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HANKEL DETERMINANT OF GENERALISED CLASSES OF STARLIKE FUNCTIONS WITH RESPECT TO m-FOLD SYMMETRIC POINTS

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Abstract: Denote $\mathcal{S}$ to be the class of functions which are analytic, normalized and univalent in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. The upper bound for the functional $|a_{m+1}a_{3m+1} - a_{2m+1}^2|$ with respect to m-fold symmetric points are determined.

Keywords and Phrases: Starlike functions, Convex functions, q^{th} Hankel determinant, m-fold symmetric points.

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1. Introduction

Let $\mathcal{S}$ denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. In [4], q^{th} Hankel determinant for $q \geq 1$ and $n \geq 0$ is defined by Noonan and Thomas as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}$$

Easily, one can observe that the Fekete and Szego functional is $H_2(1)$, Fekete and Szego then further generalised the estimate $|a_3 - \mu a_2^2|$, where μ is real. For our discussion in this paper, we consider the Hankel determinant in the case $q = 2$ and $n = 2$,

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}$$

The class $\mathcal{S}^*$ and $\mathcal{C}$ are defined as follows.

2. Preliminaries

Definition 2.1. [1] *Let $f \in \mathcal{S}$ be given by* (1)*. Then $f \in \mathcal{S}^*$ if and only if*

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \ z \in \mathbb{E} \tag{2}$$

Definition 2.2. [1] *Let $f \in \mathcal{S}$ be given by* (1)*. Then $f \in \mathcal{C}$ if and only if*

$$Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0, \ z \in \mathbb{E} \tag{3}$$

It follows that $f \in \mathcal{C}$ if and only if $zf'(z) \in \mathcal{S}^*$.

Definition 2.3. [8] *A function $f \in \mathcal{S}$ is called univalent starlike functions with respect to symmetric points if and only if*

$$Re\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0, \ z \in \mathbb{E} \tag{4}$$

and the class of functions satisfying (4) *may be denoted by $\mathcal{S}^*_{\mathcal{S}}$.*
A function $f \in \mathcal{S}$ is said to be convex functions with respect to symmetric points if and only if

$$Re\left(\frac{(zf'(z))'}{f(z) - f(-z)}\right) > 0, \ z \in \mathbb{E} \tag{5}$$

and the class of such functions is denoted by $\mathcal{K}_{\mathcal{S}}$.
$\mathcal{C}_{\mathcal{S}}$ is the class of close to convex functions $f \in \mathcal{S}$ with respect to symmetric points if there exists a function

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}_{\mathcal{S}}^* \tag{6}$$

such that

$$Re\left(\frac{zf'(z)}{g(z) - g(-z)}\right) > 0 \tag{7}$$

The class $\mathcal{C}_{\mathbf{1}(\mathcal{S})}$ consisting of functions $f(z)$ in $\mathcal{S}$ with respect to symmetric points is obtained by replacing $g(z)$ by

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{K}_{\mathcal{S}} \tag{8}$$

in the condition (7). Obviously $\mathcal{C}_{\mathbf{1}(\mathcal{S})} \subset \mathcal{C}_{\mathcal{S}}^*$.
A function f in $\mathcal{S}$ belongs to $\mathcal{S}_{\mathcal{S}}^*(\mathcal{A}, \mathcal{B})$ [7], if

$$\left(\frac{2zf'(z)}{f(z) - f(-z)}\right) \prec \frac{1 + Az}{1 + Bz}, \ -1 \leq B < A \leq 1 \tag{9}$$

The class $\mathcal{K}_{\mathcal{S}}(\mathcal{A}, \mathcal{B})$ consists of functions $f(z)$ in $\mathcal{S}$ which satisfying the condition

$$\left(\frac{2(zf'(z))'}{f(z) - f(-z)}\right) \prec \frac{1 + Az}{1 + Bz}, \ -1 \leq B < A \leq 1 \tag{10}$$

It is obvious that $f(z) \in \mathcal{K}_{\mathcal{S}}(\mathcal{A}, \mathcal{B}) \implies zf'(z) \in \mathcal{S}_{\mathcal{S}}^*(\mathcal{A}, \mathcal{B})$.

Let $\alpha \geq 0$ and $\frac{f(z)f'(z)}{z} \neq 0$. Then $\mathcal{C}_{\mathcal{S}}^*(\alpha)$ is the class of functions $f \in \mathcal{S}$ with respect to symmetric points if there exists a function $g \in \mathcal{S}_{\mathcal{S}}^*$ such that

$$Re\left(\frac{(1 - \alpha)f(z)}{g(z) - g(-z)} + \frac{\alpha zf'(z)}{g(z) - g(-z)}\right) > 0 \tag{11}$$

If g is replaced by h in the condition (11) then the corresponding class may be denoted by $\mathcal{C}_{\mathbf{1}(\mathcal{S})}^*(\alpha)$.

The classes $\mathcal{J}_{\mathcal{S}}^*(\alpha)$ and $\mathcal{J}_{\mathbf{1}(\mathcal{S})}^*(\alpha)$ represent the subclasses of functions $f \in \mathcal{S}$ which satisfy the following conditions, respectively

$$Re\left(\frac{zf'(z)}{g(z) - g(-z)} + \frac{\alpha z^2 f''(z)}{g(z) - g(-z)}\right) > 0, \ g \in \mathcal{S}_{\mathcal{S}}^* \tag{12}$$

$$Re\left(\frac{zf'(z)}{h(z)-h(-z)}+\frac{\alpha z^2 f''(z)}{h(z)-h(-z)}\right)>0,\ h\in\mathcal{K}_{\mathcal{S}} \tag{13}$$

We have the following observations
(*i*) $f(z)\in\mathcal{C}^*_{\mathcal{S}}(\alpha)\implies zf'(z)\in\mathcal{J}^*_{\mathcal{S}}(\alpha)$
(*ii*) $f(z)\in\mathcal{C}^*_{\mathbf{1}(\mathcal{S})}(\alpha)\implies zf'(z)\in\mathcal{J}^*_{\mathbf{1}(\mathcal{S})}(\alpha)$

Let $0<\delta\le 1$, $-1\le D\le B<A\le C\le 1$, $g\in\mathcal{S}^*_{\mathcal{S}}(\mathcal{A},\mathcal{B})$ and $h\in\mathcal{K}_{\mathcal{S}}(\mathcal{A},\mathcal{B})$. Then we shall also deal with the following classes.

$$\mathcal{C}^*_S(\alpha,\delta,A,B,C,D)=\left\{f\in\mathcal{S},\ \frac{2(1-\alpha)f(z)}{g(z)-g(-z)}+\frac{2\alpha zf'(z)}{g(z)-g(-z)}\prec\left(\frac{1+Cz}{1+Dz}\right)^{\delta}\right\} \tag{14}$$

$$\mathcal{C}^*_{1(S)}(\alpha,\delta,A,B,C,D)=\left\{f\in\mathcal{S},\ \frac{2(1-\alpha)f(z)}{h(z)-h(-z)}+\frac{2\alpha zf'(z)}{h(z)-h(-z)}\prec\left(\frac{1+Cz}{1+Dz}\right)^{\delta}\right\} \tag{15}$$

$$\mathcal{J}^*_S(\alpha,\delta,A,B,C,D)=\left\{f\in\mathcal{S},\ \frac{2zf'(z)}{g(z)-g(-z)}+\frac{2\alpha z^2f''(z)}{g(z)-g(-z)}\prec\left(\frac{1+Cz}{1+Dz}\right)^{\delta}\right\} \tag{16}$$

$$\mathcal{J}^*_{\mathbf{1}(S)}(\alpha,\delta,A,B,C,D)=\left\{f\in\mathcal{S},\ \frac{2zf'(z)}{h(z)-h(-z)}+\frac{2\alpha z^2f''(z)}{h(z)-h(-z)}\prec\left(\frac{1+Cz}{1+Dz}\right)^{\delta}\right\} \tag{17}$$

For $\delta=1$, we write
(*i*) $\mathcal{C}^*_S(\alpha,1,A,B,C,D)\equiv\mathcal{C}^*_S(\alpha,A,B,C,D)$
(*ii*) $\mathcal{C}^*_{1(S)}(\alpha,1,A,B,C,D)\equiv\mathcal{C}^*_{1(S)}(\alpha,A,B,C,D)$
(*iii*) $\mathcal{J}^*_S(\alpha,1,A,B,C,D)\equiv\mathcal{J}^*_S(\alpha,A,B,C,D)$
(*iv*) $\mathcal{J}^*_{\mathbf{1}(S)}(\alpha,1,A,B,C,D)\equiv\mathcal{J}^*_{\mathbf{1}(S)}(\alpha,A,B,C,D)$
Throughout this paper we assume that
$z\in\mathbb{E}$, $0\le\alpha$, $0<\delta\le 1$, $-1\le D\le B<A\le C\le 1$

$g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in\mathcal{S}^*_{\mathcal{S}}$, $h(z)=z+\sum_{n=2}^{\infty}c_nz^n\in\mathcal{K}_{\mathcal{S}}$

$G(z)=\frac{g(z)-g(-z)}{2}=z+b_{m+1}z^{m+1}+b_{3m+1}z^{3m+1}$,

$H(z)=\frac{h(z)-h(-z)}{2}=z+c_{m+1}z^{m+1}+c_{3m+1}z^{3m+1}$,

$P(z)=1+\sum_{k=1}^{\infty}p_{km}z^{km}$, $Q(z)=1+\sum_{k=1}^{\infty}q_{km}z^{km}$.

Definition 2.4. [1] *Let $m \in \mathbb{N}= \{1, 2, 3, ...\}$. A domain $\mathbb{E}$ is said to be m-fold symmetric if a rotation of $\mathbb{E}$ about the origin through an angle $\frac{2\pi}{m}$ carries $\mathbb{E}$ on itself. It follows that a function $f(z)$ analytic in $\mathbb{E}$ is said to be m-fold symmetric $(m \in \mathbb{N})$ if*

$$f(e^{\frac{2\pi i}{m}} z) = e^{\frac{2\pi i}{m}} f(z)$$

In particular, every $f(z)$ is 1-fold symmetric and every odd $f(z)$ is 2-fold symmetric. We denote by $\mathcal{S}_m$ the class of m-fold symmetric univalent functions in $\mathbb{E}$. A simple argument shows that $f \in \mathcal{S}_m$ is characterized by having a power series of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}. \tag{18}$$

Lemma 2.1. [6] *If $p \in \mathcal{P}$, $|p_k| \leq 2$, $k \in \mathbb{N}$, where the caratheodary class $\mathcal{P}$ is the family of all functions p analytic in $\mathbb{E}$ for which*

$$Re\{p(z)\} > 0, \; p(z) = 1 + p_1 z + p_2 z^2 + ...$$

Lemma 2.2. [2] *Let $p \in \mathcal{P}$, then*

$$2p_2 = p_1^2 + x(4 - p_1^2). \tag{19}$$

and

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)t. \tag{20}$$

for some x and t satisfying $|x| \leq 1$ and $|t| \leq 1$.

3. Main Results

Theorem 3.1. *Let $0 \leq \alpha < 1$, and $f \in \mathcal{C}_{\mathcal{S}}^*(\alpha)$, then*

$$|a_{m+1}a_{3m+1} - a_{2m+1}^2| \leq \frac{4}{(1 + 2\alpha m)^2} \tag{21}$$

Proof. Since, $f \in \mathcal{C}_{\mathcal{S}}^*(\alpha)$ it follows that

$$(1 - \alpha)f(z) + \alpha z f'(z) = P(z)G(z) \tag{22}$$

Identifying the terms in (22), we get

$$\begin{cases} a_{m+1} = \frac{1}{(1+\alpha m)}[b_{m+1} + p_m] \\ a_{2m+1} = \frac{1}{(1+2\alpha m)}[p_m b_{m+1} + p_{2m}] \\ a_{3m+1} = \frac{1}{(1+3\alpha m)}[p_{2m} b_{m+1} + b_{3m+1} + p_{3m}] \end{cases} \tag{23}$$

As $g \in \mathcal{S}^*_{\mathcal{S}}$, by definition

$$zg'(z) = P(z)G(z) \tag{24}$$

$$\implies b_{m+1}(m+1)z^{m+1} + b_{2m+1}(2m+1)z^{2m+1} + b_{3m+1}(3m+1)z^{3m+1}$$
$$= b_{m+1}z^{m+1} + b_{3m+1}z^{3m+1} + p_m z^{m+1} + p_m b_{m+1}z^{2m+1} + p_{2m}z^{2m+1}$$
$$+ p_{2m}b_{m+1}z^{3m+1} + p_{3m}z^{3m+1}$$

Equating the coefficients in (24), we obtain

$$\begin{cases} b_{m+1} = \frac{p_m}{m} \\ b_{2m+1} = \frac{p_m^2 + mp_{2m}}{m(2m+1)} \\ b_{3m+1} = \frac{1}{3m}[p_{2m}b_{m+1} + p_{3m}] \end{cases} \tag{25}$$

from (23) and (25), we obtain

$$a_{m+1} = \frac{(1+m)p_m}{m(1+\alpha m)} \tag{26}$$

$$a_{2m+1} = \frac{1}{m(1+2\alpha m)}[p_m b_{m+1} + p_{2m}] \tag{27}$$

$$a_{3m+1} = \frac{1}{3m^2(1+3\alpha m)}[p_{2m}b_{m+1} + b_{3m+1} + p_{3m}] \tag{28}$$

$$\implies |a_{m+1}a_{3m+1} - a_{2m+1}^2| = \left|\frac{1}{C(\alpha)}\left\{(1+m)(1+3m)(1+2\alpha m)^2\left[p_m^2 p_{2m} + mp_m p_{3m}\right] - 3m(1+\alpha m)(1+3\alpha m)\left[p_m^4 + 2mp_m^2 p_{2m} + m^2 p_{2m}^2\right]\right\}\right|$$

where $C(\alpha) = 3m^3(1+\alpha m)(1+2\alpha m)^2(1+3\alpha m)$.
Using Lemma 2.2, we get

$$|a_{m+1}a_{3m+1} - a_{2m+1}^2| \leq \frac{1}{C(\alpha)}\left\{(1+m)(1+3m)(1+2\alpha m)^2\left[\frac{p_m^2}{2}(p_m^2 + |x|(4-p_m^2))\right]\right.$$
$$+ \frac{mp_m}{4}\left[p_m^3 + 2(4-p_m^2)p_m|x| - p_m(4-p_m^2)|x|^2 + 2(4-p_m^2)(1-|x|^2)|z|\right]$$
$$\left.+3m(1+\alpha m)(1+3\alpha m)[p_m^4 + mp^2[p_m^2 + |x|(4-p_m^2)] + \frac{m^2}{4}\left[p_m^2 + |x|(4-p_m^2)^2\right]\right\}.$$

Assume $|p_m| = p$ and $p \in [0, 2]$. Using triangle inequality and $|z| \le 1$, we have

$$|a_{m+1}a_{3m+1} - a_{2m+1}^2| \le \frac{1}{C(\alpha)}\Big\{(1+m)(1+3m)(1+2\alpha m)^2\Big[\frac{p^2}{2}\Big(p^2+\delta(4-p^2)\Big)\Big]$$

$$+\frac{mp}{4}\Big[p^3+2(4-p^2)p\delta - p(4-p^2)\delta^2 + 2(4-p^2)(1-\delta^2)\Big]$$

$$+3m(1+\alpha m)(1+3\alpha m)\Big[p^4+mp(p^2+\delta(4-p^2))\Big]+\frac{m^2}{4}\Big[p^2+2p^2\delta^2(4-p^2)+\delta^2(4-p^2)^2\Big]\Big\}.$$

$$\Longrightarrow |a_{m+1}a_{3m+1} - a_{2m+1}^2| \le \frac{1}{C_1(\alpha)}\Big\{(1+m)(1+3m)(1+2\alpha m)^2\Big[p^4(m+2)$$

$$+2p^2(m+1)(4-p^2)\delta - mp^2(4-p^2)\delta^2 + 2mp(4-p^2)(1-\delta^2)\Big]$$

$$+3m(1+\alpha m)(1+3\alpha m)\Big[4p^4+4mp^3+m^2p^2+\Big(2m^2p^2+4mp+m^2(4-p^2)^2\delta\Big)(4-p^2)^2\delta\Big]\Big\}.$$

$\equiv \frac{1}{C_1(\alpha)}F(\delta)$, where $\delta = |x| \le 1$.
where $C_1(\alpha) = 4C(\alpha)$.
Using fundamental theorem of calculus,
By elementary calculation, it is seen that, $F(\delta)$ is an increasing function.
Therefore max $F(\delta)$ =$F(1)$.
Consequently

$$|a_{m+1}a_{3m+1} - a_{2m+1}^2| \le \frac{1}{C_1(\alpha)}G(p). \tag{29}$$

where $G(p) = (1+m)(1+3m)(1+2\alpha m)^2(8mp) + 3m(1+\alpha m)(1+3\alpha m)(16m^2)$.
obviously $G(p) \le 48m^3(1+\alpha m)(1+3\alpha m)$.

Theorem 3.2. *Let* $0 \le \alpha < 1$ *and* $f \in \mathcal{C}^*_{\mathbf{1}(\mathcal{S})}(\alpha)$, *then*

$$|a_{m+1}a_{3m+1} - a_{2m+1}^2| \le \frac{16}{(1+2\alpha m)^2} \tag{30}$$

Proof. Since, $f \in \mathcal{C}^*_{\mathbf{1}(\mathcal{S})}(\alpha)$ it follows that

$$(1-\alpha)f(z) + \alpha z f'(z) = P(z)H(z) \tag{31}$$

Identifying the terms in (31), we get

$$\begin{cases} a_{m+1} = \frac{1}{(1+\alpha m)}[c_{m+1} + p_m] \\ a_{2m+1} = \frac{1}{(1+2\alpha m)}[p_m c_{m+1} + p_{2m}] \\ a_{3m+1} = \frac{1}{(1+3\alpha m)}[p_{2m} c_{m+1} + c_{3m+1} + p_{3m}] \end{cases} \tag{32}$$

As $h \in \mathcal{K}_{\mathcal{S}}$, by definition

$$(zh'(z))' = P(z)H'(z) \tag{33}$$

Equating the coefficients in (33), we obtain

$$\begin{cases} c_{m+1} = \frac{p_m}{(m+1)^2} \\ c_{2m+1} = \frac{p_{2m}}{(2m+1)^2} \\ c_{3m+1} = \frac{p_{3m}}{(3m+1)^2} \end{cases} \tag{34}$$

from (32) and (34), we obtain

$$a_{m+1} = \frac{p_m(m^2+2m+2)}{(1+\alpha m)(m+1)^2}. \tag{35}$$

$$a_{2m+1} = \frac{1}{(1+2\alpha m)}\left[\frac{p_m^2}{(1+m)^2} + p_{2m}\right]. \tag{36}$$

$$a_{3m+1} = \frac{1}{(1+3\alpha m)}\left[\frac{p_{3m}}{(1+3m)^2} + p_{2m}\frac{p_m}{(1+m)^2} + p_{3m}\right]. \tag{37}$$

$$\implies |a_{m+1}a_{3m+1} - a_{2m+1}^2| = \left|\frac{1}{C(\alpha)}\left\{(m^2+2m+2)(1+2\alpha m)^2\Big[p_m p_{3m}(1+m)^2 + p_m^2 p_{2m}(1+3m)^2 + p_m p_{3m}(1+3m)^2(1+m)^2\Big] - (1+\alpha m)(1+3\alpha m)(1+3m)^2\Big[p_m^4 + 2p_m^2 p_{2m}(1+m)^2 + p_{2m}^2(1+m)^4\Big]\right\}\right|.$$

where $C(\alpha) = (1+\alpha m)(1+2\alpha m)(1+3\alpha m)(1+m)^4(1+3m)^2$

Using Lemma 2.2, we get

$$|a_{m+1}a_{3m+1} - a_{2m+1}^2| = \frac{1}{C(\alpha)}\left\{(m^2+2m+2)(1+2\alpha m)^2\Big[p_m^3 + 2(4-p_m^2)p_m|x| - p_m(4-p_m^2)|x|^2 + 2(4-p_m^2)(1-|x|^2)|z|\Big]\times\right.$$

$$\left[\frac{p_m(1+m)^2}{4}(9m^2+6m+2)+\frac{p_m^2(1+3m)^2}{2}\left(p_m^2+x(4-p_m^2)\right)\right]$$
$$+(1+\alpha m)(1+3\alpha m)(1+3m)^2\left[p_m^4+p_m^2(1+m)^2(p_m^2+|x|(4-p_m^2))\right]$$
$$+\frac{(1+m)^4}{4}\left(p_m^2+|x|(4-p_m^2)\right)^2\Bigg\}.$$

where $C(\alpha)=(1+\alpha m)(1+2\alpha m)(1+3\alpha m)(1+m)^4(1+3m)^2$
Assume $|p_m|=p$ and $p\in[0,2]$. Using triangle inequality and $|x|\le 1$, we have

$$|a_{m+1}a_{3m+1}-a_{2m+1}^2|\le\frac{1}{C_1(\alpha)}\Bigg\{(m^2+2m+2)(1+2\alpha m)^2$$
$$\left[\left(p^3+2(4-p^2)p\delta-p(4-p^2)\delta^2+2(4-p^2)(1-\delta^2)\delta\right)\times\left(\frac{p(1+m)^2}{4}(9m^2+6m+2)\right)\right.$$
$$\left.+2p^2(1+3m)^2(p^2+(4-p^2)\delta)\right]+(1+\alpha m)(1+3\alpha m)(1+3m)^2$$
$$\left[4p^4+4(1+m)^2(p^4+p^2(4-p^2)\delta)+(1+m)^4(p^2+2p\delta(4-p^2)+(4-p^2)^2\delta^2)\right]\Bigg\}.$$
$$\implies |a_{m+1}a_{3m+1}-a_{2m+1}^2|\le\frac{1}{C_1(\alpha)}\Bigg\{(m^2+2m+2)(1+2\alpha m)^2\left[\left(p^4+2(4-p^2)p^2\delta\right.\right.$$
$$\left.+p^2(4-p^2)\delta^2+2p\delta(4-p^2)(1-\delta^2)\right)\times\left((1+m)^2(9m^2+6m+2)\right)$$
$$\left.+2(1+3m)^2(p^4+(4-p^2)p^2\delta)\right]+(1+\alpha m)(1+3\alpha m)(1+3m)^2$$
$$\left[4p^4+4(1+m)^2(p^4+p^2(4-p^2)\delta)+(1+m)^4(p^2+2p\delta(4-p^2)+(4-p^2)^2\delta^2)\right]\Bigg\}.$$
$$\equiv\frac{1}{C_1(\alpha)}F(\delta)$$

where $C_1(\alpha)=4C(\alpha)$.
Using fundamental theorem of calculus,
By elementary calculation, it is seen that, $F(\delta)$ is an increasing function.
Therefore max $F(\delta)=F(1)$.
Consequently

$$|a_{m+1}a_{3m+1}-a_{2m+1}^2|\le\frac{1}{C_1(\alpha)}G(p) \tag{38}$$

where

$$G(p)=(m^2+2m+2)(1+2\alpha m)^2\left[8p(1+m)^2(9m^2+6m+2)\right]+$$

$(1+\alpha m)(1+3\alpha m)(1+3m)^2\left[16(1+3m)^4\right].$

Obviously $G(p) \leq 16(1+\alpha m)(1+3\alpha m)(1+m)^4(1+3m)^2$.
Thus, we have

$$|a_{m+1}a_{3m+1} - a_{2m+1}^2| \leq \frac{16}{(1+2\alpha m)^2}.$$

Remark 3.1. *Let f given by (1) be in the class $\mathcal{C}^*_{\mathbf{1}(\mathcal{S})}(\alpha)$ and $0 \leq \alpha < 1$. Putting $m = 1$, we get*

$$|a_2a_4 - a_3^2| \leq \frac{16}{(1+2\alpha)^2}.$$

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