8r^3 - 11r^2 + 12r - 1 = 0$ lying in $(0, 1)$.

Theorem 3.2. *The $\mathcal{UCD}-$ radius of $f \in \mathcal{S}$ is the real root of equation $r^4 - 5r^3 + 3r^2 - 15r + 1 = 0$ lying in $(0, 1)$.*

Proof. Since $f \in \mathcal{S}$, we have $|a_n| \leq n,\ n \geq 2$ and hence

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(2n-1)|a_n| r_0^{n-1} \\
\leq &\sum_{n=2}^{\infty} n(2n-1) n r_0^{n-1} \\
= &2\sum_{n=2}^{\infty} n^3 r_0^{n-1} - \sum_{n=2}^{\infty} n^2 r_0^{n-1} \\
= &2\left[\frac{1 + 4r_0 + r_0^2}{(1-r_0)^4} - 1\right] - \left[\frac{1 + r_0}{(1-r_0)^3} - 1\right]
\end{aligned}$$

$$=\frac{2(1+4r_0+r_0^2)-(1-r_0^2)-(1-r_0)^3}{(1-r_0)^4}.$$

For $\frac{1}{r_0}f(r_0z)$ to be in the subclass $\mathcal{UCD}$, we must have

$$\frac{2(1+4r_0+r_0^2)-(1-r_0^2)-(1-r_0)^3}{(1-r_0)^4}=1$$
$$\implies 2(1+4r_0+r_0^2)-(1-r_0^2)-(1-r_0)^3=(1-r_0)^4$$
$$\implies r_0^4-r_0^3+3r_0^2-15r_0+1=0.$$

Thus r_0 is the root of the equation $r^4-5r^3+3r^2-15r+1=0$ lying in $(0,1)$.

Theorem 3.3. *The $\mathcal{SD}-$ radius of $f\in\mathcal{S}$ is the real root of equation $2r^3-6r^2+7r-1=0$ lying in $(0,1)$.*

Proof. Let $f\in\mathcal{S}$. Then $|a_n|\le n,\ n\ge 2$.

$$\sum_{n=2}^{\infty} n|a_n|r_0^{n-1}\le\sum_{n=2}^{\infty} n^2r_0^{n-1}=\frac{1+r_0}{(1-r_0)^3}-1.$$

Now, $\frac{1}{r_0}f(r_0z)\in\mathcal{SD}$ if

$$\frac{1+r_0}{(1-r_0)^3}-1=1$$
$$\implies 2r_0^3-6r_0^2+7r_0-1=0.$$

Hence the $\mathcal{SD}-$ radius of f is the root of the equation $2r^3-6r^2+7r-1=0$ lying in $(0,1)$.

Theorem 3.4. *Let $\alpha>0$. The $\mathcal{UCD}(\alpha)$-radius of $f\in\mathcal{S}$ is the real root of equation $2r^4-8r^3+11r^2-6(1+\alpha)r+1=0$ lying in $(0,1)$.*

Proof. Since $f\in\mathcal{S}$, we have $|a_n|\le n, n\ge 2$ and hence

$$\sum_{n=2}^{\infty} n[n\alpha+(1-\alpha)]|a_n|r_0^{n-1}$$
$$\le\sum_{n=2}^{\infty} n[n\alpha+(1-\alpha)]|a_n|r_0^{n-1}$$
$$=\alpha\left[\frac{1+4r_0+r_0^2}{(1-r_0)^4}-1\right]+(1-\alpha)\left[\frac{1}{(1-r_0)^2}-1\right]$$
$$=\alpha\frac{1+4r_0+r_0^2}{(1-r_0)^4}+\frac{1-\alpha}{(1-r_0)^2}-1.$$

If $\frac{1}{r_0}f(r_0z) \in \mathcal{UCD}(\alpha)$ then we must have

$$\alpha(1+4r_0+r_0^2)+(1-\alpha)(1-r_0)^2 = 2(1-r_0)^4$$
$$\implies 2r_0^4-8r_0^3+11r_0^2-6(1-\alpha)r_0+1=0.$$

Let $f(r)=2r^4-8r^3+11r^2-6(1-\alpha)r+1$ we have $f(0)=1$ and $f(1)=-6\alpha<0$. Then $f(0)f(1)<0$ if $\alpha>0$. The $\mathcal{UCD}(\alpha)$- radius of $f\in\mathcal{S}$ is the root of the equation $2r^4-8r^3+11r^2-6(1-\alpha)r+1=0$ in the interval $(0,1)$ whenever $\alpha>0$.

Theorem 3.5. *If $\alpha>0$, then $\mathcal{SD}(\alpha)-$ radius of $f\in\mathcal{S}$ is the real root of equation $2r^3-6r^2+(5+2\alpha)r-1=0$.*
Proof. Since $|a_n|\le n$ for $f\in\mathcal{S}$,

$$\begin{aligned}&\sum_{n=2}^{\infty}[1+\alpha(n-1)]|a_n|r_0^{n-1}\\ \le&\sum_{n=2}^{\infty}[1+\alpha(n-1)]nr_0^{n-1}\\ =&(1-\alpha)\sum_{n=2}^{\infty}nr_0^{n-1}+\alpha\sum_{n=2}^{\infty}n^2r_0^{n-1}\\ =&\frac{1-\alpha}{(1-r_0)^2}+\frac{\alpha(1+r_0)}{(1-r_0)^3}-1.\end{aligned}$$

If $\frac{1}{r_0}f(r_0z)\in\mathcal{SD}(\alpha)$ then we must have

$$\frac{1-\alpha}{(1-r_0)^2}+\frac{\alpha(1+r_0)}{(1-r_0)^3}-1=1$$

calculation gives, $2r_0^3-6r_0^2+(5+2\alpha)r_0-1=0$.
Let $f(r)=2r^3-6r^2+(5+2\alpha)r-1=0$. Then $f(0)=-1$, $f(1)=2\alpha$. Also, $f(0)f(1)<0$ implies $\alpha>0$. This implies the $\mathcal{SD}(\alpha)-$ radius of f is r_4 where r_4 is the root of the equation $2r^3-6r^2+(5+2\alpha)r-1=0$ provided $\alpha>0$.

Theorem 3.6. *The $\mathcal{SD}(\alpha,\beta)$-radius of $f\in\mathcal{S}$ is r_5 where r_5 is the root of equation $(2-\beta)r^4-4(2-\beta)r^3+(11-6\beta-\alpha)r^2-(4\beta+6\alpha+10)r+1-\beta=0$ lying in $(0,1)$ provided $7\alpha+8\beta+4>0$.*
Proof. Let $f\in\mathcal{S}$. Then $|a_n|\le n$ for $f\in\mathcal{S}$.

Therefore

$$\begin{aligned}&\sum_{n=2}^{\infty}[\alpha n+(1-\alpha)]|a_n|r_0^{n-1}\\ \leq&\sum_{n=2}^{\infty}[\alpha n+(1-\alpha)]nr_0^{n-1}\\ =&\frac{\alpha(1+4r_0+r_0^2)}{(1-r_0)^4}+\frac{1-\alpha}{(1-r_0)^2}.\end{aligned}$$

Thus $\frac{1}{r_0}f(r_0z)\in\mathcal{SD}(\alpha,\beta)$ if

$$\begin{aligned}&\alpha(1+4r_0+r_0^2)+(1-\alpha)(1-r_0)^2=(2-\beta)(1-r_0)^4\\ \Longrightarrow\ &(2-\beta)r_0^4-4(2-\beta)r_0^3+(11-\alpha-6\beta)r_0^2-(4\beta+6\alpha+10)r_0+(1-\beta)=0.\end{aligned}$$

Let $f(r)=(2-\beta)r^4-4(2-\beta)r^3+(11-\alpha-6\beta)r^2-(4\beta+6\alpha+10)r+(1-\beta)$, then $f(0)=1-\beta$, $f(1)=-4-8\beta-7\alpha$ and hence $f(0)f(1)<0$ implies $7\alpha+8\beta+4>0$. Hence the $\mathcal{SD}(\alpha,\beta)$ radius of $f\in\mathcal{S}$ is the root of equation $(2-\beta)r^4-4(2-\beta)r^3+(11-6\beta-\alpha)r^2-(4\beta+6\alpha+10)r+1-\beta=0$ in $(0,1)$ provided $7\alpha+8\beta+4>0$.

4. Conclusion

We conclude that the radii of the largest disk inside the unit disk for which the functions in the class of normalized, analytic and univalent functions $\mathcal{S}$ belong to certain standard subclasses of it are the unique roots in $(0,1)$ of certain polynomial equations.

References

[1] Adolf S. B., Subramanian K. G., On a subclass of Noshiro type analytic functions with negative coefficient (Presented at the 13th Ramanujan Mathematical Society Conference), (1998).

[2] Baricz A., Obradovic M., Ponnusamy S., The radius of Univalence of the reciprocal of a product of two analytic functions, J. Analysis, 21 (2013), 1-19.

[3] Duren P. L., Univalent functions, Springer Verlag, New York, Inc (1983).

[4] Goodman A. W., Univalent Functions Vol 1 and 2, Mariner Publishing, Tampa, Fla. USA. (1983).

[5] Gavrilov V. I., Remarks on the radius of univalence of holomorphic functions, Mathematicheskie Zametki., 7 (1970), 295-298.

[6] Khalida I. N., Sadia R., Generalized q-Starlike Functions, Studia Scientiarum Mathematicarum Hungarica, 54 (4) (2017), 509–522.

[7] Obradovic M., Ponnusamy S., Tuneski N., Radius of univalence of certain combination of univalent and analytic functions, Bull. Malay. Math. Sci. Soc., 35(2) (2012), 325-334.

[8] Obradovic M., Ponnusamy S., Univalence of the average of two analytic functions, arxiv:1203:2713v1. 13 pp.

[9] Prajapat J. K., Raina R. K., Certain Subclasses of Analytic Functions Involving Salagean Operator, Italian J. Pure Appld. Math., 27 (2010), 91-98.

[10] Rosy T., Studies on subclasses of starlike and convex functions, Ph.D. Thesis submitted to University of Madras, (2001).

[11] Rosy T., Stephen B. A., Subramanian K. G., Silverman H., Classes of convex functions, Int. J of Math. and Mathematical Sci., 23(12) (2000), 819-825.

[12] Ruscheweyh S., Extension of Szegö's theorem on the sections of univalent functions, SIAM J. Math. Anal., 19(6) (1988), 1442-1449.

[13] Yamashita S., Radii of univalence starlikeness and convexity, Bull. of the Aust. Math.Soc. 25(3) (1982), 453-457.