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EDGE-COLORING VERTEX-WEIGHTING OF SOME PRODUCT GRAPHS

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Abstract: Let G be a graph. A $k-$vertex weighting of a graph G is a mapping $w : V(G) \to \{1, 2, 3, \ldots, k\}$. A $k-$vertex weighting induces an edge labeling $f_w : E(G) \to \mathbb{N}$ such that $f_w(uv) = w(u) + w(v)$. Such a labeling is called an edge-coloring $k-$weighting if $f_w(e) \neq f_w(e')$ for any two adjacent edges e and e'. Denote by $\mu'(G)$ the minimum k for G to admit an edge-coloring $k-$vertex weighting. In this paper, we determine $\mu'(G)$ for some product graphs.

Keywords and Phrases: Edge coloring, Vertex weighting, Cartesian product.

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1. Introduction and Preliminaries

Let G be a graph. For general notations and definitions we follow [1].

In [6], edge-coloring vertex-weighting introduced by WC Shiu et al.

A mapping $w : V(G) \to \{1, 2, 3, \ldots, k\}$ induces a edge labeling $f_w : E(G) \to \mathbb{N}$ such that $f_w(uv) = w(u)+w(v)$ is the sum of the weighting of the adjacent vertices. Such a labeling is called an *edge-coloring k-vertex-weighting* if $f_w(e) \neq f_w(e')$ for any two adjacent edges e and e'. Denote by $\mu'(G)$ the minimum k for G to admit an edge-coloring $k-$vertex weighting.

Note 1. $\mu'(G) = 1$ if and only if every component of G is a $\mathcal{K}_2$.
Note 2. Suppose w is an edge-coloring $k-$vertex weighting of G. If u and v have a common neighbor in G, then $w(u) \neq w(v)$. This is also a sufficient condition for an edge-coloring vertex-weighting.
Note 3. let $\chi'(G)$ be the chromatic index of G. Then $\mu'(G) \geq \chi'(G)$. Hence $\mu'(G) \geq \Delta(G)$, where $\Delta(G)$ is the maximum degree of G.

In [6], WC Shiu, GC Lau and HK Ng are determined edge-coloring vertex-weightings for paths, cycles, complete graphs, complete bipartite graphs, wheel graph, gear graph, Cartesian product of paths, Cartesian product of $\mathcal{P}_2$ with $\mathcal{C}_n$, double star graphs, trees, tadpole graph, lollipop graph, spider graph, theta graph and long dumbbell graph.

In this paper, we determined edge-coloring vertex-weightings for $\mathcal{P}_m \Box \mathcal{C}_n$, $\mathcal{K}_m \Box \mathcal{K}_n$ for some positive integers m and n.

2. Main Results

Theorem 2.1. *For $n \geq 3$, $\mu'(\mathcal{K}_2 \Box \mathcal{K}_n) = n$.*
Proof. Let the vertices of $\mathcal{K}_2 \Box \mathcal{K}_n$ be v_{ij}, $i \in \{1, 2\}$ and $1 \leq j \leq n$. First, we prove that $\mu'(\mathcal{K}_2 \Box \mathcal{K}_n) \leq n$. Assign the weighting to vertices of $\mathcal{K}_2 \Box \mathcal{K}_n$ by $w(v_{ij}) = j$, $i \in \{1, 2\}$ and $1 \leq j \leq n$. Next, we prove that $\mu'(\mathcal{K}_2 \Box \mathcal{K}_n) \geq n$. Since, $\Delta(\mathcal{K}_2 \Box \mathcal{K}_n) = n$, and by Note 3, $\mu'(\mathcal{K}_2 \Box \mathcal{K}_n) \geq n$. Thus, $\mu'(\mathcal{K}_2 \Box \mathcal{K}_n) = n$. This completes the proof.

Theorem 2.2. *For $m \geq 3$ and $n \geq 3$, $\mu'(\mathcal{K}_m \Box \mathcal{K}_n) = mn$.*
Proof. Let the vertices of $\mathcal{K}_m \Box \mathcal{K}_n$ be v_{ij}, $1 \leq i \leq m$ and $1 \leq j \leq n$. First, we prove that $\mu'(\mathcal{K}_m \Box \mathcal{K}_n) \leq mn$. Assign the weighting to vertices of $\mathcal{K}_m \Box \mathcal{K}_n$ by $w(v_{ij}) = (i-1)n + j$, $1 \leq i \leq m$ and $1 \leq j \leq n$.

Next, our aim is to prove $\mu'(\mathcal{K}_m \Box \mathcal{K}_n) \geq mn$. By Note 2, vertices of copy of K_n in the first row will receive the weighting $v_{1j} = j$, $1 \leq j \leq n$, since $\mathcal{K}_n$ is complete graph all vertices receive distinct label. In the second row, the copy of $\mathcal{K}_n$, suppose the vertex v_{21} label by any of $\{1, 2, 3, \ldots, n\}$ it contradict Note 2, since the vertices v_{21} and $v_{1j} = j$, $1 \leq j \leq n$ have a common neighbor. In general, any two vertices in $\mathcal{K}_m \Box \mathcal{K}_n$ have a common neighbor. So, no two distinct vertices receive same weighting. There are mn vertices in $\mathcal{K}_m \Box \mathcal{K}_n$. Hence, $\mu'(\mathcal{K}_m \Box \mathcal{K}_n) \geq mn$. Thus, $\mu'(\mathcal{K}_m \Box \mathcal{K}_n) = mn$. This completes the proof.

Theorem 2.3. *For $n \geq 4$, $n \not\equiv 2 \pmod 4$, $\mu'(\mathcal{P}_3 \Box \mathcal{C}_n) = 4$.*
Proof. Let the vertices of $\mathcal{P}_3 \Box \mathcal{C}_n$ be v_{ij}, $1 \leq i \leq 3$ and $1 \leq j \leq n$.
Case 1. $n \equiv 0 \pmod 4$.
First, we prove that $\mu'(\mathcal{P}_3 \Box \mathcal{C}_n) \leq 4$. Assign weight 1 to vertices v_{1j}, j is congruent

to 1, 2 modulo 4, and to vertices v_{3j}, j is congruent to 0, 3 modulo 4. Assign weight 2 to vertices v_{1j}, j is congruent to 0, 3 modulo 4, and to vertices v_{3j}, j is congruent to 1, 2 modulo 4.

Assign weight 3 to vertices v_{2j}, j is congruent to 0, 1 modulo 4, and assign weight 4 to vertices v_{2j}, j is congruent to 2, 3 modulo 4. Since, $\Delta(\mathcal{P}_3 \,\square\, \mathcal{C}_n) = 4$ and by Note 3, $\mu'(\mathcal{P}_3 \,\square\, \mathcal{C}_n) \geq 4$. Thus, $\mu'(\mathcal{P}_3 \,\square\, \mathcal{C}_n) = 4$.

Case 2. $n \equiv 1 (\mod 4)$.

First, we prove that $\mu'(\mathcal{P}_3 \,\square\, \mathcal{C}_n) \leq 4$.

Assign weight 1 to vertices v_{1j}, j is congruent to 1, 2 modulo 4, $j \neq n$, to vertex $v_{2(n-1)}$, and to vertices v_{3j}, j is congruent to 0, 3 modulo 4, $j \neq n-2$. Assign weight 2 to vertices v_{1j}, j is congruent to 0, 3 modulo 4, to the vertex v_{21}, and to vertices v_{3j}, j is congruent to 1, 2 modulo 4, $j \notin \{2, n\}$. Assign weight 3 to the vertex v_{1n}, to vertices v_{2j}, j is congruent to 0, 1 modulo 4, $j \neq n-1$, and to the vertices v_{32}, $v_{3(n-2)}$ and assign weight 4 to vertices v_{2j}, j is congruent to 2, 3 modulo 4, and the vertex v_{3n}. Since, $\Delta(\mathcal{P}_3 \,\square\, \mathcal{C}_n) = 4$ and by Note 3, $\mu'(\mathcal{P}_3 \,\square\, \mathcal{C}_n) \geq 4$. Thus, $\mu'(\mathcal{P}_3 \,\square\, \mathcal{C}_n) = 4$.

Case 3. $n \equiv 3 (\mod 4)$.

First, we prove that $\mu'(\mathcal{P}_3 \,\square\, \mathcal{C}_n) \leq 4$. Assign weight 1 to vertices v_{1j}, j is congruent to 1, 2 modulo 4, $j \neq n-1$, to vertex $v_{2(n-2)}$, and to vertices v_{3j}, j is congruent to 0, 3 modulo 4, $j \neq n-3$. Assign weight 2 to vertices v_{1j}, j is congruent to 0, 3 modulo 4, to the vertex v_{2n}, and to vertices v_{3j}, j is congruent to 1, 2 modulo 4, $j \notin \{1, n-1\}$. Assign weight 3 to the vertex $v_{1(n-1)}$, to vertices v_{2j}, j is congruent to 0, 1 modulo 4, $j \neq n-2$, and to the vertices v_{31}, $v_{3(n-3)}$ and assign weight 4 to vertices v_{2j}, j is congruent to 2, 3 modulo 4, $j \neq n$, and the vertex $v_{3(n-1)}$. Since, $\Delta(\mathcal{P}_3 \,\square\, \mathcal{C}_n) = 4$ and by Note 3, $\mu'(\mathcal{P}_3 \,\square\, \mathcal{C}_n) \geq 4$. Thus, $\mu'(\mathcal{P}_3 \,\square\, \mathcal{C}_n) = 4$. This completes the proof.

Theorem 2.4. *For $n \geq 4$, $n \not\equiv 2 (mod\ 4)$, $\mu'(\mathcal{P}_4 \,\square\, \mathcal{C}_n) = 4$.*

Proof. Let the vertices of $\mathcal{P}_4 \,\square\, \mathcal{C}_n$ be v_{ij}, $1 \leq i \leq 4$ and $1 \leq j \leq n$.

Case 1. $n \equiv 0 (\mod 4)$.

First, we prove that $\mu'(\mathcal{P}_4 \,\square\, \mathcal{C}_n) \leq 4$. Assign weight 1 to vertices v_{1j}, j is congruent to 1, 2 modulo 4, and to vertices v_{3j}, j is congruent to 0, 3 modulo 4. Assign weight 2 to vertices v_{1j}, j is congruent to 0, 3 modulo 4, and to vertices v_{3j}, j is congruent to 1, 2 modulo 4. Assign weight 3 to vertices v_{2j}, j is congruent to 0, 1 modulo 4, and to vertices v_{4j}, j is congruent to 2, 3 modulo 4, and assign weight 4 to vertices v_{2j}, j is congruent to 2, 3 modulo 4, to vertices v_{4j}, j is congruent to 0, 1 modulo 4. Since, $\Delta(\mathcal{P}_4 \,\square\, \mathcal{C}_n) = 4$ and by Note 3, $\mu'(\mathcal{P}_4 \,\square\, \mathcal{C}_n) \geq 4$. Thus, $\mu'(\mathcal{P}_4 \,\square\, \mathcal{C}_n) = 4$.

Case 2. $n \equiv 1 (\mod 4)$.

First, we prove that $\mu'(\mathcal{P}_4 \,\square\, \mathcal{C}_n) \leq 4$.

Subcase 2.1. $n = 5$.
Assign weight 1 to vertices $v_{11}, v_{12}, v_{24}, v_{34}, v_{41}, v_{42}$.
Assign weight 2 to vertices $v_{13}, v_{14}, v_{21}, v_{31}, v_{43}, v_{44}$.
Assign weight 3 to vertices $v_{15}, v_{25}, v_{32}, v_{33}$. Assign weight 4 to vertices $v_{22}, v_{23}, v_{35}, v_{45}$. Since, $\Delta(\mathcal{P}_3 \square \mathcal{C}_5) = 4$ and by Note 3, $\mu'(\mathcal{P}_4 \square \mathcal{C}_5) \geq 4$. Thus, $\mu'(\mathcal{P}_4 \square \mathcal{C}_5) = 4$.
Subcase 2.2. $n \geq 9$.

Assign weight 1 to vertices v_{1j}, j is congruent to 1, 2 modulo 4, $j \neq n$, to vertex $v_{2(n-1)}$, to vertices v_{3j}, j is congruent to 0, 3 modulo 4, $j \neq n-2$, and to the vertices $v_{41}, v_{4(n-3)}$. Assign weight 2 to vertices v_{1j}, j is congruent to 0, 3 modulo 4, to the vertex v_{21}, to vertices v_{3j}, j is congruent to 1, 2 modulo 4, $j \notin \{2, n\}$, and to vertices $v_{43}, v_{4(n-1)}$.

Assign weight 3 to the vertex v_{1n}, to vertices v_{2j}, j is congruent to 0, 1 modulo 4, $j \notin \{1, n-1\}$, to the vertices v_{32}, $v_{3(n-2)}$ and to vertices v_{4j}, j is congruent to 2, 3 modulo 4, $j \notin \{3, n-3\}$, assign weight 4 to vertices v_{2j}, j is congruent to 2, 3 modulo 4, the vertex v_{3n}, and to vertices v_{4j}, j is congruent to 0, 1 modulo 4, $j \neq n-1$. Since, $\Delta(\mathcal{P}_4 \square \mathcal{C}_n) = 4$ and by Note 3, $\mu'(\mathcal{P}_4 \square \mathcal{C}_n) \geq 4$. Thus, $\mu'(\mathcal{P}_4 \square \mathcal{C}_n) = 4$.
Case 3. $n \equiv 3 (\mod 4)$.
First, we prove that $\mu'(\mathcal{P}_4 \square \mathcal{C}_n) \leq 4$. Assign weight 1 to vertices v_{1j}, j is congruent to 1, 2 modulo 4, $j \neq n-1$, to vertex $v_{2(n-2)}$, to vertices v_{3j}, j is congruent to 0, 3 modulo 4, $j \neq n-3$, and to vertices $v_{4(n-4)}, v_{4n}$. Assign weight 2 to vertices v_{1j}, j is congruent to 0, 3 modulo 4, to the vertex v_{2n}, to vertices v_{3j}, j is congruent to 1, 2 modulo 4, $j \notin \{1, n-1\}$, and to vertices $v_{42}, v_{4(n-2)}$. Assign weight 3 to the vertex $v_{1(n-1)}$, to vertices v_{2j}, j is congruent to 0, 1 modulo 4, $j \neq n-2$, to the vertices v_{31}, $v_{3(n-3)}$, and to vertices v_{4j}, j is congruent to 2, 3 modulo 4, $j \notin \{2, n-4, n\}$, assign weight 4 to vertices v_{2j}, j is congruent to 2, 3 modulo 4, $j \neq n$, the vertex $v_{3(n-1)}$, and to vertices v_{4j}, j is congruent to 0, 1 modulo 4, $j \neq v_{4(n-2)}$. Since, $\Delta(\mathcal{P}_4 \square \mathcal{C}_n) = 4$ and by Note 3, $\mu'(\mathcal{P}_4 \square \mathcal{C}_n) \geq 4$. Thus, $\mu'(\mathcal{P}_4 \square \mathcal{C}_n) = 4$.

Theorem 2.5. *For* $m \in \{3, 4\}$, $\mu'(\mathcal{P}_m \square \mathcal{C}_3) = \Delta(\mathcal{P}_m \square \mathcal{C}_3) + 1 = 5$.
Proof. Let the vertices of $\mathcal{P}_m \square \mathcal{C}_3$ be v_{ij}, $1 \leq i \leq m$ and $1 \leq j \leq 3$.
Case 1. $m = 3$.
First, we prove that $\mu'(\mathcal{P}_3 \square \mathcal{C}_3) \leq 5$. Assign weight 1 to the vertices v_{11}, v_{33}, assign weight 2 to the vertices v_{12}, v_{31}, assign weight 3 to the vertices v_{13}, v_{23}, assign weight 4 to the vertex v_{21}, assign weight 5 to the vertices v_{22}, v_{32}. Since, $\Delta(\mathcal{P}_m \square \mathcal{C}_3) = 4$, $\mu'(\mathcal{P}_m \square \mathcal{C}_3) \geq 4$. Suppose assume that $\mu'(\mathcal{P}_m \square \mathcal{C}_3) = 4$. In the Cartesian product of $\mathcal{P}_3 \square \mathcal{C}_3$, the first row will be the copy of $\mathcal{C}_3$, by note 2, the vertices v_{11}, v_{12}, v_{13} will receive different weight, without loss of generality, assume that v_{11}, v_{12}, v_{13} receives 1, 2, and 3 respectively.

Next, assign weight to v_{21}. Since, v_{21} adjacent to v_{11}, we can assign weight to the vertex v_{21} by either 1 or 4, suppose v_{21} receive 2 or 3 it contradicts note 2.

Subcase 1.1. Assume that $w(v_{21}) = 1$, similarly, assign weight to v_{22} by 2 or 4 and assign weight to v_{23} by 3 or 4. Without loss of generality, assume that $w(v_{21}) = 1$, $w(v_{22}) = 2$, $w(v_{23}) = 3$. Now, assign weight to v_{31}, since, v_{21} adjacent to v_{11}, v_{22}, v_{23} v_{31} and v_{11}, v_{22}, v_{23} received colors 1, 2 and 3 respectively. By note 2, v_{31} receive weight 4. Next assign weight to v_{32}. Since, v_{22} adjacent to v_{12}, v_{21}, v_{23}, v_{32} and v_{12}, v_{21}, v_{23} received colors 1, 2 and 3 respectively, so v_{32} must receive weight 4, by note 2. Hence, $w(v_{31}) = w(v_{31}) = 4$, which contradicts that v_{31} and v_{32} have the common neighbor v_{33}. Thus, $w(v_{32}) = 5$.

Subcase 1.2. Assume that $w(v_{21}) = 4$. By note 2, $w(v_{22}) = 2$, $w(v_{23}) = 3$. Similarly, by note 2, $w(v_{31}) = 4$, and $w(v_{32}) = 1$. The adjacent vertices of v_{23} are v_{13}, v_{21}, v_{22} receives weights 3, 4 and 2 respectively, then $w(v_{33})$ must be 1 which contradicts that v_{33} and v_{32} have a common neighbor v_{31}. Hence, $w(v_{33}) = 5$. Same way we can prove for other possible cases.

Case 2. $m = 4$.

First, we prove that $\mu'(\mathcal{P}_4 \square \mathcal{C}_3) \leq 5$. Assign weight 1 to the vertices v_{11}, v_{33}, v_{43} assign weight 2 to the vertices v_{12}, v_{31}, assign weight 3 to the vertices v_{13}, v_{23}, v_{42} assign weight 4 to the vertices v_{21}, v_{31} assign weight 5 to the vertices v_{22}, v_{32}. Since, $\Delta(\mathcal{P}_m \square \mathcal{C}_3) = 4$, $\mu'(\mathcal{P}_m \square \mathcal{C}_3) \geq 4$. Suppose $\mu'(\mathcal{P}_m \square \mathcal{C}_3) = 4$, then we will get the contradictions as we discussed previous case. Hence, $\mu'(\mathcal{P}_m \square \mathcal{C}_3) = \Delta(\mathcal{P}_m \square \mathcal{C}_3) + 1 = 5$. This completes the proof.

3. Conclusion

We determined edge-coloring vertex-weightings for $\mathcal{P}_m \square \mathcal{C}_n$, $\mathcal{K}_m \square \mathcal{K}_n$ for some positive integers m and n.

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