South East Asian J. of Mathematics and Mathematical Sciences
Vol. 17, No. 3 (2021), pp. 29-44

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

SPACE OF INVARIANT BILINEAR FORMS UNDER A REPRESENTATION OF $SL_2(3)$

Dilchand Mahto and Jagmohan Tanti*

Department of Mathematics,
Central University of Jharkhand,
Ranchi - 835205, Jharkhand, INDIA

E-mail : dilchandiitk@gmail.com

*Department of Mathematics,
Babasaheb Bhimrao Ambedkar University,
Lucknow - 226025, Uttar Pradesh, INDIA

E-mail : jagmohan.t@gmail.com

(**Received:** *May 05, 2021* **Accepted:** *Nov. 01, 2021* **Published:** *Dec. 30, 2021)*

Abstract: Let n be a positive integer. In this paper we compute the space of invariant bilinear forms under an n degree representation of the special linear group $SL_2(3)$ and its dimension over the complex field $\mathbb{C}$. We discuss the existence of a non-degenerate invariant bilinear form explicitly.

Keywords and Phrases: Bilinear forms, Representation theory, Vector space, Direct sums.

2020 Mathematics Subject Classification: 20C33, 15A63, 15A03.

1. Introduction

Representation theory enables the study of a group as operators on certain vector spaces. Since last several years the search of non-degenerate invariant bilinear forms has remained of great importance among researcher. Such types of studies acquire an important place in quantum mechanics and other branches of physical sciences.

Let G be a finite group and $\mathbb{V}$, a vector space over a field $\mathbb{F}$, then we have following.

Definition 1.1. *A homomorphism $\rho : G \to GL(\mathbb{V})$ is called a representation of the group G. $\mathbb{V}$ is also called a representing space of G. The dimension of $\mathbb{V}$ over $\mathbb{F}$ is called degree of the representation.*

Definition 1.2. *A bilinear form $\mathbb{B}$ on $\mathbb{V}$ is said to be invariant under the representation ρ if*

$$\mathbb{B}(\rho(g)x, \rho(g)y) = \mathbb{B}(x, y),\ \forall\ g \in G \text{ and } x,y \in \mathbb{V}.$$

For the basic properties of a bilinear form one can refer to [7].
Let Ξ denotes the space of bilinear forms on the vector space $\mathbb{V}$ over $\mathbb{F}$.

Definition 1.3. *The space of invariant bilinear forms under the representation ρ is given by*

$$\Xi_G = \{\mathbb{B} \in \Xi \mid \mathbb{B}(\rho(g)x, \rho(g)y) = \mathbb{B}(x, y),\ \forall\ g \in G \text{ and } x,y \in \mathbb{V}\}.$$

It is easy to see that Ξ_G is a subspace of Ξ.
The representation $(\rho,\ \mathbb{V})$ is irreducible of degree n if and only if $\{0\}$ and $\mathbb{V}$ are the only invariant sub-spaces of $\mathbb{V}$ under ρ. Let r be the number of conjugacy classes of G. If $\mathbb{F}$ is algebraically closed and char($\mathbb{F}$) is 0 or relative prime to $|G|$, by Frobenius (see [1], Theorem 5.9, p. 318) there are r irreducible representations ρ_i (say), $1 \leq i \leq r$ of G and χ_i (say) is the corresponding character of ρ_i. Also by Maschke's theorem (see [1], Corollary 4.9, p. 316) every n degree representation of G can be written as a direct sum of copies of irreducible representations. For $\rho = \oplus_{i=1}^{r} k_i\rho_i$ an n degree representation of G, the coefficient of ρ_i is k_i, $1 \leq i \leq r$, so that $\sum_{i=1}^{r} d_i k_i = n$, and $\sum_{i=1}^{r} d_i^2 = |G|$, where d_i is the degree of ρ_i and $d_i||G|$ with $d_{i'} \geq d_i$ when $i' > i$. It is already well understood in the literature that the invariant space Ξ_G under ρ can be expressed by the set $\Xi'_G = \{X \in \mathbb{M}_n(\mathbb{F}) \mid C^t_{\rho(g)} X C_{\rho(g)} = X, \forall g \in G\}$ with respect to an ordered basis $\underline{e}$ of $\mathbb{V}$, where $\mathbb{M}_n(\mathbb{F})$ is the set of square matrices of order n with entries from $\mathbb{F}$ and $C_{\rho(g)} = [\rho(g)]_{\underline{e}}$ is the matrix representation of the linear transformation $\rho(g)$ with respect to $\underline{e}$.

Definition 1.4 *For $\mathbb{Z}_3$ the complete residue system* $\pmod 3$ *in standard form, we define the special linear group*

$$SL_2(3) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}_3 \,\&\, ad - bc = 1 \right\}.$$

For more information about $SL_2(3)$ one can refer to a general text book of Algebra by Serge Lang [9]. It is easy to see that $ord(SL_2(3)) = 24$.
In this paper our investigation pertains to the following two questions.

1. What is the dimension of $\Xi_{SL_2(3)}$ for degree n representation?
2. What are the necessary and sufficient conditions for the existence of a non-degenerate invariant bilinear form?

These questions have been studied by many people in the distinct perspectives. Gongopadhyay and Kulkarni [4] investigated the existence of T-invariant non-degenerate symmetric (resp. skew-symmetric) bilinear forms. Kulkarni and Tanti [8] investigated the dimension of the space of T-invariant bilinear forms. Gongopadhyay, Mazumder and Sardar [6] investigated the case for an invertible linear map $T : V \to V$, when does the vector space V over F admit a T-invariant non-degenerate c-hermitian form. Chen [2] discussed about the matrix representations of the real numbers. Sergeichuk [13] studied the systems of forms and linear mappings by associating with them self-adjoint representations of a category with involution. Frobenius [3] proved that every endomorphism of a finite dimensional vector space V is self-adjoint for at least one non-degenerate symmetric bilinear form on V. Later, Stenzel [11] determined when an endomorphism could be skew-self adjoint for a non-degenerate quadratic form, or self-adjoint or skew-self adjoint for a symplectic form on complex vector spaces. However his results were later generalized to an arbitrary field [5]. Pazzis [10] tackled the case of the automorphisms of a finite dimensional vector space that are orthogonal (resp. symplectic) for at least one non-degenerate quadratic form (resp. symplectic form) over an arbitrary field of characteristics 2.
In this paper, we investigate the dimensions of the space of invariant bilinear forms and establish a characterization criteria for the existence of a non-degenerate invariant bilinear form of $SL_2(3)$ over $\mathbb{C}$. Our investigations are summarized in two main theorems stated in section 5.

2. Preliminaries

The group $SL_2(3)$ is a subgroup of the general linear group $GL_2(3)$. The order of it's center is 2. $SL_2(3)$ contains cyclic subgroups of orders $1, 2, 3, 4$ and 6. By sylow's theorem the quaternions form a normal subgroup and there are 4 subgroups of order 3, thus 8 elements of order 3 and 6. It has one subgroup of order 1, one of order 2, four of order 3, three of order 4, four of order 6, one of order 8 and one of order 24. We noted that here all r irreducible representations of $SL_2(3)$, where r is the number of conjugacy classes of $SL_2(3)$ ([12], Ch. 8, p. 61).

Definition 2.1. *The character of ρ is a function $\chi : G \to \mathbb{F}$, $\chi(g) = tr(\ [\rho(g)]_{\underline{e}}\ )$ and is also called character of the group G.*

Theorem 2.1. *(Maschke's Theorem): If char($\mathbb{F}$) does not divide $|G|$, then every*

representation of G is a direct sum of irreducible representations.
Proof. See [1], Corollary 4.9, p. 316.

Theorem 2.2. *Two representations $(\rho, \mathbb{V})$ and $(\rho', \mathbb{V})$ of G are isomorphic if and only if their character tables are same i.e, $\chi(g) = \chi'(g)$ for all $g \in G$.*
Proof. See [1], Corollary 5.13, p. 319.

In the rest part of this section we take $\mathbb{F} = \mathbb{C}$.

2.1. Irreducible representations of $SL_2(3)$
In this subsection $(\rho_i, \mathbb{V}_{\rho_i})$ stands for an irreducible representation of $SL_2(3)$ with degree $d_i = 1, 2$ or 3 over $\mathbb{C}$ and $C_{\rho_i(g)}$ is the matrix representation of $\rho_i(g)$, where $1 \leq i \leq 7$. Since ρ_i is an homomorphism from $SL_2(3)$ to $GL(\mathbb{V}_{\rho_i}) \cong GL(d_i, \mathbb{C})$. So by the fundamental theorem of homomorphism $\frac{G}{Ker(\rho_i)} \cong \rho_i(G)$. The possible order of $Ker(\rho_i)$ is 1 or 2 or 8 or 24, thus order of $\rho_i(SL_2(3))$ is 1 or 3 or 12 or 24. Note that the representation of $SL_2(3)$ has been well studied in the literature (for detail see [12], Exercise 8.11, p. 67). Here we construct all irreducible representations of $SL_2(3)$ induced from those of the quaternion group $Q_8 = \langle x,\ y,\ z \mid x^2 = \mathbf{1}, y^2 = x, y^2 = z^2, z^{-1}yz = xy\rangle$, where $x = -1, y = -\mathrm{i}$ and $z = \mathrm{j}$. We have the maximal subnormal series $1 \lhd \mathbb{Z}_2 \lhd \mathbb{Z}_4 \lhd Q_8 \lhd SL_2(3)$. The presentation of $SL_2(3)$ is given as below.

$$SL_2(3) = \left\langle x,\ y,\ z,\ t \mid x^2 = \mathbf{1},\ y^2 = x,\ y^2 = z^2,\ z^{-1}yz = xy,\ t^3 = \mathbf{1},\ t^{-1}yt = z^{-1},\ t^{-1}zt = zy \right\rangle.$$

Since Q_8 have five irreducible representations. Let $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ of degree one and σ_5 of degree 2, there exists a sub-representation of dimension 2 under σ_5. Let $V_{\sigma_5} = \mathbb{C} \oplus \mathbb{C}z$. It is an irreducible sub-representation of $\mathbb{C}[Q_8]$ of degree 2 with $\{1, z\}$ as an order basis. The action $\sigma_5 : Q_8 \times V_{\sigma_5} \to V_{\sigma_5}$ is defined by $\sigma_5(g, w) = w(g)$ and so we have

$$\sigma_5(y, 1) = 1(-\mathrm{i}) = -\mathrm{i}.1 + 0.z,\ \sigma_5(z, 1) = 1(z) = 0.1 + 1.z$$
$$\sigma_5(y, z) = z(-\mathrm{i}) = \mathrm{i}z = 0.1 + \mathrm{i}.z,\ \sigma_5(z, z) = zz = -1 = -1.1 + 0.z.$$

Thus the matrix representations of linear operators $\sigma_5(y)$ and $\sigma_5(z)$ with respect to the prescribed basis are

$$C_{\sigma_5(y)} = \begin{bmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{bmatrix} \text{ and } C_{\sigma_5(z)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

With these discussions we record all the representation matrices for Q_8 in the following table.

	$\mathbf{1}$	x	y	z	yz	y^{-1}	z^{-1}	$(yz)^{-1}$
σ_1	1	1	1	1	1	1	1	1
σ_2	1	1	1	-1	-1	1	-1	-1
σ_3	1	1	-1	-1	1	-1	-1	1
σ_4	1	1	-1	1	-1	-1	1	-1
σ_5	$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$	$\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$	$\begin{bmatrix}-\mathrm{i} & 0\\0 & \mathrm{i}\end{bmatrix}$	$\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$	$\begin{bmatrix}0 & \mathrm{i}\\\mathrm{i} & 0\end{bmatrix}$	$\begin{bmatrix}\mathrm{i} & 0\\0 & -\mathrm{i}\end{bmatrix}$	$\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$	$\begin{bmatrix}0 & -\mathrm{i}\\-\mathrm{i} & 0\end{bmatrix}$

Since t is an extension element to form $SL_2(3)$ from Q_8, so we only need a matrix representation of ρ_i at t. The matrix representation table of Q_8 helps us to construct representation of $SL_2(3)$.
For $2 \leq i \leq 4$, the conjugate $\sigma_i^s(h) = \sigma_i(shs^{-1})$, where $h, shs^{-1} \in Q_8$ and $s \in SL_2(3) - Q_8$, is not isomorphic to σ_i. From proposition and corollary [see [12], Proposition 24, p. 61 and Corollary, p. 60], the induced representation $\mathrm{Ind}_{Q_8}^{SL_2(3)}(\sigma(h)) = \sigma_2(h) \oplus \sigma_3(h) \oplus \sigma_4(h)$ is irreducible. Similarly $\sigma_i^s \cong \sigma_i$, for $i = 1, 5$, as their character tables are same. The index of Q_8 in $SL_2(3)$ is 3, therefore σ_1 and σ_5 extends to three irreducible representations of $SL_2(3)$. The values at x, y, z of an extension representation remains same.
Let the representation ρ_i be an extension of σ_1, also t is an extension element to form $SL_2(3)$, so $\rho_i(t)$, $1 \leq i \leq 3$ assume the values at cube roots of unity, i.e, we have

$$\rho_1(t) = 1,\ \rho_2(t) = \omega \ and\ \rho_3(t) = \omega^2,$$

where ω is a primitive cube root of unity.
Their matrix representations are as follows.

$$C_{\rho_1(t)} = 1,\ C_{\rho_2(t)} = \omega \ and\ C_{\rho_3(t)} = \omega^2.$$

$$C_{\rho_1(y)} = 1,\ C_{\rho_2(y)} = 1 \ and\ C_{\rho_3(y)} = 1.$$

$$C_{\rho_1(z)} = 1,\ C_{\rho_2(z)} = 1 \ and\ C_{\rho_3(z)} = 1.$$

$$C_{\rho_1(x)} = 1,\ C_{\rho_2(x)} = 1 \ and\ C_{\rho_3(x)} = 1.$$

Similarly let the representation ρ_i, $4 \leq i \leq 6$ be an extension of σ_5, we have the conditions $\rho_i(t^{-1}yt) = \rho_i(z^{-1})$ and $\rho_i(t^{-1}zt) = \rho_i(zy)$. Also as $\rho_i(t)^3$ is the identity operator, the three possible matrix representations are

$$\begin{bmatrix}\frac{-1+\mathbf{i}}{2} & \frac{-1-\mathbf{i}}{2}\\ \frac{1-\mathbf{i}}{2} & \frac{-1-\mathbf{i}}{2}\end{bmatrix},\ \omega\begin{bmatrix}\frac{-1+\mathbf{i}}{2} & \frac{-1-\mathbf{i}}{2}\\ \frac{1-\mathbf{i}}{2} & \frac{-1-\mathbf{i}}{2}\end{bmatrix}\ and\ \omega^2\begin{bmatrix}\frac{-1+\mathbf{i}}{2} & \frac{-1-\mathbf{i}}{2}\\ \frac{1-\mathbf{i}}{2} & \frac{-1-\mathbf{i}}{2}\end{bmatrix}.$$

Utilising the other two conditions, we get the three induced irreducible representations of degree two as ρ_4, ρ_5 *and* ρ_6, from σ_5, whose matrix representations are as follows.

$$C_{\rho_4(x)} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, C_{\rho_4(y)} = \begin{bmatrix} -\mathbf{i} & 0 \\ 0 & \mathbf{i} \end{bmatrix}, C_{\rho_4(z)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ and } C_{\rho_4(t)} = \begin{bmatrix} \frac{-1+\mathbf{i}}{2} & \frac{-1-\mathbf{i}}{2} \\ \frac{1-\mathbf{i}}{2} & \frac{-1-\mathbf{i}}{2} \end{bmatrix}.$$

$$C_{\rho_5(x)} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, C_{\rho_5(y)} = \begin{bmatrix} -\mathbf{i} & 0 \\ 0 & \mathbf{i} \end{bmatrix}, C_{\rho_5(z)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ and } C_{\rho_5(t)} = \omega \begin{bmatrix} \frac{-1+\mathbf{i}}{2} & \frac{-1-\mathbf{i}}{2} \\ \frac{1-\mathbf{i}}{2} & \frac{-1-\mathbf{i}}{2} \end{bmatrix}.$$

$$C_{\rho_6(x)} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, C_{\rho_6(y)} = \begin{bmatrix} -\mathbf{i} & 0 \\ 0 & \mathbf{i} \end{bmatrix}, C_{\rho_6(z)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ and } C_{\rho_6(t)} = \omega^2 \begin{bmatrix} \frac{-1+\mathbf{i}}{2} & \frac{-1-\mathbf{i}}{2} \\ \frac{1-\mathbf{i}}{2} & \frac{-1-\mathbf{i}}{2} \end{bmatrix}.$$

Let $\rho_7 = \mathrm{Ind}_{Q_8}^{SL_2(3)}(\sigma)$, then the corresponding matrix representations at $x, y, z \in Q_8$ are given by

$$C_{\rho_7(x)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, C_{\rho_7(y)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, C_{\rho_7(z)} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

To evaluate $C_{\rho_7(t)}$, we have homomorphic conditions, $\rho_7(t^{-1}yt) = \rho_7(z^{-1})$, $\rho_7(t^{-1}zt) = \rho_7(zy)$ and $\rho_7(t)^3$ is the identity operator. Using these conditions, we get its matrix representation as

$$C_{\rho_7(t)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Remark 2.1. *If* $p \equiv 1 \pmod{12}$ *is a rational prime number, then with* γ *a primitive root* $\pmod p$*, the representation of* $SL_2(3)$ *will be same if considered over* $\mathbb{F}_p$ *a field of order* p *by substituting for* $\mathbf{i} = \gamma^{\frac{p-1}{4}}$ *and* $\omega = \gamma^{\frac{p-1}{3}}$.

Remark 2.2. *The representation of* $SL_2(3)$ *is equally good when considered over an algebraically closed field with characteristic* $\neq 2, 3$.

Now as

$$\rho = k_1\rho_1 \oplus k_2\rho_2 \oplus \cdots \oplus k_7\rho_7, \tag{2.1}$$

where for every $1 \leq i \leq 7$, $k_i\rho_i$ stands for the direct sum of k_i copies of the irreducible representation ρ_i.

Let χ be the corresponding character of the representation ρ, then

$$\chi = k_1\chi_1 + k_2\chi_2 + \cdots + k_7\chi_7,$$

where χ_i is the irreducible character of ρ_i, for every i, $1 \le i \le 7$. Degree of the character χ is being calculated at the identity element of a group. i.e,

$$deg(\rho) = \chi(1) = tr(\rho(1)).$$

$$\implies d_1k_1 + d_2k_2 + \cdots + d_7k_7 = n. \tag{2.2}$$

3. Existence of Non-degenerate Invariant Bilinear forms under an n Degree Representation

In this section we see degenerate and non-degenerate invariant bilinear forms. An element in the space of invariant bilinear forms under representation of a finite group is either non-degenerate or degenerate, it means $\Xi_{SL_2(3)}$ may consists both type of invariant bilinear form. If all elements of the space is degenerate then the space is called a degenerate invariant space. From subsection 2.1, for $1 \le i \le 7$, the association of k_i's with ρ_i's are in a well defined manner. We discuss the existence of non-degenerate invariant bilinear form and use it to prove the next lemmas.

Note 3.1. *If $B \in \mathbb{M}_n(\mathbb{C})$ then $B = [B^{i,j}]$, where $B^{i,j} = [b^{ij}_{\alpha,\beta}]$ is a sub-matrix of order $d_ik_i \times d_jk_j$, $1 \le i,j \le 7$, $1 \le \alpha \le d_ik_i$, $1 \le \beta \le d_jk_j$. Thus*

$$B = \begin{bmatrix} B^{1,1} & B^{1,2} & \cdots & B^{1,7} \\ B^{2,1} & B^{2,2} & \cdots & B^{2,7} \\ \vdots & \vdots & \ddots & \vdots \\ B^{7,1} & B^{7,2} & \cdots & B^{7,7} \end{bmatrix} \text{ and } B^{i,j} = \begin{bmatrix} b^{ij}_{1,1} & b^{ij}_{1,2} & \cdots & b^{ij}_{1,d_jk_j} \\ b^{ij}_{2,1} & b^{ij}_{2,2} & \cdots & b^{ij}_{2,d_jk_j} \\ \vdots & \vdots & \ddots & \vdots \\ b^{ij}_{d_ik_i,1} & b^{ij}_{d_ik_i,2} & \cdots & b^{ij}_{d_ik_i,d_jk_j} \end{bmatrix}.$$

Theorem 3.1. *If $\Xi'_{SL_2(3)}$ is the space of invariant bilinear forms under an n degree representation ρ, then the (i,j)th block sub-matrix of $X \in \Xi'_{SL_2(3)}$ is given by*

$$X^{i,j} = \begin{cases} X^{ij}_{d_ik_i \times d_jk_j}, & \text{if } (i,j) \in A \\ \mathbb{O}, & \text{if } (i,j) \notin A. \end{cases}$$

Where $\mathbb{O}$ represents the zero sub-matrix, $A = \{(1,1),(2,3),(3,2),(4,4),\ (5,6),$ $(6,5),\ (7,7)\}$ and for $(i,j) = (1,1),(2,3),(3,2)$ with $d_i = d_j = 1$, we have

$$X^{ij}_{d_ik_i \times d_jk_j} = X^{ij}_{k_i \times k_j} = \begin{bmatrix} x^{ij}_{1,1} & x^{ij}_{1,2} & \cdots & x^{ij}_{1,k_j} \\ x^{ij}_{2,1} & x^{ij}_{2,2} & \cdots & x^{ij}_{2,k_j} \\ \vdots & \vdots & \ddots & \vdots \\ x^{ij}_{k_i,1} & x^{ij}_{k_i,2} & \cdots & x^{ij}_{k_i,k_j} \end{bmatrix},$$

whereas for $(i,j)=(4,4),(5,6),(6,5)$ *with* $d_i=d_j=2$, *we have*

$$X^{ij}_{d_ik_i\times d_jk_j}=X^{ij}_{2k_i\times 2k_j}=\begin{bmatrix} x^{ij}_{1,2}I_2^- & x^{ij}_{1,4}I_2^- & \cdots & x^{ij}_{1,2k_j}I_2^- \\ x^{ij}_{3,2}I_2^- & x^{ij}_{3,4}I_2^- & \cdots & x^{ij}_{3,2k_j}I_2^- \\ \vdots & \vdots & \ddots & \vdots \\ x^{ij}_{(2k_i-1),2}I_2^- & x^{ij}_{(2k_i-1),4}I_2^- & \cdots & x^{ij}_{(2k_i-1),2k_j}I_2^- \end{bmatrix}.$$

For $(i,j)=(7,7)$ *with* $d_7=3$, *it is*

$$X^{ij}_{d_ik_i\times d_jk_j}=X^{ij}_{3k_i\times 3k_j}=\begin{bmatrix} x^{ij}_{1,1}I_3 & x^{ij}_{1,4}I_3 & \cdots & x^{ij}_{1,(3k_j-2)}I_3 \\ x^{ij}_{4,1}I_3 & x^{ij}_{4,4}I_3 & \cdots & x^{ij}_{4,(3k_j-2)}I_3 \\ \vdots & \vdots & \ddots & \vdots \\ x^{ij}_{(3k_i-2),1}I_3 & x^{ij}_{(3k_i-2),4}I_3 & \cdots & x^{ij}_{(3k_i-2),(3k_j-2)}I_3 \end{bmatrix},$$

where $I_2^-=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Proof. From the Definition 1.3, we have $\Xi'_{SL_2(3)}=\{X\in\mathbb{M}_n(\mathbb{C})\,|\,C^t_{\rho(g)}XC_{\rho(g)}=X,\forall g\in SL_2(3)\}$ and $C_{\rho(g)}$ is the matrix representation of the linear operator $\rho(g)=\oplus_{i=1}^7k_i\rho_i(g)$ with respect to the basis $\underline{e}$, then we have

$$C_{\rho(g)}=\begin{bmatrix} C_{k_1\rho_1(g)} & 0 & \cdots & 0 \\ 0 & C_{k_2\rho_2(g)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_{k_7\rho_7(g)} \end{bmatrix}, \textit{ where for } 1\le i\le 7,\ C_{k_i\rho_i(g)}=\begin{bmatrix} C_{\rho_i(g)} & 0 & \cdots & 0 \\ 0 & C_{\rho_i(g)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_{\rho_i(g)} \end{bmatrix}.$$

An element $X\in\mathbb{M}_n(\mathbb{C})$ is invariant under ρ if and only if $C^t_{\rho(g)}XC_{\rho(g)}=X,\forall g\in G$, i.e

$$\begin{bmatrix} C^t_{k_1\rho_1(g)} & 0 & \cdots & 0 \\ 0 & C^t_{k_2\rho_2(g)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C^t_{k_7\rho_7(g)} \end{bmatrix}\begin{bmatrix} X^{1,1} & X^{1,2} & \cdots & X^{1,7} \\ X^{2,1} & X^{2,2} & \cdots & X^{2,7} \\ \vdots & \vdots & \ddots & \vdots \\ X^{7,1} & X^{7,2} & \cdots & X^{7,7} \end{bmatrix}\begin{bmatrix} C_{k_1\rho_1(g)} & 0 & \cdots & 0 \\ 0 & C_{k_2\rho_2(g)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_{k_7\rho_7(g)} \end{bmatrix}$$

$$= \begin{bmatrix} X^{1,1} & X^{1,2} & \cdots & X^{1,7} \\ X^{2,1} & X^{2,2} & \cdots & X^{2,7} \\ \vdots & \vdots & \ddots & \vdots \\ X^{7,1} & X^{7,2} & \cdots & X^{7,7} \end{bmatrix}.$$

The block sub-matrices are conformable partition among the above matrices

$$\begin{bmatrix} C^t_{k_1\rho_1(g)}X^{1,1}C_{k_1\rho_1(g)} & C^t_{k_1\rho_1(g)}X^{1,2}C_{k_2\rho_2(g)} & \cdots & C^t_{k_1\rho_1(g)}X^{1,7}C_{k_7\rho_7(g)} \\ C^t_{k_2\rho_2(g)}X^{2,1}C_{k_1\rho_1(g)} & C^t_{k_2\rho_2(g)}X^{2,2}C_{k_2\rho_2(g)} & \cdots & C^t_{k_2\rho_2(g)}X^{2,7}C_{k_7\rho_7(g)} \\ \vdots & \vdots & \ddots & \vdots \\ C^t_{k_7\rho_7(g)}X^{7,1}C_{k_1\rho_1(g)} & C^t_{k_7\rho_7(g)}X^{7,2}C_{k_1\rho_1(g)} & \cdots & C^t_{k_7\rho_7(g)}X^{7,7}C_{k_7\rho_7(g)} \end{bmatrix} = \begin{bmatrix} X^{1,1} & X^{1,2} & \cdots & X^{1,7} \\ X^{2,1} & X^{2,2} & \cdots & X^{2,7} \\ \vdots & \vdots & \ddots & \vdots \\ X^{7,1} & X^{7,2} & \cdots & X^{7,7} \end{bmatrix}.$$

Equating the corresponding sub-matrices, we have $C^t_{k_i\rho_i(g)}X^{i,j}C_{k_j\rho_j(g)} = X^{i,j}$, $\forall g \in G \,\&\, 1 \leq i, j \leq 7$.

Case 1. When $(i,j) \notin A$, i.e $(\rho_i, \mathbb{V}_{\rho_i})$ and $(\rho_j, \mathbb{V}_{\rho_j})$ are not isomorphic as well as not dual to each other.

By Schur lemma [[1], Ch. 9, p. 326, [12], p. 13], any linear transformation ϕ between the representing spaces $\mathbb{V}_{\rho_i}$ and $\mathbb{V}_{\rho_j}$ for which $\rho_j(g) \circ \phi = \phi \circ \rho_i(g), for\, all\, g \in SL_2(3)$, we have ϕ is identically zero thus the corresponding invariant bilinear form over the spaces $\mathbb{V}_{\rho_i}$ and $\mathbb{V}_{\rho_j}$ is also identically zero. Therefore for $(i,j) \notin A$ we have $X^{i,j} = \mathbb{O}$.

Case 2. When $(i,j) \in A$ and $(\rho_i, \mathbb{V}_{\rho_i})$ is self dual, $(i,j) = (1,1), (4,4), (7,7)$. By Frobenius-Schur indicator [[12], Theorem 31, p. 106], if ρ_i is self dual then a non-degenerate invariant bilinear form exists and it is unique up to homothety [see the definition of homothety in [12], Proposition 38, p. 108]. Two cases arise

Case 2.1. If ρ_i is a real representation, $(i,j) = (1,1)$ and $(7,7)$.
Then non-degenerate symmetric invariant bilinear form exists and it is scalar multiple of I_{d_i} (which is unique up to homothety). Thus we have

$$X^{11} = \begin{bmatrix} x^{11}_{1,1} & x^{11}_{1,2} & \cdots & x^{11}_{1,k_j} \\ x^{11}_{2,1} & x^{11}_{2,2} & \cdots & x^{11}_{2,k_j} \\ \vdots & \vdots & \ddots & \vdots \\ x^{11}_{k_i,1} & x^{11}_{k_i,2} & \cdots & x^{11}_{k_i,k_j} \end{bmatrix} \, and\, X^{77} = \begin{bmatrix} x^{77}_{1,1}I_3 & x^{77}_{1,4}I_3 & \cdots & x^{77}_{1,(3k_j-2)}I_3 \\ x^{77}_{4,1}I_3 & x^{77}_{4,4}I_3 & \cdots & x^{77}_{4,(3k_j-2)}I_3 \\ \vdots & \vdots & \ddots & \vdots \\ x^{77}_{(3k_i-2),1}I_3 & x^{77}_{(3k_i-2),4}I_3 & \cdots & x^{77}_{(3k_i-2),(3k_j-2)}I_3 \end{bmatrix}.$$

Case 2.2. If ρ_i is a quaternionic representation, $(i,j) = (4,4)$.
Then non-degenerate skew-symmetric invariant bilinear form exists and it is scalar

multiple of I_2^- (which is unique up to homothety). Thus we have

$$X^{44} = \begin{bmatrix} x^{44}_{1,2} I_2^- & x^{44}_{1,4} I_2^- & \cdots & x^{44}_{1,2k_j} I_2^- \\ x^{44}_{3,2} I_2^- & x^{44}_{3,4} I_2^- & \cdots & x^{44}_{3,2k_j} I_2^- \\ \vdots & \vdots & \ddots & \vdots \\ x^{44}_{(2k_i-1),2} I_2^- & x^{44}_{(2k_i-1),4} I_2^- & \cdots & x^{44}_{(2k_i-1),2k_j} I_2^- \end{bmatrix}.$$

Case 3. When $(i,j) \in A$ and $(\rho_i, \mathbb{V}_{\rho_i})$ are not self dual, $(i,j) = (2,2), (3,3), (5,5), (6,6)$.
Here representation ρ_i is complex and one of its character value is not real then ρ_i does not exist non-zero invariant bilinear form [[12], Proposition 38, p. 108]. Thus for $(i,j) = (2,2), (3,3), (5,5), (6,6)$, we have $X^{i,j} = 0$.

Case 4. When $(i,j) \in A$, $(\rho_i, \mathbb{V}_{\rho_i})$ and $(\rho_j, \mathbb{V}_{\rho_j})$ are not isomorphic irreducible representations but dual to each other, $(i,j) = (2,3), (3,2), (5,6), (6,5)$.
By Schur lemma [[1], Theorem 9.6, p. 326] a linear transformation $\phi : \mathbb{V}_{\rho_i} \to \mathbb{V}_{\rho_j}$ for which $\rho_j(g) \circ \phi = \phi \circ \rho_i(g), for\, all\, g \in SL_2(3)$ is either an isomorphism or $\phi = 0$. Frobenius-Schur [[12], Proposition 38, p. 108] can not apply here due to the fact that $(\rho_i, \mathbb{V}_{\rho_i})$ and $(\rho_j, \mathbb{V}_{\rho_j})$ are non-isomorphic and dual to each other. From subsection 2.1, for $(i,j) = (2,3), (3,2), (5,6), (6,5)$, we have a non-zero invariant bilinear form corresponding to ϕ which is non-degenerate symmetric I_1 and skew-symmetric I_2^- according as $(i,j) = (2,3), (3,2)$ and $(i,j) = (5,6), (6,5)$ respectively (unique up to homothety).
Thus for $(i,j) = (2,3), (3,2)$ it is

$$X^{ij} = \begin{bmatrix} x^{ij}_{1,1} & x^{ij}_{1,2} & \cdots & x^{ij}_{1,k_j} \\ x^{ij}_{2,1} & x^{ij}_{2,2} & \cdots & x^{ij}_{2,k_j} \\ \vdots & \vdots & \ddots & \vdots \\ x^{ij}_{k_i,1} & x^{ij}_{k_i,2} & \cdots & x^{ij}_{k_i,k_j} \end{bmatrix},$$

and for $(i,j) = (5,6), (6,5)$ we have

$$X^{ij} = \begin{bmatrix} x^{ij}_{1,2} I_2^- & x^{ij}_{1,4} I_2^- & \cdots & x^{ij}_{1,2k_j} I_2^- \\ x^{ij}_{3,2} I_2^- & x^{ij}_{3,4} I_2^- & \cdots & x^{ij}_{3,2k_j} I_2^- \\ \vdots & \vdots & \ddots & \vdots \\ x^{ij}_{(2k_i-1),2} I_2^- & x^{ij}_{(2k_i-1),4} I_2^- & \cdots & x^{ij}_{(2k_i-1),2k_j} I_2^- \end{bmatrix}.$$

Combined all above cases if $X \in \Xi'_G$ then

$$X^{i,j} = \begin{cases} X^{ij}_{d_ik_i \times d_jk_j}, & \text{if } (i,j) \in \text{A} \\ 0, & \text{if } (i,j) \notin A. \end{cases}$$

This completes the proof of theorem.

Corollary 3.1. *$X \in \Xi'_G$, is invariant bilinear form under ρ if and only if $X^{ij}_{d_ik_i \times d_jk_j} = C^t_{k_i\rho_i(g)} X^{ij}_{d_ik_i \times d_jk_j} C_{k_j\rho_j(g)}$, $\forall g \in G$, for every $(i,j) \in A$.*
Proof. This consequence is easy to see from the proof of Theorem 3.1.

3.1. Characterization of Invariant Bilinear forms under an n Degree Representation of $SL_2(3)$

Lemma 3.1. *If $X \in \Xi'_{SL_2(3)}$ is non-singular then $k_2 = k_3$ and $k_5 = k_6$.*
Proof. Suppose X is non-singular then rows or columns of X are linearly independent, so sub-matrix $X^{ij}_{d_ik_i \times d_jk_j}$ is a non-singular for $(i,j) \in (2,3)\,\&\,(5,6)$, this completes the proof.

Note that the converse part of Lemma 3.1 is not true as a square sub-matrix of X may be singular.

Lemma 3.2. *If $X \in \Xi'_G$ with $k_2 = k_3$, $k_5 = k_6$, then for $(i,j) \in A$, $X^{ij}_{d_ik_i \times d_jk_j}$ is a non-singular sub-matrix, if and only if X is non-singular.*
Proof. With reference to the Theorem 3.1, for every $X \in \Xi'_G$, we have

$$X = \begin{bmatrix} X^{11}_{k_1 \times k_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & X^{23}_{k_3 \times k_2} & 0 & 0 & 0 & 0 \\ 0 & X^{32}_{k_2 \times k_3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X^{44}_{2k_4 \times 2k_4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & X^{56}_{2k_5 \times 2k_5} & 0 \\ 0 & 0 & 0 & 0 & X^{65}_{2k_5 \times 2k_5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & X^{77}_{3k_7 \times 3k_7} \end{bmatrix}.$$

Suppose X is non-singular then rows of X are linearly independent, as well as $k_2 = k_3$ and $k_5 = k_6$, this shows $X^{ij}_{d_ik_i \times d_jk_j}$ is non-singular for $(i,j) \in A$.

Converse part: Since $k_2 = k_3$, $k_5 = k_6$ and $X^{ij}_{d_ik_i \times d_jk_j}$ is non-singular for $(i,j) \in A$ this implies that rows (columns) of X are linearly independent.

For proving the next lemma we will choose only those $X \in \mathbb{M}_n(\mathbb{C})$ whose all block sub-matrices X^{ij} are zero except $(i,j) \in A$ and from Theorem 3.1, the block sub-matrix $X^{ij} = X^{ij}_{d_ik_i \times d_jk_j}$ is non-singular.

Lemma 3.3. *For $n \in \mathbb{Z}^+$, every n degree representation of $SL_2(3)$ has a non-degenerate invariant bilinear form if and only if $k_2 = k_3$ and $k_5 = k_6$.*

Proof. From equation (2.2) we have $k_1 + k_2 + k_3 + 2k_4 + 2k_5 + 2k_6 + 3k_7 = n$ and $X \in \mathbb{M}_n(\mathbb{C})$ such that

$$X = \begin{bmatrix} X^{11}_{k_1\times k_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & X^{23}_{k_2\times k_3} & 0 & 0 & 0 & 0 \\ 0 & X^{32}_{k_3\times k_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X^{44}_{2k_4\times 2k_4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & X^{56}_{2k_5\times 2k_6} & 0 \\ 0 & 0 & 0 & 0 & X^{65}_{2k_6\times 2k_5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & X^{77}_{3k_7\times 3k_7} \end{bmatrix}.$$

Suppose $k_2 = k_3$ & $k_5 = k_6$, then for every $(i,j) \in A$, the block sub-matrix $X^{ij}_{d_ik_i \times d_jk_j}$ of X can be chosen (from the Theorem 3.1) to be non-singular with $X^{ij}_{d_ik_i \times d_jk_j} = C^t_{k_i\rho_i(g)} X^{ij}_{d_ik_i \times d_jk_j} C_{k_j\rho_j(g)}$, $\forall g \in G$. This implies that rows (columns) of X are linearly independent. Therefore $X \in \Xi'_G$ and is non-singular.

Converse part: Suppose X is a non-degenerate invariant bilinear form of n degree representation of $SL_2(3)$, then from Lemma 3.1 we have $k_2 = k_3$ & $k_5 = k_6$.

Remark 3.1. *Since $\mathbb{C}$ contains infinitely many non zero elements, hence if there is one non-degenerate invariant bilinear form in the space Ξ_G, it has infinitely many.*

Thus from Lemma 3.3, we find that n degree representation of $SL_2(3)$ consists of a non-degenerate invariant bilinear form.

Lemma 3.4. *Let $G = SL_2(3)$ and $\rho = \oplus_{i=1}^7 k_i\rho_i$ be an n degree representation of G, then ρ has only degenerate invariant bilinear forms if and only if either $k_2 \neq k_3$*

or $k_5 \neq k_6$.

Proof. Its proof is obvious and easy to see.

Definition 3.1. *The space Ξ_G of invariant bilinear forms is called degenerate if it's all elements are degenerate.*

4. Dimension of the Space of Invariant Bilinear forms under a Representation of the Group $SL_2(3)$

The space of invariant bilinear forms under an n degree representation is generated by the finitely many vectors so its dimension is always finite. In this section we will give a formula for computation of this dimension over $\mathbb{C}$.

Theorem 4.1. *If Ξ_G is the space of invariant bilinear forms under an n degree representation $\rho = \oplus_{i=1}^{7} k_i\rho_i$ of $SL_2(3)$, then $dim(\Xi_G) = 2k_2k_3 + 2k_5k_6 + \sum_{i=1}^{3} k_{3i-2}^2$.*

Proof. For every $X \in \Xi'_G$, we have

$$X = \begin{bmatrix} X^{11}_{k_1\times k_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & X^{23}_{k_2\times k_3} & 0 & 0 & 0 & 0 \\ 0 & X^{32}_{k_3\times k_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X^{44}_{2k_4\times 2k_4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & X^{56}_{2k_5\times 2k_6} & 0 \\ 0 & 0 & 0 & 0 & X^{65}_{2k_6\times 2k_5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & X^{77}_{3k_7\times 3k_7} \end{bmatrix}.$$

with $X^{ij}_{d_ik_i\times d_jk_j} = C^t_{k_i\rho_i(g)} X^{ij}_{d_ik_i\times d_jk_j} C_{k_j\rho_j(g)}, \forall g \in G$, for $(i,j) \in A$ and to generate these sub-matrices from Theorem 3.1 it needs k_ik_j vectors from $\mathbb{M}_{d_ik_i\times d_jk_j}(\mathbb{C})$. This completes the proof.

Corollary 4.1. *The space of invariant symmetric bilinear forms under an n degree representation $\rho = \oplus_{i=1}^{7} k_i\rho_i$ of $SL_2(3)$ has dimension $= k_2k_3 + k_5k_6 + \frac{k_1(k_1+1)}{2} + \frac{k_4(k_4-1)}{2} + \frac{k_7(k_7+1)}{2}$.*

Proof. The proof is obvious from the Theorem 3.1 and proof of Theorem 4.1.

Corollary 4.2. *The space of invariant skew-symmetric bilinear forms under an n degree representation $\rho = \oplus_{i=1}^{7} k_i\rho_i$ of $SL_2(3)$ has dimension $= k_2k_3 + k_5k_6 +$*

$\frac{k_1(k_1-1)}{2}+\frac{k_4(k_4+1)}{2}+\frac{k_7(k_7-1)}{2}$.

Proof. The proof is obvious from the Theorem 3.1 and the proof of Theorem 4.1.

5. Main Results

In this section, we are given the proofs of the two main theorems.

Theorem 5.1. *For $G = SL_2(3)$, the space Ξ_G, under an n degree representation $(\rho, \mathbb{V})$ over $\mathbb{C}$ is isomorphic to the direct sum of the sub-spaces $\mathbb{W}_{(i,j)\in A}$ of $\mathbb{M}_n(\mathbb{C})$, i.e., $\Xi'_G = \bigoplus_{(i,j)\in A} \mathbb{W}_{(i,j)\in A}$. Where $A = \{(i,j)\,|\,\rho_i$ and ρ_j dual to each other$\}$ and $\mathbb{W}_{(i,j)\in A} = \{X \in \mathbb{M}_n(\mathbb{C})\,|$ only non-zero sub-matrix is X^{ij} of order $d_ik_i \times d_jk_j$ satisfying $X^{ij} = C^t_{k_i\rho_i(g)} X^{ij} C_{k_j\rho_j(g)}$, $\forall g \in SL_2(3)\}$. Also over $\mathbb{C}$, the dimension of $\mathbb{W}_{(i,j)\in A} = k_ik_j$.*

Theorem 5.2. *An n degree representation $\rho = \oplus_{i=1}^{7} k_i\rho_i$ of $SL_2(3)$ admits a non-degenerate invariant bilinear form if and only if the multiplicity of irreducible representation ρ_i having one of the values of its character is not real, and is equal to its dual multiplicity.*

Proof of Theorem 5.1. Let X be an element of Ξ'_G then we have $C^t_{\rho(g)} X C_{\rho(g)} = X$ *and*

$$X = \begin{bmatrix} X^{11}_{k_1\times k_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & X^{23}_{k_2\times k_3} & 0 & 0 & 0 & 0 \\ 0 & X^{32}_{k_3\times k_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X^{44}_{2k_4\times 2k_4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & X^{56}_{2k_5\times 2k_6} & 0 \\ 0 & 0 & 0 & 0 & X^{65}_{2k_6\times 2k_5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & X^{77}_{3k_7\times 3k_7} \end{bmatrix}.$$

Existence:
Let $X \in \Xi'_G$ then for $(i,j) \in A$, there exists at least one $X_{(i,j)} \in \mathbb{W}_{(i,j)\in A}$, such that $\sum_{(i,j)\in A} X_{(i,j)} = X$.
Uniqueness:
For $(i,j) \in A$, suppose there exists $Y_{(i,j)} \in \mathbb{W}_{(i,j)\in A}$, such that $\sum_{(i,j)\in A} Y_{(i,j)} = X$, then $\sum_{(i,j)\in A} X_{(i,j)} = \sum_{(i,j)\in A} Y_{(i,j)}$ i.e., $Y_{(i',j')} - X_{(i',j')} = \sum_{(i,j)\neq(i',j')}(X_{(i,j)} -$

$Y_{(i,j)}$). Therefore $Y_{(i',j')} - X_{(i',j')} \in \sum_{(i,j)\neq(i',j')} \mathbb{W}_{(i,j)\in A}$ hence $Y_{(i',j')} - X_{(i',j')} = 0$ or $Y_{(i',j')} = X_{(i',j')}$ for all $(i', j') \in A$.
Thus we have

$$\Xi'_G = \oplus_{(i,j)\in A}\mathbb{W}_{(i,j)\in A} \; and \; dim(\Xi'_G) = \sum_{(i,j)\in A} dim(\mathbb{W}_{(i,j)\in A}). \tag{5.1}$$

Now from Theorem 3.1, $\mathbb{W}_{(i,j)\in A} = \{X \in \mathbb{M}_n(\mathbb{C}) \,|\,$ the only non-zero block sub-matrix $X^{ij} = X^{ij}_{d_ik_i\times d_jk_j}$ satisfying $X^{ij} = C^t_{k_i\rho_i(g)}X^{ij}C_{k_j\rho_j(g)}, \; \forall g \in G$ and rest blocks are zero$\}$, also we see that for $(i, j) \in A$, the sub-matrices $X^{ij} = X^{ij}_{d_ik_i\times d_jk_j}$ in $\mathbb{W}_{(i,j)\in A}$ have k_ik_j free variables & $\mathbb{W}_{(i,j)\in A} \cong \mathbb{M}_{k_i\times k_j}(\mathbb{C})$. Thus $\Xi'_G \cong \oplus_{(i,j)\in A}\mathbb{M}_{k_i\times k_j}(\mathbb{C})$ and $dim(\mathbb{W}_{(i,j)\in A}) = k_ik_j$.

Thus substituting this in equation (5.1) we get the dimension of Ξ'_G.

Proof of Theorem 5.2. From the subsection 2.1, for $i = 2, 3, 5, 6$ we see that one of the values of χ_i of ρ_i is not real. Also we have the dual of ρ_2 and ρ_5 are ρ_3 and ρ_6 respectively, their corresponding well defined multiplicities in ρ are k_2, k_3, k_5, k_6, so enough to prove $\Xi_{SL_2(3)}$ admits non-degenerate invariant bilinear form if and only if $k_2 = k_3$ & $k_5 = k_6$. Hence, the proof is completes from Lemmas 3.1 to 3.3.

Remark 5.1. *Thus we get the necessary and sufficient condition for the existence of a non-degenerate invariant bilinear form under an n degree representation of a* $SL_2(3)$ *group over* $\mathbb{C}$ *(in particular over the cyclotomic number field* $\mathbb{Q}(\zeta_{12})$*, with* $\zeta_{12} = e^{2\pi i/12}$*).*

5.1. Space of Degenerate Invariant Bilinear forms

From Theorem 5.2 and Lemma 3.3, for every $n \in \mathbb{Z}^+$, an n degree representation of $SL_2(3)$ has a non-degenerate invariant bilinear form if and only if $k_2 = k_3$ & $k_5 = k_6$. Implies whenever $k_2 \neq k_3$ or $k_5 \neq k_6$ then every element in the invariant space $\Xi_{SL_2(3)}$ is degenerate.

Thus here we have completely characterized the representations of $SL_2(3)$ to admit a non-degenerate invariant bilinear form over complex field.

Remark 5.2. *With reference to the Remarks 2.1 and 2.2, we expect that all the results also hold equally good when considered over either an algebraically closed field with characteristic* $\neq 2, 3$ *or a field of characteristic* $\equiv 1 \pmod{12}$.

Acknowledgment

The first author would like to thank UGC, India for providing the research fellowship (Grant no. 19/06/2016(i)EU-V).

References

[1] Artin M., Algebra, First Edition, Prentice Hall Inc., (1991), 307-344.

[2] Chen Y., Matrix representations of the real numbers, Linear Algebra Appl., 536 (2018), 174-185.

[3] Frobenius G., Uber die mit einer Matrix vertauschbaren matrizen, Sitzungsber. Preuss. Akad. Wiss., (1910), 3–15.

[4] Gongopadhyay K. and Kulkarni R. S., On the existence of an invariant non-degenerate bilinear form under a linear map, Linear Algebra Appl., 434(1) (2011), 89-103.

[5] Gow R. and Laffey T. J., Pairs of alternating forms and products of two skew-symmetric matrices, Linear Algebra Appl., 63 (1984), 119–132.

[6] Gongopadhyay K., Mazumder S. and Sardar S. K., Conjugate Real Classes in General Linear Groups, Journal of Algebra and Its Applications, 18(03) (2019), 1950054.

[7] Hoffman K. and Kunze R., Linear Algebra, First Edition, Prentice Hall Inc, (1961).

[8] Kulkarni R. S. and Tanti J., Space of invariant bilinear forms, Indian Academic of sciences, 128(4) (2018), 47.

[9] Lang S., Algebra, Third edition, Springer-Verlag, (2002).

[10] Pazzis C. H., When does a linear map belong to at least one orthogonal or symplectic group? Linear Algebra Appl., 436(5) (2012), 1385–1405.

[11] Stenzel H., Uber die Darstellbarkeit einer Matrix als Produkt von zwei symmetrischer matrizen, als Produkt von zwei alternierenden matrizen und als Produkt von einer symmetrischen und alternierenden matrix, Mat. Zeitschrift, 15 (1922), 1–25.

[12] Serre J. P., Linear representations of finite groups, Springer-Verlag, (1977).

[13] Sergeichuk V. V., Classification problems for systems of forms and linear map, Izv. Akad. Nauk SSSR Ser. Mat., 51(6) (1987), 1170–1190.