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A FUZZY BRK TOPOLOGICAL ACTION ON HOMOMORPHISM FUNCTION UNDER SUBGROUP

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Abstract: In this paper, we show how to extend fuzzy BRK topological action to a homomorphic function under subgroup S. Some theorems and properties of a homomorphism in $fBRKtA$ on subgroup are also discussed.

Keywords and Phrases: $fBRKtg$, $fBRKtA$, $fBRKtAS_g$.

2020 Mathematics Subject Classification: 57M10, 54C05.

1. Introduction and Preliminaries

Zadeh [17] introduced the idea of a fuzzy set, which provides a general topology known as fuzzy topological spaces. Foster's structure of a fuzzy topological space combined with a fuzzy community [3]. The elements of a theory of fuzzy topological groups have been formulated by Rosenfeld [10]. The meaning was modified by Ma [8] and Ya [16] to ordinary topological group is a special case of a ftg. Bandaru introduced BRK-algebra in 2012, which is a generalisation of the $BCK/BCI/BCH/Q/QS/BM$-algebras [5, 6, 7, 9]. Sivakumar et al. proposed a topology for the BRK-algebra [11] and investigated its properties. From an algebraic aspect, Haddadi [4], Roventa [10], and Spircu [10] investigate fuzzy behaviour of fuzzy submonoids and fuzzy subgroups. Boixader et al. [2]. In the paper [13], the fuzzy behaviours in BRK-topological spaces are clarified, and $fBRKtA$ is extended to a subgroup S. Here, we study about a homomorphic function in $fBRKtA$ of subgroup S and their properties.

From 2019 and 2020, Sivakumar et al.; Sivakumar and Kousalya (2020) and (2021); Sivakumar et al. (year of publication) introduced and studied topological BRK algebras, fuzzy topological BRK subalgebra, fuzzy BRK topological action and their respective subgroups and fuzzy topological BRK groups in topological and fuzzy topological algebras.

Definition 1.1. [15] *Let G_{BRK} be a group and $(G_{BRK}, \star, 0, \Gamma)$ be a $fBRKts$. Then $(G_{BRK}, \star, 0, \Gamma)$ is called fuzzy BRK topological group (briefly, $fBRKtg$) if the maps*

$$g : (G_{BRK} \times G_{BRK}, \Gamma \times \Gamma) \to (G_{BRK}, \star, 0, \Gamma) \text{ defined by } g(u, v) = u \star v$$

and

$$h : (G_{BRK}, \star, 0, \Gamma) \to (G_{BRK}, \star, 0, \Gamma) \text{ defined by } h(u) = u^{-1}$$

are $fBRK$ Cts.

2. Homomorphic Properties of $fBRKtAS_g$

Definition 2.1. *Let θ be a mapping from I to J.*

(i) Let ${G_{BRK}}'$ be a $fBRKtg$ $({G_{BRK}}', \star, 0, \Gamma)$ on F $fBRKtA$ under $(S_{BRK}, \star, 0, \Gamma)$. Then the inverse image of F under θ denoted by $\theta^{-1}(F)$ is a $fBRKtg$ in $(G_{BRK}, \star, 0, \Gamma)$ defined by $\theta^{-1}(F) = \mu_{\theta^{-1}(F)}$ where $\mu_{\theta^{-1}(F)}(i) = \mu_F(\theta(i))$;

(ii) Let G_{BRK} be a $fBRKtg$ $(G_{BRK}, \star, 0, \Gamma)$ on E $fBRKtA$ under $(S_{BRK}, \star, 0, \Gamma)$. Then the image of E under θ denoted by $\theta(E)$, where

$$\mu_{\theta(E)}(j) = \begin{cases} \sup \mu_E(i) : i \in \theta^{-1}(j) & \text{if } \theta^{-1}(j) \neq 0; \\ 0 & \text{, otherwise.} \end{cases}$$

Then $\mu_{\theta^{-1}(F)}(i \star s) = \mu_F(\theta(i) \star s)$ $\forall s \in S_{BRK}$. Also

$$\mu_{\theta(E)}(j \star s) = \begin{cases} \sup \mu_E(i \star s) : (i \star s) \in (\theta^{-1}(j) \star s) & \forall s \in S_{BRK} \text{ if } \theta^{-1}(j) \neq 0; \\ 0 & \text{, otherwise.} \end{cases}$$

Theorem 2.1. *Let* $({G_{BRK}}', \star, 0, \Gamma)$ *be a* $fBRKtg$ *on* F $fBRKtA$ *under* $(S_{BRK}, \star, 0, \Gamma)$ *and* $\theta : (G_{BRK}, \star, 0, \Gamma) \to ({G_{BRK}}', \star, 0, \Gamma)$ *be an onto homomorphism on groups. Then* $fBRKtg$ $(G_{BRK}, \star, 0, \Gamma)$ *acts on the* $fBRKtg$ $\theta^{-1}(F)$ *under* $(S_{BRK}, \star, 0, \Gamma)$.

Proof. There exists a map $\mu' : ({G_{BRK}}', \star, 0, \Gamma) \times (S_{BRK}, \star, 0, \Gamma) \to (S_{BRK}, \star, 0, \Gamma)$ such that $(g \star' h) \star' s = g \star' (h \star' s)$ and $0 \star' s = s$ for all $g, h \in {G_{BRK}}'$ and $s \in S_{BRK}$. For $a, b \in G_{BRK}$, define $\mu_{\theta^{-1}(F)}(a \star s) = \mu_F(\theta(a) \star s)$. Using μ' and θ, there is a map $\mu : (G_{BRK}, \star, 0, \Gamma) \times (S_{BRK}, \star, 0, \Gamma) \to (S_{BRK}, \star, 0, \Gamma)$ defined by $\mu(a, s) = \theta(a) \star s$ with (i) $(a \star b) \star s = a \star (b \star s)$ and $0 \star s = s$ for all $a, b \in G_{BRK}$ and $s \in S_{BRK}$. Thus $(G_{BRK}, \star, 0, \Gamma)$ acts on the $fBRKtg$ $\theta^{-1}(F)$ under $(S_{BRK}, \star, 0, \Gamma)$. Ξ

Theorem 2.2. *Let* $(G_{BRK}, \star, 0, \Gamma)$ *be a* $fBRKtg$ *on* E $fBRKtA$ *under* $(S_{BRK}, \star, 0, \Gamma)$ *and* $\theta : (G_{BRK}, \star, 0, \Gamma) \to ({G_{BRK}}', \star, 0, \Gamma)$ *be an onto homomorphism on groups. Then* $fBRKtg$ $({G_{BRK}}', \star, 0, \Gamma)$ *acts on the* $fBRKtg$ $\theta(E)$ *under* $(S_{BRK}, \star, 0, \Gamma)$.

Proof. There exists a map $\mu : (G_{BRK}, \star, 0, \Gamma) \times (S_{BRK}, \star, 0, \Gamma) \to (S_{BRK}, \star, 0, \Gamma)$ such that $(a \star b) \star s = a \star (a \star s)$ and $0 \star s = s$ for all $a, b \in G_{BRK}$ and $s \in S_{BRK}$. Define a fuzzy set $\theta(E)$ on $({G_{BRK}}', \star, 0, \Gamma)$ by

$$\mu_{\theta(A)}(j) = \begin{cases} \sup \mu_E(i) : i \in \theta^{-1}(j) & , \text{ if } \theta^{-1}(j) \neq 0 \, ; \\ 0 & , \text{ otherwise} \end{cases} .$$

There exists a map $\mu' : ({G_{BRK}}', \star, 0, \Gamma) \times (S_{BRK}, \star, 0, \Gamma) \to (S_{BRK}, \star, 0, \Gamma)$ so that $(g \star' h) \star' s = g \star' (h \star' s)$ and $0 \star' s = s$ for all $g, h \in {G_{BRK}}'$ and $s \in S_{BRK}$. The $fBRKtg$ $(G_{BRK}, \star, 0, \Gamma)$ acts on the $fBRKtg$ $\theta(E)$ under $(S_{BRK}, \star, 0, \Gamma)$. Ξ

Theorem 2.3. *Let* $({G_{BRK}}', \star, 0, \Gamma)$ *be a* $fBRKtg$ *on* F $fBRKtA$ *under* $(S_{BRK}, \star, 0, \Gamma)$ *and* $\theta : (G_{BRK}, \star, 0, \Gamma) \to ({G_{BRK}}', \star, 0, \Gamma)$ *be an onto homomorphism on groups. Then* $fBRKtg$ $(G_{BRK}, \star, 0, \Gamma)$ *acts on the* $fBRKtg$ $\theta^{-1}(F)$ *under* $(S_{BRK}, \star, 0, \Gamma)$.

Proof. Let ${G_{BRK}}'$ be a group acting on a fuzzy group F under $(S_{BRK}, \star, 0, \Gamma)$. Then $fBRKtg$ $(G_{BRK}, \star, 0, \Gamma)$ acts on the $fBRKtg$ $\theta^{-1}(F)$ under $(S_{BRK}, \star, 0, \Gamma)$ by Theorem 2.1.

Let $g, h \in G_{BRK}$ and $s, t \in S_{BRK}$.

$$\begin{aligned} (b) \ \mu_{\theta^{-1}(F)}(g \star (s \star t)) &= \mu_F(\theta(g) \star (s \star t)) \\ &= \mu_F(\theta(g) \star (s \star t)) \\ &\geq \min\{\mu_F(\theta(g) \star s), \mu_F(\theta(g) \star t)\} \\ &= \min\{\mu_{\theta^{-1}(F)}(g \star s), \mu_{\theta^{-1}(F)}(g \star t)\}. \end{aligned}$$

$$\begin{aligned}(c)\ \mu_{\theta^{-1}(F)}((g\star h)\star s) &= \mu_F((\theta(g\star h))\star s)\\ &= \mu_F((\theta(g)\star\theta(h))\star s)\\ &\geq \min\{\mu_F(\theta(g)\star s), \mu_F(\theta(h)\star s)\}\\ &= \min\{\mu_{\theta^{-1}(F)}(g\star s), \mu_{\theta^{-1}(F)}(h\star s)\}.\\ (d)\ \mu_{\theta^{-1}(F)}(g\star s^{-1}) &= \mu_F(\theta(g)\star s^{-1})\\ &= \mu_F(\theta(g)\star s)\\ &= \mu_{\theta^{-1}(F)}(g\star s)\end{aligned}$$

Therefore the $fBRKtg$ $(G_{BRK}, \star, 0, \Gamma)$ acts the $fBRKtg$ $\theta^{-1}(F)$ under $(S_{BRK}, \star, 0, \Gamma)$. Ξ

Theorem 2.4. *Let* $\theta : (G_{BRK}, \star, 0, \Gamma) \to (G_{BRK}{}', \star, 0, \Gamma)$ *be an epimorphism and* $(G_{BRK}, \star, 0, \Gamma)$ *be a* $fBRKtg$ *on* μ_E $fBRKtA$ *under* $(S_{BRK}, \star, 0, \Gamma)$*. Then* $fBRKtg$ $(G_{BRK}{}', \star, 0, \Gamma)$ *acts on the* $fBRKtg$ $\theta(\mu_E)$ *under* $(S_{BRK}, \star, 0, \Gamma)$*.*

Proof. Let $(G_{BRK}{}', \star, 0, \Gamma)$ be a $fBRKtg$ on μ_F $fBRKtA$ under $(S_{BRK}, \star, 0, \Gamma)$. Then $fBRKtg$ $(G_{BRK}{}', \star, 0, \Gamma)$ acts on the $fBRKtg$ $\theta(\mu_E)$ under S_{BRK} by Definition 2.1. Also $\theta(\mu_E)$ is a $fBRKtg$ of $(G_{BRK}{}', \star, 0, \Gamma)$.

Let g, h be in $G_{BRK}{}'$ and $s, t \in S_{BRK}$. It follows that

$$\begin{aligned}(b)\ \theta(\mu_E)(g\star(s\star t)) &= \sup_{x\in X}\{\mu_E(u\star(s\star t)) : x\star s\in\theta^{-1}(g)\star s \text{ if } \theta^{-1}(u)\star s\neq 0\}\\ &\geq \sup_{x\in X}\min\{\mu_E((u\star s)\star(u\star t))\}\\ &\geq \min\{\theta(\mu_E)(g\star s), \theta(\mu_E)(g\star t)\}\end{aligned}$$

$$\begin{aligned}(c)\ \theta(\mu_E)((g\star h)\star s) &= \sup_{y\in X,\ x\in X\ \text{is fixed}}\{\mu_E((u\star v)\star s) : x\star s\in\theta^{-1}(g\star h)\star s\\ &\qquad \text{if } \theta^{-1}(v)\star s\neq 0\}\\ &\geq \sup_{y\in X\ x\in X\ \text{is fixed}}\min\{\mu_E((u\star s)\star(v\star s))\}\\ &\geq \min\{\theta(\mu_E)(g\star s), \theta(\mu_E)(h\star s)\}\end{aligned}$$

$$\begin{aligned}(d)\ \theta(\mu_E)(g\star s^{-1}) &= \sup_{x\in X}(\mu_E(u\star s))\\ &= \sup_{x\in X}(\mu_E(u\star s))\\ &= \theta(\mu_E)(g\star s)\end{aligned}$$

Then $fBRKtg$ $(G_{BRK}{}', \star, 0, \Gamma)$ acts on the $fBRKtg$ $\theta(\mu_E)$ under $(S_{BRK}, \star, 0, \Gamma)$. Ξ

Theorem 2.5. *If a $fBRKtg$ $(G_{BRK}, \star, 0, \Gamma)$ acts all non-empty level subset $U(\mu_E, t)$ under $(S_{BRK}, \star, 0, \Gamma)$, then the $fBRKtg$ $(G_{BRK}, \star, 0, \Gamma)$ acts $fBRKtAS_g$ μ_E under $(S_{BRK}, \star, 0, \Gamma)$.*
Proof. Then $U(\mu_E, t) = \{s \in S : \inf_{u \in G}\{\mu_E(u \star s)\} \geq \alpha\}$.

Let $s, t \in U(\mu_E, t_0)$. Then $\inf_{u \in G}\{\mu_E(u \star t)\} \geq \alpha$ and $\inf_{u \in G}\{\mu_E(u \star s)\} \geq \alpha$.

$$\begin{aligned} \mu_E(\inf_{u \in G} u \star (s \star t)) &\geq \min\{\mu_E(\inf_{u \in G} u \star s), \mu_E(\inf_{u \in G} u \star t)\} \\ &\geq \min\{\alpha, \alpha\} \\ &= \alpha. \end{aligned}$$

Thus $s \star t \in U(\mu_E, t_0)$.

Further $s \in U(\mu_E, t_0)$ and iff $\mu_E(\inf_{x \in G}(u \star s^{-1})) = \mu_E(\inf_{x \in G}(u \star s)) \geq \alpha$ which implies $s^{-1} \in U(\mu_E, t_0)$. Therefore $U(\mu_E, t_0)$ is a subgroup of S_{BRK}. Then $fBRKtg$ G_{BRK} acts on $fBRKtAS_g$ $U(\mu_E, t)$ under S_{BRK}.

But $\mu_E = \cup_{t \in [0,1]} U(\mu_E, t)$ and every level fuzzy subgroup $U(\mu_E, t_1)$ is contained in other level fuzzy subgroup $U(\mu_E, t_2)$ $\forall\ t_1, t_2 \in [0, 1]$. So $\mu_E = \cup_{t \in [0,1]} U(\mu_E, t)$ is a $fBRKtg$ on S_{BRK}. Since the group $(G_{BRK}, \star, 0, \Gamma)$ acts on each level set $U(\mu_E, t)$ on $(S_{BRK}, \star, 0, \Gamma)$, then $fBRKtg$ $(G_{BRK}, \star, 0, \Gamma)$ acts on $fBRKtAS_g$ μ_E under $(S_{BRK}, \star, 0, \Gamma)$. Ξ

Theorem 2.6. *A set of necessary and sufficient conditions for $(G_{BRK}, \star, 0, \Gamma)$ be a $fBRKtg$ on $fBRKtA$ μ_E under $(S_{BRK}, \star, 0, \Gamma)$ is that $\mu_E(u \star (s \star t^{-1})) \geq \min\{\mu_E(u \star s), \mu_E(u \star t)\}$ $\forall\ x, y \in G_{BRK}$ and $s, t \in S_{BRK}$, where every element of $(G_{BRK}, \star, 0, \Gamma)$ has its own inverse under addition.*
Proof. Let $(G_{BRK}, \star, 0, \Gamma)$ be a $fBRKtg$ on μ_E $fBRKtA$ under $(S_{BRK}, \star, 0, \Gamma)$. Then

$$\begin{aligned} \mu_E(g \star (s \star t^{-1})) &\geq \min\{\mu_E(g \star s), \mu_E(g \star t^{-1})\} \\ &= \min\{\mu_E(g \star s), \mu_E(g \star t)\}\ \forall\ u, v \in G\ \&\ s, t \in S. \end{aligned}$$

For the converse, suppose that the condition holds.

Thus $\mu_E(g \star (s \star t^{-1})) \geq \min\{\mu_E(g \star s), \mu_E(g \star t)\}$ $\forall\ x, y \in G_{BRK}$ and $s, t \in S_{BRK}$. It follows that

$$\begin{aligned} \mu_E(0 \star 0) &= \mu_E((g \star u^{-1}) \star (s \star s^{-1})) \\ &\geq \min\{\mu_E(g \star s), \mu_E(g \star s)\} \\ &= \mu_E(g \star s)\ \forall\ x, y \in G\ \&\ s, t \in S. \end{aligned}$$

Since μ_E is a $fBRKtg$ on S_{BRK}, then it gives that

$$\begin{aligned}\mu_E(g\star 0) &= \mu_E(g\star(s\star s^{-1}))\\ &\geq \min\{\mu_E(g\star s),\mu_E(g\star s)\}\\ &= \mu_E(g\star s)\end{aligned}$$

$$\begin{aligned}\mu_E(0\star s) &= \mu_E((g\star u^{-1})\star s)\\ &\geq \min\{\mu_E(g\star s),\mu_E(g\star s)\}\\ &= \mu_E(g\star s)\end{aligned}$$

$$\begin{aligned}\mu_E(h\star 0) &= \mu_E(h\star(t\star t^{-1}))\\ &\geq \min\{\mu_E(h\star t),\mu_E(h\star t)\}\\ &= \mu_E(h\star t)\end{aligned}$$

$$\begin{aligned}\mu_E(0\star t) &= \mu_E((h\star y^{-1})\star t)\\ &\geq \min\{\mu_E(h\star t),\mu_E(h\star t)\}\\ &= \mu_E(h\star t)\ \forall\ u,v\in G\ \&\ s,t\in S\end{aligned}$$

Thus $\mu_E(0\star s)\geq \mu_E(h\star s)\ \forall\ s\in S_{BRK}$.

Further $\mu_E((0\star y^{-1})\star(s\star t))\geq \min\{\mu_E(0\star s),\mu_E(h^{-1}\star t)\}\geq \mu_E(h^{-1}\star t)\ \forall\ s,t\in S_{BRK}$. It implies that $\mu_E(h^{-1}\star s)\geq \mu_E(h\star s)\ \forall\ v\in G_{BRK}$. Therefore $\mu_E(h^{-1}\star s)=\mu_E(h\star s)\ \forall\ v\in G_{BRK}$.

Also $\mu_E((g\star h)\star(s\star t))\geq \min\{\mu_E(g\star s),\mu_E(h\star t)\}=\min\{\mu_E(g\star s),\mu_E(h\star s)\}\ \forall\ u,v\in G_{BRK}$ and $s,t\in S_{BRK}$. Hence $(G_{BRK},\star,0,\Gamma)$ be a $fBRKtg$ on $fBRKtA$ μ_E under $(S_{BRK},\star,0,\Gamma)$. Ξ

3. Conclusion

In this paper, $fBRKtA$ in a subgroup $(S_{BRK},\star,0,\Gamma)$ of $(G_{BRK},\star,\ 0,\Gamma)$ has been extended homomorphic images and epimorphism in a $fBRKtg$. Also, we have studied about the level cut in a $fBRKtA$ under $(S_{BRK},\star,0,\Gamma)$.

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