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DISCRETE LAPLACE TRANSFORMS OF SINE FUNCTION BY NABLA OPERATOR

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Abstract: In this paper, we define difference operator providing some results also we derived Laplace transform of sine series.A definition for the Laplace transform corresponding to the nabla difference operator is given.

Keywords and Phrases: Generalized Laplace Transform, Inverse Difference Operator, Nabla Operator and Sine Series.

2020 Mathematics Subject Classification: 39A13, 39A60, 39A70.

1. Introduction

Many applications are obtained using difference equation and its corresponding difference operator ∇. The Laplace transform can also be used to solve differential equation and is used extensively in electrical engineering.The theory of difference equation is developed the difference operator $\nabla_\ell u(k) = u(k) - u(k-l), k\in N$, where N is the set of natural numbers. The Laplace Transform of $f(t)$ is defined by $L(f(t)) = \int\limits_0^\infty e^{-st} f(t)dt$ provided the integral exists, s is a parameter.

Definition 1.1. *If n and ℓ are any two positive integers then the generalized positive polynomial factorial is $k_\ell^{(n)} = k(k-\ell)(k-2\ell)...(k-(n-1)\ell), k_\ell^{(0)} = 1$ and*

$k_\ell^{(1)} = k$.

Definition 1.2. *For a given function $u(k)$, the generalized Laplace transform is*

$$L_\ell u(k) = \ell \nabla_\ell^{-1} u(k) e^{-sk}\Big|_0^\infty \tag{1}$$

Definition 1.3. *Let $\ell > 0$ and $u(k)$ and $w(k)$ are real valued bounded functions. Then*

$$\nabla_\ell^{-1}(u(k)w(k)) = u(k)\nabla_\ell^{-1}w(k) - \nabla_\ell^{-1}(\nabla_\ell^{-1}w(k-\ell)\nabla_\ell u(k)) \tag{2}$$

Definition 1.4. *Let $\ell > 0$, $s > 0$ and $1 - e^{-s\ell} \neq 0$, then*

$$\nabla_\ell^{-1} e^{sk} = \frac{e^{sk}}{1 - e^{-s\ell}} \tag{3}$$

Definition 1.5. *Let $\ell > 0$, $s > 0$and $1 - e^{s\ell} \neq 0$, then*

$$\nabla_\ell^{-1} e^{-sk} = \frac{e^{-sk}}{1 - e^{s\ell}} \tag{4}$$

Definition 1.6. *Let $\ell > 0$ and n is a natural number, then*

$$\nabla_\ell k_\ell^{(n)} = n\ell(k-\ell)_\ell^{(n-1)} \tag{5}$$

2. Generalised Laplace Transform of Sine Series

Generalised Laplace Transform of sine series are derived below.

Lemma 2.1. *Let $k \in (0, \infty)$ and $\ell > 0$, then*

$$\nabla_\ell^{-1} \ell e^{k_\ell^{(1)}} sin k_\ell^{(1)} e^{-sk}\Big|_0^\infty = \frac{1}{1 + (s-1)^2} \tag{6}$$

Proof.

$$\nabla_\ell^{-1} \ell e^{k_\ell^{(1)}} sin k_\ell^{(1)} e^{-sk}\Big|_0^\infty = \ell \nabla_\ell^{-1}(k_\ell^{(1)} - \frac{k_\ell^{(3)}}{3!} + \frac{k_\ell^{(5)}}{5!} - \cdots) e^{-(s-1)k}\Big|_0^\infty \tag{7}$$

Now,

$$\ell(\nabla_\ell^{-1} k_\ell^{(1)}(e^{-(s-1)k})\Big|_0^\infty = \ell(k_\ell^{(1)} \frac{e^{-(s-1)k}}{1 - e^{(s-1)\ell}} - \nabla_\ell^{-1}(\frac{e^{-(s-1)(k-\ell)}}{1 - e^{(s-1)\ell}})\Big|_0^\infty (from (2) \& (4))$$

$$= \ell(k_\ell^{(1)} \frac{e^{-(s-1)k}}{1 - e^{(s-1)\ell}} - \frac{e^{-(s-1)(k-\ell)}}{(1 - e^{(s-1)\ell})^2}\ell)\Big|_0^\infty$$

$$\ell(\nabla_\ell^{-1} k_\ell^{(1)}(e^{-(s-1)k})\Big|_0^\infty = \frac{1}{(s-1)^2} \qquad as \quad \ell \to 0 \tag{8}$$

Also,

$$\ell(\nabla_\ell^{-1}(\frac{k_\ell^{(3)}}{3!}e^{-(s-1)k}))\Big|_0^\infty = \frac{\ell}{6}(k_\ell^{(3)}\frac{e^{-(s-1)k}}{1-e^{(s-1)\ell}} - \nabla_\ell^{-1}(\frac{e^{-(s-1)(k-\ell)}}{1-e^{(s-1)\ell}}3\ell(k-\ell)_\ell^{(2)}))\Big|_0^\infty$$

$$= \frac{\ell}{6}(k_\ell^{(3)}\frac{e^{-(s-1)k}}{1-e^{(s-1)\ell}} - 3\ell((k-\ell)_\ell^{(2)}\frac{e^{-(s-1)(k-\ell)}}{(1-e^{(s-1)\ell})^2}$$

$$- \nabla_\ell^{-1}(\frac{e^{-(s-1)(k-2\ell)}}{(1-e^{(s-1)\ell})^2}2\ell(k-2\ell)_\ell^{(1)})))\Big|_0^\infty$$

$$= \frac{\ell}{6}(k_\ell^{(3)}\frac{e^{-(s-1)k}}{1-e^{(s-1)\ell}} - 3\ell((k-\ell)_\ell^{(2)}\frac{e^{-(s-1)(k-\ell)}}{(1-e^{(s-1)\ell})^2}$$

$$-2\ell((k-2\ell)_\ell^{(1)}\frac{e^{-(s-1)(k-2\ell)}}{(1-e^{(s-1)\ell})^3} - \nabla_\ell^{-1}(\frac{e^{-(s-1)(k-3\ell)}}{(1-e^{(s-1)\ell})^3}\ell)))\Big|_0^\infty (from (2)\&(4))$$

$$\ell(\nabla_\ell^{-1}(\frac{k_\ell^{(3)}}{3!}e^{-(s-1)k}))\Big|_0^\infty = \frac{1}{(s-1)^4} \qquad as \quad \ell\to 0 \tag{9}$$

Continuing this process,we get

$$\ell(\nabla_\ell^{-1}\frac{k_\ell^{(5)}}{5!}e^{-(s-1)k})\Big|_0^\infty = \frac{1}{(s-1)^6} \qquad as \quad \ell\to 0 \tag{10}$$

Substituting (8), (9), (10) in ((7), we get

$$\nabla_\ell^{-1}\ell e^{k_\ell^{(1)}} sin k_\ell^{(1)} e^{-sk}\Big|_0^\infty = \frac{1}{(s-1)^2} - \frac{1}{(s-1)^4} + \frac{1}{(s-1)^6} - \cdots$$

This completes the proof of (6).

Lemma 2.2. *Let* $k\in(0,\infty)$ *and* $\ell > 0$, *then*

$$\nabla_\ell^{-1}\ell e^{-k_\ell^{(1)}} sin k_\ell^{(1)} e^{-sk}\Big|_0^\infty = \frac{1}{1+(s+1)^2} \tag{11}$$

Proof.

$$\nabla_\ell^{-1}\ell e^{-k_\ell^{(1)}} sin k_\ell^{(1)} e^{-sk}\Big|_0^\infty = \ell\nabla_\ell^{-1}(k_\ell^{(1)} - \frac{k_\ell^{(3)}}{3!} + \frac{k_\ell^{(5)}}{5!} - \cdots)e^{-(s+1)k}\Big|_0^\infty \tag{12}$$

Now,

$$\ell(\nabla_\ell^{-1}\, k_\ell^{(1)}(e^{-(s+1)k}))\Big|_0^\infty = \ell(k_\ell^{(1)}\frac{e^{-(s+1)k}}{1-e^{(s+1)\ell}} - \nabla_\ell^{-1}(\frac{e^{-(s+1)(k-\ell)}}{1-e^{(s+1)\ell}})\Big|_0^\infty (from (2)\&(4))$$

$$= \ell(k_\ell^{(1)}\frac{e^{-(s+1)k}}{1-e^{(s+1)\ell}} - \frac{e^{-(s+1)(k-\ell)}}{(1-e^{(s+1)\ell})^2}\ell)\Big|_0^\infty$$

$$\ell(\nabla_\ell^{-1}k_\ell^{(1)}(e^{-(s+1)k})\Big|_0^\infty = \frac{1}{(s+1)^2} \qquad as \quad \ell\to 0 \tag{13}$$

Also ,

$$\ell(\nabla_\ell^{-1}(\frac{k_\ell^{(3)}}{3!}e^{-(s+1)k}))\Big|_0^\infty = \frac{\ell}{6}(k_\ell^{(3)}\frac{e^{-(s+1)k}}{1-e^{(s+1)\ell}} - \nabla_\ell^{-1}(\frac{e^{-(s+1)(k-\ell)}}{1-e^{(s+1)\ell}}3\ell(k-\ell)_\ell^{(2)}))\Big|_0^\infty$$
$$= \frac{\ell}{6}(k_\ell^{(3)}\frac{e^{-(s+1)k}}{1-e^{(s+1)\ell}} - 3\ell((k-\ell)_\ell^{(2)}\frac{e^{-(s+1)(k-\ell)}}{(1-e^{(s+1)\ell})^2}$$
$$- \nabla_\ell^{-1}(\frac{e^{-(s+1)(k-2\ell)}}{(1-e^{(s+1)\ell})^2}2\ell(k-2\ell)_\ell^{(1)})))\Big|_0^\infty$$
$$= \frac{\ell}{6}(k_\ell^{(3)}\frac{e^{-(s+1)k}}{1-e^{(s+1)\ell}} - 3\ell((k-\ell)_\ell^{(2)}\frac{e^{-(s+1)(k-\ell)}}{(1-e^{(s+1)\ell})^2}$$
$$- 2\ell((k-2\ell)_\ell^{(1)}\frac{e^{-(s+1)(k-2\ell)}}{(1-e^{(s+1)\ell})))^3} - \nabla_\ell^{-1}(\frac{e^{-(s+1)(k-3\ell)}}{(1-e^{(s+1)\ell})^3}\ell)))\Big|_0^\infty (from (2)\&(4))$$

$$\ell(\nabla_\ell^{-1}(\frac{k_\ell^{(3)}}{3!}e^{-(s+1)k}))\Big|_0^\infty = \frac{1}{(s+1)^4} \qquad as \quad \ell\to 0 \tag{14}$$

Continuing this process,we get

$$\ell(\nabla_\ell^{-1}\frac{k_\ell^{(5)}}{5!}e^{-(s+1)k})\Big|_0^\infty = \frac{1}{(s+1)^6} \qquad as \quad \ell\to 0 \tag{15}$$

Substituting (13), (14), (15) in (12, we get

$$\nabla_\ell^{-1}\ell e^{k_\ell^{(1)}} sink_\ell^{(1)} e^{-sk}\Big|_0^\infty = \frac{1}{(s+1)^2} - \frac{1}{(s+1)^4} + \frac{1}{(s+1)^6} - \cdots$$

This completes the proof of (11).

Lemma 2.3. *Let* $k\in(0,\infty)$ *and* $\ell > 0$*, then*

$$\nabla_\ell^{-1}\ell e^{-ak_\ell^{(1)}} sinbk_\ell^{(1)} e^{-sk}\Big|_0^\infty = \frac{b}{b^2+(s+a)^2} \tag{16}$$

Proof.

$$\nabla_\ell^{-1}\ell e^{-ak_\ell^{(1)}} sinbk_\ell^{(1)} e^{-sk}\Big|_0^\infty = \ell\nabla_\ell^{-1}(bk_\ell^{(1)} - \frac{b^3k_\ell^{(3)}}{3!} + \frac{b^5k_\ell^{(5)}}{5!} - \cdots)e^{-(s+a)k}\Big|_0^\infty \tag{17}$$

Now,

$$\ell\,\nabla_\ell^{-1}\,bk_\ell^{(1)}e^{-(s+a)k}\Big|_0^\infty = \ell b(k_\ell^{(1)}\frac{e^{-(s+a)k}}{1-e^{(s+a)\ell}} - l\,\nabla_\ell^{-1}\,\tfrac{e^{-(s+a)(k-\ell)}}{(1-e^{(s+a)\ell})})\Big|_0^\infty (from (2)\&(4))$$

$$\ell\nabla_\ell^{-1}bk_\ell^{(1)}e^{-(s+a)k}\Big|_0^\infty = \frac{b}{(s+a)^2} \qquad as \quad \ell\to 0 \tag{18}$$

Also ,

$$\ell\,\nabla_\ell^{-1}\,\tfrac{b^3k_\ell^{(3)}}{3!}e^{-(s+a)k}\Big|_0^\infty = \ell\frac{b^3}{3!}(k_\ell^{(3)}\frac{e^{-(s+a)k}}{1-e^{(s+a)\ell}} - \nabla_\ell^{-1}(\frac{e^{-(s+a)(k-\ell)}}{(1-e^{(s+a)\ell})}3\ell((k-\ell)_\ell^{(2)}))\Big|_0^\infty$$

$$= \ell\frac{b^3}{6}(k_\ell^{(3)}\frac{e^{-(s+a)k}}{1-e^{(s+a)\ell}} - 3\ell((k-\ell)_\ell^{(2)}\frac{e^{-(s+a)(k-\ell)}}{(1-e^{(s+a)\ell})^2}$$

$$-2\ell((k-\ell)_\ell^{(1)}\frac{e^{-(s+a)(k-2\ell)}}{(1-e^{(s+a)\ell}))^3} - \nabla_\ell^{-1}(\tfrac{e^{-(s+a)(k-3\ell)}}{(1-e^{(s+a)\ell})^3}\ell))))\Big|_0^\infty$$

$$\ell\nabla_\ell^{-1}\frac{b^3k_\ell^{(3)}}{3!}e^{-(s+a)k}\Big|_0^\infty = \frac{b^3}{(s+a)^4} \qquad as \quad \ell{\to}0 \tag{19}$$

Proceeding like this,we get

$$\ell\nabla_\ell^{-1}\frac{b^5k_\ell^{(5)}}{5!}e^{-(s+a)k}\Big|_0^\infty = \frac{b^5}{(s+a)^6} \qquad as \quad \ell{\to}0 \tag{20}$$

Substituting (18), (19), (20) in (17), we get

$$\nabla_\ell^{-1}\,\ell e^{-ak_\ell^{(1)}}sinbk_\ell^{(1)}e^{-sk}\Big|_0^\infty = \frac{b}{(s+a)^2} - \frac{b^3}{(s+a)^4} + \frac{b^5}{(s+a)^6} - \cdots$$

This gives the proof of (16).

Lemma 2.4. *Let* $k{\in}(0,\infty)$ *and*$\ell > 0$, *then*

$$\nabla_\ell^{-1}\ell e^{ak_\ell^{(1)}}sinbk_\ell^{(1)}e^{-sk}\Big|_0^\infty = \frac{b}{b^2+(s-a)^2} \tag{21}$$

Proof.

$$\nabla_\ell^{-1}\ell e^{ak_\ell^{(1)}}sinbk_\ell^{(1)}e^{-sk}\Big|_0^\infty = \ell\nabla_\ell^{-1}(bk_\ell^{(1)} - \frac{b^3k_\ell^{(3)}}{3!} + \frac{b^5k_\ell^{(5)}}{5!} - \cdots)e^{-(s-a)k}\Big|_0^\infty \tag{22}$$

Now,

$$\ell\,\nabla_\ell^{-1}\,bk_\ell^{(1)}e^{-(s-a)k}\Big|_0^\infty = \ell b(k_\ell^{(1)}\frac{e^{-(s-a)k}}{1-e^{(s-a)\ell}} - l\,\nabla_\ell^{-1}\,\tfrac{e^{-(s-a)(k-\ell)}}{(1-e^{(s-a)\ell})})\Big|_0^\infty (from(2)\&(4))$$

$$\ell\nabla_\ell^{-1}bk_\ell^{(1)}e^{-(s-a)k}\Big|_0^\infty = \frac{b}{(s-a)^2} \qquad as \quad \ell{\to}0 \tag{23}$$

Also ,

$$\ell\,\nabla_\ell^{-1}\,\tfrac{b^3k_\ell^{(3)}}{3!}e^{-(s-a)k}\Big|_0^\infty = \ell\frac{b^3}{3!}(k_\ell^{(3)}\frac{e^{-(s-a)k}}{1-e^{(s-a)\ell}} - \nabla_\ell^{-1}(\frac{e^{-(s-a)(k-\ell)}}{(1-e^{(s-a)\ell})}3\ell((k-\ell)_\ell^{(2)}))\Big|_0^\infty$$

$$= \ell\frac{b^3}{6}(k_\ell^{(3)}\frac{e^{-(s-a)k}}{1-e^{(s-a)\ell}} - 3\ell((k-\ell)_\ell^{(2)}\frac{e^{-(s-a)(k-\ell)}}{(1-e^{(s-a)\ell})^2}$$
$$- 2\ell((k-\ell)_\ell^{(1)}\frac{e^{-(s-a)(k-2\ell)}}{(1-e^{(s-a)\ell}))^3} - \nabla_\ell^{-1}(\frac{e^{-(s-a)(k-3\ell)}}{(1-e^{(s-a)\ell})^3}\ell))))\Big|_0^\infty$$

$$\ell\nabla_\ell^{-1}\frac{b^3k_\ell^{(3)}}{3!}e^{-(s-a)k}\Big|_0^\infty = \frac{b^3}{(s-a)^4} \qquad as \quad \ell\to 0 \tag{24}$$

Proceeding like this,we get

$$\ell\nabla_\ell^{-1}\frac{b^5k_\ell^{(5)}}{5!}e^{-(s-a)k}\Big|_0^\infty = \frac{b^5}{(s-a)^6} \qquad as \quad \ell\to 0 \tag{25}$$

Substituting (23), (24), (25) in (22), we get
$\nabla_\ell^{-1}\,\ell e^{ak_\ell^{(1)}} sinbk_\ell^{(1)}e^{-sk}\Big|_0^\infty = \frac{b}{(s-a)^2} - \frac{b^3}{(s-a)^4} + \frac{b^5}{(s-a)^6} - \cdots$
This gives the proof of (21).

Lemma 2.5. *Let $k\in(0,\infty)$ and $\ell > 0$, then*

$$\nabla_\ell^{-1}\ell e^{k_\ell^{(1)}} sinak_\ell^{(1)}e^{-sk}\Big|_0^\infty = \frac{a}{a^2+(s-1)^2} \tag{26}$$

Proof.

$$\nabla_\ell^{-1}\ell e^{k_\ell^{(1)}} sinak_\ell^{(1)}e^{-sk}\Big|_0^\infty = \ell\nabla_\ell^{-1}(ak_\ell^{(1)} - \frac{a^3k_\ell^{(3)}}{3!} + \frac{a^5k_\ell^{(5)}}{5!} - \cdots)e^{-(s-1)k}\big|_0^\infty \tag{27}$$

Now,
$$\nabla_\ell^{-1}(ak_\ell^{(1)}e^{-(s-1)k})\Big|_0^\infty = \ell a(k_\ell^{(1)}\frac{e^{-(s-1)k}}{1-e^{(s-1)\ell}} - \ell\,\nabla_\ell^{-1}\frac{e^{-(s-1)(k-\ell)}}{(1-e^{(s-1)\ell})})\big|_0^\infty (from (2)\&(4))$$

$$\nabla_\ell^{-1}(ak_\ell^{(1)}e^{-(s-1)k})\Big|_0^\infty = \frac{a}{(s-1)^2} \qquad as \quad \ell\to 0 \tag{28}$$

Also ,
$$\ell\,\nabla_\ell^{-1}(\tfrac{a^3k_\ell^{(3)}}{3!}e^{-(s-1)k})\big|_0^\infty = \frac{\ell a^3}{6}(k_l^{(3)}\frac{e^{-(s-1)k}}{1-e^{(s-1)l}} - \nabla_\ell^{-1}(\frac{e^{-(s-1)(k-l)}}{(1-e^{(s-1)l})}3\ell((k-\ell)_\ell^{(2)})))\Big|_0^\infty$$
$$= \frac{\ell a^3}{6}(k_\ell^{(3)}\tfrac{e^{-(s-1)k}}{1-e^{(s-1)\ell}} - 3\ell(((k-\ell)_\ell^{(2)}\frac{e^{-(s-1)(k-\ell)}}{(1-e^{(s-1)\ell)^2}}$$
$$- \nabla_\ell^{-1}(\frac{e^{-(s-1)(k-2\ell)}}{(1-e^{(s-1)\ell)^2}}2\ell((k-2\ell)_\ell^{(1)})))\Big|_0^\infty$$

$$= \frac{\ell a^3}{6}(k_\ell^{(3)}\frac{e^{-(s-1)k}}{1-e^{(s-1)\ell}} - 3\ell((k-\ell)_\ell^{(2)}\frac{e^{-(s-1)(k-\ell)}}{(1-e^{(s-1)\ell})^2}$$

$$-2\ell((k-2\ell)_\ell^{(1)}\frac{e^{-(s-1)(k-2\ell)}}{(1-e^{(s-1)\ell})^3} - \nabla_\ell^{-1}(\frac{e^{-(s-1)(k-3\ell)}}{(1-e^{(s-1)\ell})^3}\ell))))\Big|_0^\infty (from(2)\&(4))$$

$$\ell\nabla_\ell^{-1}(\frac{a^3 k_\ell^{(3)}}{3!}e^{-(s-1)k})\Big|_0^\infty = \frac{a^3}{(s-1)^4} \qquad as \quad \ell{\to}0 \tag{29}$$

Continuing this process,we get

$$\ell\nabla_\ell^{-1}(\frac{a^5 k_\ell^{(5)}}{5!}e^{-(s-1)k})\Big|_0^\infty = \frac{a^5}{(s-1)^6} \qquad as \quad \ell{\to}0 \tag{30}$$

Substituting (28), (29), (30) in (27), we get

$\nabla_\ell^{-1}\,\ell e^{k_\ell^{(1)}} sinak_\ell^{(1)} e^{-sk}\Big|_0^\infty = \frac{a}{(s-1)^2} - \frac{a^3}{(s-1)^4} + \frac{a^5}{(s-1)^6} - \cdots$

which gives (26).

Lemma 2.6. *Let $k{\in}(0,\infty)$ and $\ell > 0$, then*

$$\nabla_\ell^{-1}\ell e^{-k_\ell^{(1)}} sinak_\ell^{(1)} e^{-sk}\Big|_0^\infty = \frac{a}{a^2+(s+1)^2} \tag{31}$$

Proof.

$$\nabla_\ell^{-1}\ell e^{-k_\ell^{(1)}} sinak_\ell^{(1)} e^{-sk}\Big|_0^\infty = \ell\nabla_\ell^{-1}(ak_\ell^{(1)} - \frac{a^3k_\ell^{(3)}}{3!} + \frac{a^5k_\ell^{(5)}}{5!} - \cdots)e^{-(s+1)k}|_0^\infty \tag{32}$$

Now,

$\nabla_\ell^{-1}(ak_\ell^{(1)}e^{-(s+1)k})\big|_0^\infty = \ell a(k_\ell^{(1)}\frac{e^{-(s+1)k}}{1-e^{(s+1)\ell}} - \ell\,\nabla_\ell^{-1}\frac{e^{-(s+1)(k-\ell)}}{(1-e^{(s+1)\ell})})\Big|_0^\infty (from(2)\&(4))$

$$\nabla_\ell^{-1}(ak_\ell^{(1)}e^{-(s+1)k})\Big|_0^\infty = \frac{a}{(s+1)^2} \qquad as \quad \ell{\to}0 \tag{33}$$

Also ,

$\ell\,\nabla_\ell^{-1}(\frac{a^3k_\ell^{(3)}}{3!}e^{-(s+1)k})\big|_0^\infty = \frac{la^3}{6}(k_l^{(3)}\frac{e^{-(s+1)k}}{1-e^{(s+1)l}} - \nabla_l^{-1}(\frac{e^{-(s+1)(k-l)}}{(1-e^{(s+1)l})}3\ell((k-\ell)_\ell^{(2)})))\Big|_0^\infty$

$$= \frac{\ell a^3}{6}(k_\ell^{(3)}\frac{e^{-(s+1)k}}{1-e^{(s+1)\ell}} - 3\ell(((k-\ell)_\ell^{(2)}\frac{e^{-(s+1)(k-\ell)}}{(1-e^{(s+1)\ell})^2}$$

$$-\nabla_\ell^{-1}(\frac{e^{-(s+1)(k-2\ell)}}{(1-e^{(s+1)\ell})^2}2\ell((k-2\ell)_\ell^{(1)})))\big|_0^\infty$$

$$= \frac{\ell a^3}{6}(k_\ell^{(3)}\frac{e^{-(s+1)k}}{1-e^{(s+1)\ell}} - 3\ell((k-\ell)_\ell^{(2)}\frac{e^{-(s+1)(k-\ell)}}{(1-e^{(s+1)\ell})^2}$$
$$-2\ell((k-2\ell)_\ell^{(1)}\frac{e^{-(s+1)(k-2\ell)}}{(1-e^{(s+1)\ell})^3} - \nabla_\ell^{-1}(\frac{e^{-(s+1)(k-3\ell)}}{(1-e^{(s+1)\ell})^3}\ell))))\Big|_0^\infty (from(2)\&(4))$$

$$\ell \underset{\ell}{\overset{-1}{\nabla}}(\frac{a^3k_\ell^{(3)}}{3!}e^{-(s+1)k})\Big|_0^\infty = \frac{a^3}{(s+1)^4} \qquad as \quad \ell{\to}0 \tag{34}$$

Continuing this process,we get

$$\ell\nabla_\ell^{-1}(\frac{a^5k_\ell^{(5)}}{5!}e^{-(s+1)k})\Big|_0^\infty = \frac{a^5}{(s+1)^6} \qquad as \quad \ell{\to}0 \tag{35}$$

Substituting (33), (34), (35) in (32), we get
$\nabla_\ell^{-1}\,\ell e^{-k_\ell^{(1)}} sinak_\ell^{(1)}e^{-sk}\Big|_0^\infty = \frac{a}{(s+1)^2} - \frac{a^3}{(s+1)^4} + \frac{a^5}{(s+1)^6} - \cdots$
which gives (31).

Lemma 2.7. *Let* $k{\in}(0,\infty)$ *and* $\ell > 0$, *then*

$$\nabla_\ell^{-1}\ell sinak_\ell^{(1)}e^{-sk}\Big|_0^\infty = \frac{a}{s^2+a^2} \tag{36}$$

Proof.

$$\nabla_\ell^{-1}lsinak_\ell^{(1)}e^{-sk}\Big|_0^\infty = \ell\nabla_\ell^{-1}(ak_\ell^{(1)} - \frac{a^3K_\ell^{(3)}}{3!} + \frac{a^5k_\ell^{(5)}}{5!} - \cdots)e^{-sk}\Big|_0^\infty \tag{37}$$

Now,
$\ell\,\nabla_\ell^{-1}(ak_\ell^{(1)}e^{-sk})\Big|_0^\infty = \ell a(k_\ell^{(1)}\frac{e^{-sk}}{1-e^{s\ell}} - \nabla_\ell^{-1}(\frac{e^{-s(k-\ell)}}{1-e^{s\ell}}))\Big|_0^\infty (from(2)\&(4))$
$$= \ell a(k_l^{(1)}\frac{e^{-sk}}{1-e^{sl}} - \frac{e^{-s(k-l)}}{(1-e^{sl})^2}l)\Big|_0^\infty$$

$$\ell\nabla_\ell^{-1}(ak_\ell^{(1)}e^{-sk})\Big|_0^\infty = \frac{a}{s^2} \qquad as \quad \ell{\to}0 \tag{38}$$

Also ,
$\ell\,\nabla_\ell^{-1}(\frac{a^3k_\ell^{(3)}}{3!}e^{-sk})\Big|_0^\infty = \frac{\ell a^3}{6}(k_\ell^{(3)}\frac{e^{-sk}}{1-e^{s\ell}} - \nabla_\ell^{-1}(\frac{e^{-s(k-\ell)}}{(1-e^{s\ell})}3l((k-l)_l^{(2)})))|_0^\infty$
$$= \frac{la^3}{6}(k_l^{(3)}\frac{e^{-sk}}{1-e^{sl}} - 3l((k-l)_l^{(2)}\frac{e^{-s(k-l)}}{(1-e^{sl})^2}$$

$$-\nabla_l^{-1}\left(\frac{e^{-s(k-2l)}}{(1-e^{sl})^2}2l((k-2l)_l^{(1)}))\right)\Big|_0^\infty$$

$$=\frac{la^3}{6}\Big(k_l^{(3)}\frac{e^{-sk}}{1-e^{sl}}-3l((k-l)_l^{(2)}\frac{e^{-s(k-l)}}{(1-e^{sl})^2}-2l((k-2l)_l^{(1)}\frac{e^{-s(k-2l)}}{(1-e^{sl})^3}$$

$$-\nabla_l^{-1}\big(\frac{e^{-s(k-3l)}}{(1-e^{sl})^3}l\big)\big)\big)\Big)\Big|_0^\infty (from (2)\&(4))$$

$$\ell\nabla_\ell^{-1}\left(\frac{a^3k_\ell^{(3)}}{3!}e^{-sk}\right)\Big|_0^\infty=\frac{a^3}{s^4} \qquad as \quad \ell{\to}0 \tag{39}$$

continuing this process,we get

$$\ell\nabla_\ell^{-1}\left(\frac{a^5k_\ell^{(5)}}{5!}e^{-sk}\right)\Big|_0^\infty=\frac{a^5}{s^6} \qquad as \quad \ell{\to}0 \tag{40}$$

substituting (38), (39), (40) in (37), we get
$\nabla_\ell^{-1}\,\ell sinak_\ell^{(1)}e^{-sk}\Big|_0^\infty=\frac{a}{s^2}-\frac{a^3}{s^4}+\frac{a^5}{s^6}-\cdots$
This completes the proof of (36).

Corollary 2.8. *Let* $k\in(0,\infty)$ *and* $\ell>0$ *, then* $\nabla_\ell^{-1}\,\ell sink_\ell^{(1)}e^{-sk}\Big|_0^\infty=\frac{1}{s^2+1^2}$

3. Conclusion

The properties of generalized discrete Laplace transform of sine series are studied and using exponential and sine series interested results were occurs.Using the new properties the researcher approaches the sine series.

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