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# REPRESENTATION OF COVERING SPACES IN ALGEBRAIC TOPOLOGY

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**Abstract:** In this paper introduction to the covering spaces is given. We have proved some theorems and problems for the covering spaces. Examples related to the covering spaces are also discussed.

**Keywords and Phrases:** Homeomorphism, Evenly covered, Covering map or projection, Covering spaces, Local homeomorphism.

**2020 Mathematics Subject Classification:** 57M10, 57S05, 18F60, 54C05, 54C10, 97F50.

## 1. Introduction

The theory of covering spaces is one of the important topic in Algebraic Topology. This is common stage for the development of various branches of mathematics. Such as Differential geometry, The theory of Lie groups, and the theory Riemann surfaces [4]. In which the base space is an arbitrary topological space. For any even subspace of $\mathbb{R}^n$ has a trivial fundamental group. One of the most useful tool for some fundamental groups that are not trivial [3]. Let B be a topological space, a covering space of B consists of a space A and a continuous map $\phi$ of A onto B which satisfies a certain requirement [4]. Covering space is a pair of topological space $\phi$ : A $\rightarrow$ B, where A and B are having a simple relation to each other. The basic concepts with theorem, problems and examples for the covering space and non-covering space are discussed in this paper.

## 2. Preliminaries

In this section, some definitions, examples and remarks are discussed.

**Definition 2.1. (Homeomorphism)** *A homeomorphism is called a continuous transformation, is an equivalence relation and 1-1 correspondence between the two topological spaces that is continuous in both directions. Homeomorphism is a function $\phi : A \rightarrow B$ that is bijective, continuous and has a continuous inverse. If homeomorphism exists then A and B are homeomorphic.*

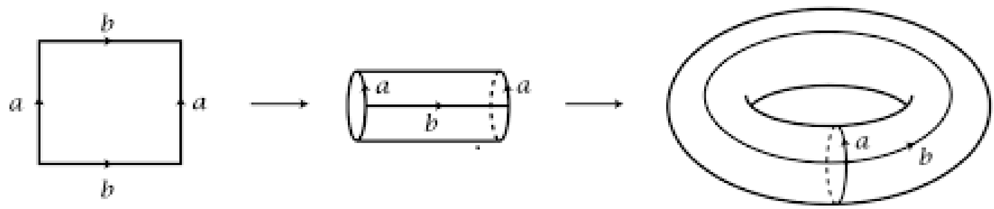


Figure 1

**Remarks 2.2.**

1. *Continuous: Preimage of open set is open.*

2. *Equivalence relation: are topologically equivalent $A \approx B$.*

**Definition 2.3. (Evenly Covered)** *Let A and B be topological spaces. Let $\phi$ : A $\rightarrow$ B be a continuous and surjective map. The open set $\lambda$ of B is said to be evenly covered by $\phi$. If the image $\phi^{-1}(\lambda)$ is a union of disjoint open sets $\mu_\alpha$ in A, for each $\alpha$ the restriction of $\phi$ to $\mu_\alpha$ is a homeomorphism of $\mu_\alpha$ onto $\lambda$ by $\phi$.*

**Definition 2.4. (Covering Space)** *The map $\phi$ is a covering map. If $\phi$ is continuous maps A onto B. (i.e.) $\phi : A \rightarrow B$. If every point$\beta$ of B has an open neighborhood $\lambda$ that is evenly covered by $\phi$. (i.e.) $\phi^{-1}(\lambda) \subseteq A$ is a disjoint union spaces, each one is homeomorphic to $\lambda$ under $\phi$. And A is said to be a covering space of B. (OR) (A, $\phi$) is covering space. It's simply called cover.*

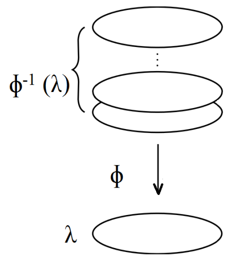


Figure 2

**Remark 2.5.** *If $\phi : A \to B$ is a covering map then $\phi$ is open map for each $\beta \in B$. The subspace $\phi^{-1}(\beta)$ of $A$ has the discrete topology for each $\mu_\alpha$ is open in $A$ and intersects the set $\phi^{-1}(\beta)$ in a single point. This point is open in $\phi^{-1}(\beta)$.*

**Examples 2.6. (Covering Spaces)**

1. Every space trivially covers itself.

2. Consider the map $\phi : \mathbb{R}^2 \to S^1 \times S^1$ plane projection to the tours. Taking any two points (x, y) in $\mathbb{R}^2$, the equation (x, y) $\to$ (exp($2\pi$ix), exp($2\pi$iy)). Now taking the inverse image, the inverse image of this tours would be infinitely many rectangles, but call the would be countable. Covering projection always has $\phi^{-1}(\lambda)$, where $\phi$ is projection. So we have the inverse image is countably many infinite rectangles.

3. Standard trivial example, should take the identity map from (A $\times$ A), let (i : A $\to$ A), let A be any space, then i is a covering map.

4. The infinite spiral is projecting down onto a circle.

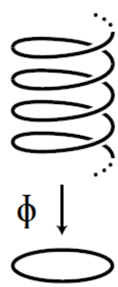


Figure 3

**Examples 2.7. (Non-Covering Spaces)**

1. The finite spiral to the circle, this is not a covering spaces. Because the end points create a problem, end points do not have a neighborhood. If take it's an inverse image $\lambda$ in the space B. This is not a homeomorphic in the covering space. So inverse image is not a homeomorphism.

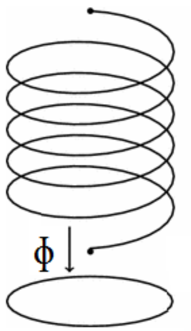


Figure 4

2. The rectangle to the line, take an open interval $\lambda$ on the real line and corresponding to the inverse image $\phi^{-1}(\lambda)$ is forms a rectangle. Also it's not a covering space, because we do not have a homeomorphism from the open set $\lambda$ to $\phi^{-1}(\lambda)$

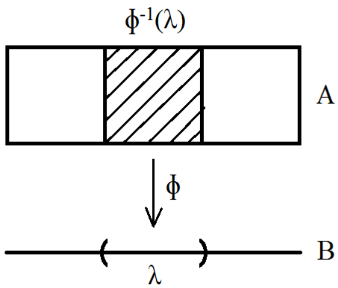


Figure 5

**Definition 2.8. (Local Homeomorphism)** *If $\phi : A \to B$ is a covering map, then $\phi$ is a local homeomorphism of A with B. That is each point of $\alpha$ of A has a open neighborhood that is mapped homeomorphically by $\phi$ onto an open subset $\lambda$ of B.*

## 3. Representation of Covering Spaces

In this section, theorem, lemmas and some problems are discussed.

**Theorem 3.1.** *The map $\phi : S^2 \to P^2$, sphere to the projective plane is covering map.*

**Proof.** $\phi : S^2 \to P^2$ is defined by a pair of antipodal points $\alpha$ and $\bar{\alpha}$, that's $\phi(\alpha) = (\alpha, \bar{\alpha}); \alpha, \bar{\alpha} \in S^2$. Thus $\phi(\alpha)$ is one dimensional subspace containing $\alpha$. This is going to be a two to one map, because the image of $\alpha$ is equals to the image of $\bar{\alpha}$. (i.e.) $\phi(\alpha) = \phi(\bar{\alpha})$. Both $\alpha$ and $\bar{\alpha}$ gets to same pair of points in the projective plane $P^2$. By the covering condition, we have, any elements in the projective plane. Take antipodal points are $\alpha$ and $\bar{\alpha}$ of $P^2$ exists in the neighborhood $\lambda$ then around it's inverse image $\phi^{-1}(\lambda)$ under the projection consist of two same around $\alpha$ and $\bar{\alpha}$ in $S^2$. And each them individually in $S^2$ looks exactly original neighborhood in the projection plane. So it's covering map.

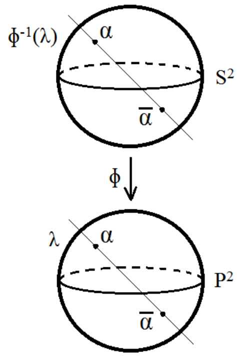


Figure 6

**Lemmas 3.2.**

1. *The composition of two local homeomorphism is a local homeomorphism.*

2. *Every homeomorphism is also a local homeomorphism.*

3. *All local homeomorphisms are need not be covering maps.*

*4. Every local homeomorphism is a continuous and open map.*

*5. Products of covering spaces are covering spaces.*

*6. Every covering map is a local homeomorphism.*

**Problem 3.3.** *Show that $\phi : (A \times B) \to A$ is map on the first co-ordinate. Let $B$ have the discrete topology, then $\phi$ is a covering map, also $\phi$ is a local homeomorphism.*

**Solution.** We know that, $\phi$ is continuous, for any open set $\lambda \in B, \phi^{-1}(\lambda) = (\lambda \times \mu \mid \mu \subseteq B)$. Since B has a discrete topology, then all $\mu \subseteq$ B are open and $\phi^{-1}(\lambda)$ is open. Thus $\phi$ is continuous surjective map. So we will show that A, itself is evenly covered, to prove $\phi$ is a covering map. We can write $\phi^{-1}(A)$ as the disjoint union of open sets $\mu_\beta =$ (A $\times$ B) for all $\beta \in$ B. $\phi$ is restricted to $\mu_\beta$ is already surjective continuous map. $\phi^{-1}$ is continuous for each $\mu_\beta$ to get homeomorphism. Let $(\alpha_1,\ \beta)$ $(\alpha_2,\ \beta) \in$ (A $\times$ B) with $\phi(\alpha_1,\ \beta) = \phi(\alpha_2,\ \beta) \Rightarrow (\alpha_1 = \alpha_2)$. This gives us $(\alpha_1,\ \beta) = (\alpha_2,\ \beta)$ finally for any open set we have the pre-image to be the open set $\lambda$. So $\phi^{-1}$ is continuous the restriction of $\phi$ to $\mu_\beta$ is a homeomorphism of $\mu_\beta$ onto A. The collection of $\{\mu_\beta\}$ will be called a partition of $\phi^{-1}(A)$. So $\phi$ is a covering map. By the definition, if $\phi :$ (A$\times$B) $\to$ A is a covering map, then $\phi$ is a local homeomorphism of (A$\times$B) has a neighborhood with A. That is each point of $(\alpha\times\beta)$ of (A$\times$B) has a neighborhood, that is mapped homeomorphically by $\phi$ onto an open subset of A. So the map $\phi$ is local homeomorphism.

**Problem 3.4.** *Let $\phi_1 : A \to B$ and $\phi_2 : B \to \Gamma$ be the covering maps, let $\phi = (\phi_2 \circ \phi_1)$ . Show that $\phi_2^{-1}(\Gamma)$ is finite for each $\gamma \in \Gamma$ then $\phi$ is a covering map.*

**Solution.** Given that $\phi_1$ and $\phi_2$ be the covering maps, let $\phi = (\phi_2 \circ \phi_1)$. We have to prove that, $\phi_2^{-1}(\Gamma)$ is finite for each $\gamma \in \Gamma$ then $\phi$ is a covering map. Let A, B and $\Gamma$ be the topological spaces.

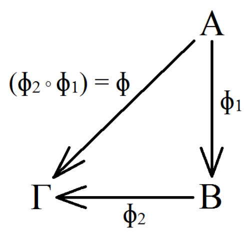


Figure 7

By the definition, $\phi_1$ and $\phi_2$ and are continuous and surjective maps. If every points of both $\beta \in \lambda$ and $\gamma \in \mu$ that is evenly covered by $\phi_1$ and $\phi_2$. The inverse image of $\phi_1^{-1}(\lambda)$ and $\phi_2^{-1}(\mu)$ as the union of disjoint open sets $\phi_1$ to $\alpha_i$ and $\phi_2$ to $\beta_i$ is a homeomorphisms of $\phi_1$ onto $\lambda$ and $\phi_2$ onto $\mu$.Now we have to show that, $\phi$ is a covering map. It's enough to prove that, $\phi_2^{-1}(\Gamma)$ is finite for each $\gamma \in \Gamma$. As $\phi_2$ is a covering map, let choose $\gamma \in \Gamma$ be an open neighborhood $\Gamma$ of $\gamma$. Such that $\phi^{-1}(\Gamma)$ union of disjoint open sets $(\beta_i)$ is a homeomorphism of $\beta_i$ onto $\lambda$. Define $\beta_i$ to be the single element of $(\phi^{-1}(\gamma) \cap \beta_i)$ for each i. Also $\phi_1$ is a covering map, we can find an open neighborhood $(\alpha_i)$ of $(\beta_i)$, such that $\phi_1^{-1}(\alpha_i) = \bigcup_i \nu_{in}$, we need common open set $\tau$. So define $\tau = \bigcap_{i=1}^{n} \phi_2\ (\alpha_i \cap \beta_i)$, each $(\alpha_i \cap \beta_i)$ is a neighborhood B which is evenly covered by $\phi_1$ and has $\phi_2\ (\alpha_i \cap \beta_i) \subseteq \Gamma$. $\tau$ So is open and evenly covered by $\phi_2$ (every partition is finite $(\alpha_i \cap \beta_i)$). Next define, $\psi_{in} = \phi^{-1}(\alpha_i \cap \beta_i)$ $\cap\ \nu_{in}$ is open. If every point $\gamma \in \Gamma$ has a neighborhood $\tau$ that is evenly covered by $\phi$ for which the inverse image $\phi^{-1}(\tau)$ is union of disjoint open sets. $\psi_{in}$ And the restriction of $\phi$ to $\psi_{in}$ is a homeomorphism of $\psi_{in}$. Hence $\psi$ is covering map.

**Problem 3.5.** *Show that the map $\phi : S^1 \to S^1$ given by the equation $\phi(Z) = Z^2$ is a covering map. Generalize to the map $\phi(Z) = Z^n$.*

**Solution.** Given map $\phi : S^1 \to S^1$ given by the equation $\phi(Z) = Z^2$ is a covering map. Here we consider $S^1 \subset \mathbb{C}$, $|Z|$ =1. The unit circle can be considered as the set of complex numbers Z. The function $Z = e^{i\theta}$ for $\theta = arg(Z) \in [0, 2\pi); Z \in S^1$. Let Z = $e^{i\theta}$, $\phi(Z) = Z^2 = (e^{i\theta})^2 = e^{i2\theta}$, $\phi$ is continuous and surjective function. Let $\lambda$ be the image of $(\theta - \frac{\pi}{2},\ \theta + \frac{\pi}{2})$, under the map $\theta \to e^{i\theta}$. So that $\lambda$ is the open semicircle centered at Z and $\lambda$ is an open neighborhood of Z, $\phi^{-1}(\lambda) = \bigcup_{i=1}^{2} \mu_i$ where the intersection of $\mu_i$'s are empty and $\mu_1$, $\mu_2$ are open. Now $\phi^{-1}(\lambda)$ consist of the quarter circle with centered at (Z) $\Rightarrow \mu_1 = exp(\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2})$, Centered at (-Z) $\Rightarrow \mu_2 = exp(-\theta + \frac{\pi}{2}, -\theta - \frac{\pi}{2})$. Clearly, $\lambda = (\mu_1 \cup \mu_2)$. Since the functions are continuous in $\mathbb{C}$ at everywhere, except at Z = 0. But the space $S^1$ does not contain the point Z at 0. By the definition, the restriction of $\phi$ to $\mu_1$ and $\mu_2$ are homeomorphisms of $\mu_1$ and $\mu_2$ onto $\lambda$. (i.e.), $\lambda$ is homeomorphic $\mu_1$ and $\mu_2$ to by $Z = Z^2$, therefore $\phi$ is covering map. In general, $\phi^{-1}(\lambda) = (\mu_1 \cup \mu_2 \cup \mu_3 ... \cup \mu_n)$, n disjoint open sets. Here the intersection of $\mu_i$'s are empty and open. The restriction of $\phi$ to $\bigcup_{i=1}^{n} \mu_i$ is homeomorphism of $\bigcup_{i=1}^{n} \mu_i$ onto $\lambda$ by $Z = Z^n$, therefore $\phi$ is covering map.

## 4. Conclusion

In this paper, the basic definitions required to slove the problem and theorems are mentioned. It is also discussed with samples for the covering space and non-covering space. Following this, the covering space is represented by some problems and theorem using the basic definitions.

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