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Bailey transform, WP-Bailey pairs and q-series transformations

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Abstract: In this paper, making use of WP-Bailey pairs and certain identities, we have established transformation formulae for basic (q-) hypergeometric series.
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1. Introduction, Notations and Definitions

Transformation theory play very important role in the theory of q-hypergeometric series. Rogers-Ramanujan type identities are established through transformation formulae and identities have great importance in the theory of partitions. In this paper, we have established certain transformation formulae for q-series by using WP-Bailey pairs. we employ usual notations

$$(a;q)_n = (1-a)(1-aq)...(1-aq^{n-1}), \quad n \geq 1,$$

$$(a;q)_0 = 1$$

$$(a;q)_\infty = \prod_{r=0}^{\infty}(1-aq^r)$$

and

$$(a_1, a_2, ..., a_r; q)_n = (a_1;q)_n(a_2;q)_n...(a_r;q)_n,$$

$$(a_1, a_2, ..., a_r; q)_\infty = (a_1;q)_\infty(a_2;q)_\infty...(a_r;q)_\infty.$$

An ${}_r\Phi_s$ basic hypergeometric series is defined by

$${}_r\Phi_s\left[\begin{array}{l} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s \end{array}\right] = \sum_{n=0}^{\infty}\frac{[a_1, a_2, ..., a_r; q]_n z^n}{[q, b_1, b_2, ..., b_s; q]_n}\left[(-)^n q^{n(n-1)/2}\right]^{1+s-r} \qquad (1.1)$$

W.N. Bailey in 1944 stated a theorem which is simple but very useful.
If

$$\beta_n = \sum_{r=0}^{n}\alpha_r u_{n-r} v_{n+r} \qquad (1.2)$$

and

$$\gamma_n = \sum_{r=0}^{\infty} \delta_{r+n} u_r v_{r+2n}, \tag{1.3}$$

where α_r, δ_r, u_r and v_r are functions of r alone, such that the series γ_n exists, then

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n. \tag{1.4}$$

Bailey himself defined the Bailey pair relative to the parameter a as the pair of sequences satisfying the relation

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}}. \tag{1.5}$$

There are six sets of sequence $\langle \alpha_n \rangle$ for which $\langle \beta_n \rangle$ are known in closed form. Out of these six sets five are due to Bailey and one is due to Slater. For detail one is referred the paper of Verma, A. [5]

A W.P. Bailey pair is a pair of sequences $\{\alpha_n(a,k), \beta_n(a,k)\}$ satisfying

$$\alpha_0(a,k) = \beta_0(a,k) = 1$$

and

$$\beta_n(a,k) = \sum_{r=0}^{n} \frac{(k/a;q)_{n-r}(k;q)_{n+r}}{(q;q)_{n-r}(aq;q)_{n+r}} \alpha_r(a,k). \tag{1.6}$$

2. Theorems

In this section we establish certain theorems which will be used in next section.

(i) Let us choose $u_r = \frac{(k/a;q)_r}{(q;q)_r}$ and $v_r = \frac{(k;q)_r}{(aq;q)_r}$ in (1.2) we get

$$\beta_n(a,k) = \sum_{r=0}^{n} \frac{(k/a;q)_{n-r}(k;q)_{n+r}}{(q;q)_{n-r}(aq;q)_{n+r}} \alpha_r(a,k), \tag{2.1}$$

which is (1.6)

Now, (1.3) gives

$$\gamma_n(a,k) = \sum_{r=0}^{\infty} \frac{(k/a;q)_r(k;q)_{r+2n}}{(q;q)_r(aq;q)_{r+2n}} \delta_{r+n}.$$

$$= \frac{(k;q)_{2n}}{(aq;q)_{2n}} \sum_{r=0}^{\infty} \frac{(k/a;q)_r(kq^{2n};q)_r}{(q;q)_r(aq^{1+2n};q)_r} \delta_{r+n}. \tag{2.2}$$

If we choose $\delta_r = \left(\frac{a^2q}{k^2}\right)^r$, then

$$\gamma_n(a,k) = \frac{(k;q)_{2n}}{(aq;q)_{2n}} \left(\frac{a^2q}{k^2}\right)^n {}_2\Phi_1\left[\begin{matrix} k/a, kq^{2n}; q; a^2q/k^2 \\ aq^{2n+1} \end{matrix}\right]. \tag{2.3}$$

Summing ${}_2\Phi_1$ series in (2.3) by using [Slater 4; App. IV (IV.2)] we have

$$\gamma_n(a,k) = \frac{(aq/k, a^2q/k; q)_\infty}{(aq, a^2q/k^2; q)_\infty} \frac{(k;q)_{2n}}{(a^2q/k;q)_{2n}} \left(\frac{a^2q}{k^2}\right)^n. \tag{2.4}$$

Using (1.4) we get following theorem.

Theorem 1.

If $\{\alpha_n(a,q), \beta_n(a,k)\}$are WP Bailey pair then

$$\sum_{n=0}^{\infty} \left(\frac{a^2q}{k^2}\right)^n \beta_n(a,k) = \frac{(aq/k, a^2q/k; q)_\infty}{(aq, a^2q/k^2; q)_\infty} \sum_{n=0}^{\infty} \frac{(k;q)_{2n}}{(a^2q/k;q)_{2n}} \left(\frac{a^2q}{k^2}\right)^n \alpha_n(a,k). \tag{2.5}$$

This is a known identity [Laughlin 1; (1.7)]

(ii) Doing all as in (2.1), (2.2) and then taking $\delta_r = a^2/k^2$ we get

$$\gamma_n(a,k) = \frac{(k;q)_{2n}}{(aq;q)_{2n}} \left(\frac{a^2q}{k^2}\right)^n {}_2\Phi_1\left[\begin{matrix} k/a, kq^{2n}; q; a^2/k^2 \\ aq^{2n+1} \end{matrix}\right]. \tag{2.6}$$

Summing the ${}_2\Phi_1$ series in (2.6) with the help of following summation theorem

$${}_2\Phi_1\left[\begin{matrix} a, b; q; c/ab \\ cq \end{matrix}\right] = \frac{(cq/a, cq/b; q)_\infty}{(cq, cq/ab; q)_\infty} \left\{\frac{ab(1+c) - c(a+b)}{ab - c}\right\} \tag{2.7}$$

[Verma, A. 5; (1.4) p. 771]

we have

$$\gamma_n(a,k) = \frac{(k;q)_{2n}(1+aq^{2n})}{(a^2q/k;q)_{2n}} \left(\frac{k}{k+a}\right) \frac{(aq/k, a^2q/k; q)_\infty}{(aq, a^2q/k^2; q)_\infty} \left(\frac{a^2}{k^2}\right)^n. \tag{2.8}$$

Now, using (1.4) we find,

Theorem 2.

If $\{\alpha_n(a,k), \beta_n(a,k)\}$ are sequences forming a WP Bailey pair then

$$\sum_{n=0}^{\infty} \left(\frac{a^2}{k^2}\right)^n \beta_n(a,k) = \frac{(aq/k, a^2q/k; q)_\infty}{(aq, a^2q/k^2; q)_\infty} \left(\frac{k}{k+a}\right) \times$$

$$\times \sum_{n=0}^{\infty} \frac{(k;q)_{2n}}{(a^2q/k;q)_{2n}}(1+aq^{2n})\left(\frac{a^2}{k^2}\right)^n \alpha_n(a,k). \tag{2.9}$$

This identity is assumed to be new.

(iii) Doing all as in (2.1), (2.2) and then taking

$$\delta_r = \frac{(q\sqrt{k}, -q\sqrt{k}, \rho_1, \rho_2; q)_r}{(\sqrt{k}, -\sqrt{k}, kq/\rho_1, kq/\rho_2; q)_r}\left(\frac{aq}{\rho_1\rho_2}\right)^r$$

we get,

$$\gamma_n(a,k) = \frac{(k;q)_{2n}}{(aq;q)_n}\left(\frac{1-kq^{2n}}{1-k}\right)\frac{(\rho_1,\rho_2;q)_n}{(kq/\rho_1, kq/\rho_2;q)_n}\left(\frac{aq}{\rho_1\rho_2}\right)^n \times$$

$$\times\ {}_6\Phi_5\left[\begin{matrix} kq^{2n}, q^{n+1}\sqrt{k}, -q^{n+1}\sqrt{k}, \rho_1 q^n, \rho_2 q^n, k/a; q; \dfrac{aq}{\rho_1\rho_2} \\ q^n\sqrt{k}, -q^n\sqrt{k}, \dfrac{kq^{n+1}}{\rho_1}, \dfrac{kq^{n+1}}{\rho_2}, aq^{2n+1} \end{matrix}\right]. \tag{2.10}$$

Now, summing the ${}_6\Phi_5$ series using [Slater 4; App. IV (IV.7)] we get after some simplification,

$$\gamma_n(a,k) = \frac{(kq, kq/\rho_1\rho_2, aq/\rho_1, aq/\rho_2;q)_\infty(\rho_1,\rho_2;q)_n}{(aq, aq/\rho_1\rho_2, kq/\rho_1, kq/\rho_2;q)_\infty(aq/\rho_1, aq/\rho_2;q)_n}\left(\frac{aq}{\rho_1\rho_2}\right)^n. \tag{2.11}$$

Using (1.4) we have following theorem

Theorem 3.

If $\{\alpha_n(a,k), \beta_n(a,k)\}$ is a WP-Bailey pair then

$$\sum_{n=0}^{\infty}\left(\frac{1-kq^{2n}}{1-k}\right)\frac{(\rho_1,\rho_2;q)_n}{(kq/\rho_1, kq/\rho_2;q)_n}\left(\frac{aq}{\rho_1\rho_2}\right)^n \beta_n(a,k)$$

$$= \frac{(kq, kq/\rho_1\rho_2, aq/\rho_1, aq/\rho_2;q)_\infty}{(aq, aq/\rho_1\rho_2, kq/\rho_1, kq/\rho_2;q)_\infty}\sum_{n=0}^{\infty}\frac{(\rho_1,\rho_2;q)_n}{(aq/\rho_1, aq/\rho_2;q)_n}\left(\frac{aq}{\rho_1\rho_2}\right)^n \alpha_n(a,k). \tag{2.12}$$

This is also a known identity [Laughlin 2; theorem 1, p.3009]

(iv) Taking $\rho_2 = \sqrt{k}$ in (2.12) we get an another theorem,

Theorem 4.

If $\{\alpha_n(a,k), \beta_n(a,k)\}$ is a WP Bailey pair then

$$\sum_{n=0}^{\infty}\frac{(-q\sqrt{k}, \rho_1;q)_n}{(-\sqrt{k}, kq/\rho_1;q)_n}\left(\frac{aq}{\rho_1\sqrt{k}}\right)^n \beta_n(a,k)$$

$$= \frac{(kq, q\sqrt{k}/\rho_1, aq/\rho_1, aq/\sqrt{k}; q)_\infty}{(aq, aq/\rho_1\sqrt{k}, kq/\rho_1, q\sqrt{k}; q)_\infty} \sum_{n=0}^{\infty} \frac{(\rho_1, \sqrt{k}; q)_n}{(aq/\rho_1, aq/\sqrt{k}; q)_n} \left(\frac{aq}{\rho_1\sqrt{k}}\right)^n \alpha_n(a,k). \tag{2.13}$$

(v) Taking $\rho_1\rho_2 \to \infty$ in (2.12) we get another theorem,
Theorem 5.
If $\{\alpha_n(a,k), \beta_n(a,k)\}$ is a WP Bailey pair then

$$\sum_{n=0}^{\infty} \left(\frac{1-kq^{2n}}{1-k}\right) q^{n^2} a^n \beta_n(a,k) = \frac{(kq;q)_\infty}{(aq;q)_\infty} \sum_{n=0}^{\infty} a^n q^{n^2} \alpha_n(a,k). \tag{2.14}$$

(vi) Taking $\rho_1 \to \infty$ in (2.13) we get another theorem,
Theorem 6.
If $\{\alpha_n(a,k), \beta_n(a,k)\}$ is a WP Bailey pair then

$$\sum_{n=0}^{\infty} \frac{(-)^n(-q\sqrt{k};q)_n q^{n(n+1)/2}}{(-\sqrt{k};q)_n} \left(\frac{a}{\sqrt{k}}\right)^n \beta_n(a,k)$$

$$= \frac{(kq, aq/\sqrt{k}; q)_\infty}{(aq, q\sqrt{k}; q)_\infty} \sum_{n=0}^{\infty} \frac{(\sqrt{k};q)_n q^{n(n+1)/2}}{(aq/\sqrt{k};q)_n} \left(\frac{a}{\sqrt{k}}\right)^n (-)^n \alpha_n(a,k) \tag{2.15}$$

3. Main Results

In this section we shall establish our main transformation formulae.
(a) Unit WP Bailey pair is

$$\alpha_n(a,k) = \frac{(a, q\sqrt{a}, -q\sqrt{a}, a/k; q)_n}{(q, \sqrt{a}, -\sqrt{a}, kq; q)_n} \left(\frac{k}{a}\right)^n$$

$$\beta_n(a,k) = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{for } \quad n > 0. \end{cases} \tag{3.1}$$

Substituting this WP-Bailey pair in (2.5) and (2.9) we get following summation respectively,
(i)

$${}_8\Phi_7 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, a/k, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}; q; \dfrac{aq}{k} \\ \sqrt{a}, -\sqrt{a}, kq, \dfrac{aq}{\sqrt{k}}, -\dfrac{aq}{\sqrt{k}}, a\sqrt{\dfrac{q}{k}}, -a\sqrt{\dfrac{q}{k}} \end{matrix} \right]$$

$$= \frac{(aq, a^2q/k^2; q)_\infty}{(aq/k, a^2q/k; q)_\infty}. \tag{3.2}$$

(ii)

$$
{}_8\Phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, a/k, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}; q; \dfrac{a}{k} \\ \sqrt{a}, -\sqrt{a}, kq, \dfrac{aq}{\sqrt{k}}, -\dfrac{aq}{\sqrt{k}}, a\sqrt{\dfrac{q}{k}}, -a\sqrt{\dfrac{q}{k}} \end{matrix}\right]
$$

$$
+a\ {}_8\Phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, a/k, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}; q; \dfrac{aq^2}{k} \\ \sqrt{a}, -\sqrt{a}, kq, \dfrac{aq}{\sqrt{k}}, -\dfrac{aq}{\sqrt{k}}, a\sqrt{\dfrac{q}{k}}, -a\sqrt{\dfrac{q}{k}} \end{matrix}\right]
$$

$$
= \frac{(aq, a^2q/k^2; q)_\infty}{(aq/k, a^2q/k; q)_\infty}\left(\frac{k+a}{k}\right). \tag{3.3}
$$

(iii) Substituting the WP-Bailey pair of (3.1) in (2.12) we get,

$$
{}_6\Phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, \rho_1, \rho_2, a/k; q; \dfrac{kq}{\rho_1\rho_2} \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq}{\rho_1}, \dfrac{aq}{\rho_2}, kq \end{matrix}\right]
$$

$$
= \frac{(aq, aq/\rho_1\rho_2, kq/\rho_1, kq/\rho_2; q)_\infty}{(kq, kq/\rho_1\rho_2, aq/\rho_1, aq/\rho_2; q)_\infty} \tag{3.4}
$$

For $\rho_1, \rho_2 \to \infty$, (3.4) yields

$$
{}_4\Phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, a/k; q; kq \\ \sqrt{a}, -\sqrt{a}, kq, 0, 0 \end{matrix}\right] = \frac{(aq; q)_\infty}{(kq; q)_\infty}. \tag{3.5}
$$

(b) The trivial WP-Bailey pair is

$$
\alpha_n(a,k) = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{for } \quad n > 0. \end{cases}
$$

$$
\beta_n(a,k) = \frac{(k, k/a; q)_n}{(q, aq; q)_n} \tag{3.6}
$$

Substituting (3.6) in (2.5) we get

(i)

$$
{}_2\Phi_1\left[\begin{matrix} k, k/a; q; a^2q/k^2 \\ aq \end{matrix}\right] = \frac{(aq/k, a^2q/k; q)_\infty}{(aq, a^2q/k^2; q)_\infty}, \tag{3.7}
$$

which is Gauss summation formula. Substituting (3.6) in (2.9) we get,

$$ {}_2\phi_1\left[\begin{array}{l} k, k/a; q; a^2/k^2 \\ aq \end{array}\right] = \frac{(aq/k, a^2q/k; q)_\infty}{(aq, a^2q/k^2; q)_\infty}\left\{\frac{k(1+a)}{k+a}\right\}. \tag{3.8} $$

(ii) Substituting the WP-Bailey pair of (3.6) in (2.12) we get,

$$ {}_6\Phi_5\left[\begin{array}{l} k, q\sqrt{k}, -q\sqrt{k}, \rho_1, \rho_2, k/a; q; \dfrac{aq}{\rho_1\rho_2} \\ \sqrt{k}, -\sqrt{k}, \dfrac{kq}{\rho_1}, \dfrac{kq}{\rho_2}, aq \end{array}\right] $$

$$ = \frac{(kq, kq/\rho_1\rho_2, aq/\rho_1, aq/\rho_2; q)_\infty}{(aq, aq/\rho_1\rho_2, kq/\rho_1, kq/\rho_2; q)_\infty}, \tag{3.9} $$

which is same as (3.4).
(c) WP-Bailey pair due to Singh [3] is

$$ \alpha_n(a,k) = \frac{(a, q\sqrt{a}, -q\sqrt{a}, y, z, a^2q/kyz; q)_n}{(q, \sqrt{a}, -\sqrt{a}, aq/y, aq/z, kyz/a; q)_n}\left(\frac{k}{a}\right)^n $$

$$ \beta_n(a,k) = \frac{(ky/a, kz/a, k, aq/yz; q)_n}{(q, aq/y, aq/z, kyz/a; q)_n}. \tag{3.10} $$

Substituting WP-Bailey pair (3.10) in (2.5) we get,

$$ {}_4\Phi_3\left[\begin{array}{l} k, ky/a, kz/a, aq/yz; q; a^2q/k^2 \\ aq/y, aq/z, kyz/a \end{array}\right] = \frac{(aq/k, a^2q/k; q)_\infty}{(aq, a^2q/k^2; q)_\infty}\times $$

$$ \times {}_{10}\Phi_9\left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, y, z, \dfrac{a^2q}{kyz}, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}; q; \dfrac{aq}{k} \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq}{y}, \dfrac{aq}{z}, \dfrac{k\rho_1\rho_2}{a}, \dfrac{aq}{\sqrt{k}}, -\dfrac{aq}{\sqrt{k}}, a\sqrt{\dfrac{q}{k}}, -a\sqrt{\dfrac{q}{k}} \end{array}\right]. \tag{3.11} $$

Substituting WP-bailey pair (3.10) in (2.9) we have

$$ \frac{(aq, a^2q/k^2; q)_\infty}{(aq/k, a^2q/k; q)_\infty}\left(\frac{k+a}{k}\right)\ {}_4\Phi_3\left[\begin{array}{l} k, ky/a, kz/a, aq/yz; q; a^2/k^2 \\ aq/y, aq/z, kyz/a \end{array}\right] $$

$$ = {}_{10}\Phi_9\left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, y, z, \dfrac{a^2q}{k\rho_1\rho_2}, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}; q; \dfrac{a}{k} \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq}{y}, \dfrac{aq}{z}, \dfrac{k\rho_1\rho_2}{a}, \dfrac{aq}{\sqrt{k}}, -\dfrac{aq}{\sqrt{k}}, a\sqrt{\dfrac{q}{k}}, -a\sqrt{\dfrac{q}{k}} \end{array}\right] $$

$$+a\ {}_{10}\Phi_9\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, y, z, \dfrac{a^2q}{k\rho_1\rho_2}, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}; q; \dfrac{aq^2}{k} \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq}{y}, \dfrac{aq}{z}, \dfrac{k\rho_1\rho_2}{a}, \dfrac{aq}{\sqrt{k}}, -\dfrac{aq}{\sqrt{k}}, a\sqrt{\dfrac{q}{k}}, -a\sqrt{\dfrac{q}{k}} \end{matrix}\right]. \tag{3.12}$$

(iii) Substituting the WP-Bailey pair of (3.10) in(2.12) we get,

$$\begin{aligned} &{}_8\Phi_7\left[\begin{matrix} k, q\sqrt{k}, -q\sqrt{k}, \rho_1, \rho_2, ky/a, kz/a, aq/yz; q\dfrac{aq}{\rho_1\rho_2} \\ \sqrt{k}, -\sqrt{k}, kq, \dfrac{kq}{\rho_1}, \dfrac{kq}{\rho_2}, \dfrac{aq}{y}, \dfrac{aq}{z}, \dfrac{kyz}{a} \end{matrix}\right] \\ &= \frac{(kq, kq/\rho_1\rho_2, aq/\rho_1, aq/\rho_2; q)_\infty}{(aq, aq/\rho_1\rho_2, kq/\rho_1, kq/\rho_2; q)_\infty}\times \\ &\times {}_8\Phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, \rho_1, \rho_2, y, z, \frac{a^2q}{kyz}; q\dfrac{kq}{\rho_1\rho_2} \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq}{\rho_1}, \dfrac{aq}{\rho_2}, \dfrac{aq}{y}, \dfrac{aq}{z}, \dfrac{kyz}{a} \end{matrix}\right] \end{aligned} \tag{3.13}$$

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