J. of Ramanujan Society of Math. and Math. Sc. **ISSN : 2319-1023**
Vol.2, No.1 (2013), pp. 73-84

Transformation formulae for poly-basic hypergeometric series

(Received May 19, 2013)

Satya Prakash Singh, Vijay Yadav and Sunil Singh*,
Department of Mathematics,T.D.P.G. College, Jaunpur - 222002,(U.P.) India.
E-mail: snsp39@yahoo.com
* Department of Mathematics, Sydenham College of Commerce and Economics, Churchagate, Mumbai

Abstract: In this paper, we have established some very interesting transformation formulae for poly-basic hypergeometric series.
Keywords: Transformation formula, Summation formula, Continued fraction, Poly-basic hypergeometric series.
AMS subject classification code: Primary 33D10, 11F27

1. Introductions, Notations and Definitions:

As usual, for a and q complex numbers with $|q| < 1$, define

$$[a;q]_0 = 1$$
$$[a;q]_n = (1-a)(1-aq)(1-aq^2)...(1-aq^{n-1}), \quad \text{for } n \in N,$$
$$[a_1, a_2, a_3, ..., a_r; q]_n = [a_1;q]_n [a_2;q]_n [a_3;q]_n ... [a_r;q]_n,$$
$$[a;q]_\infty = \prod_{r=0}^{\infty}(1-aq^r).$$

An ${}_r\Phi_s$ basic hypergeometric series is defined by

$${}_r\Phi_s \left[\begin{matrix} a_1, a_2, a_3, ..., a_r; q; z \\ b_1, b_2, b_3, ..., b_s \end{matrix} \right]$$
$$= \sum_{n=0}^{\infty} \frac{[a_1, a_2, a_3, ..., a_r; q]_n}{[q, b_1, b_2, b_3, ..., b_s; q]_n} \left[(-)^n q^{n(n-1)/2}\right]^{1+s-r} z^n. \tag{1.1}$$

A poly-basic hypergeometric series is defined as,

$$\phi \left[\begin{matrix} a_1, a_2, ..., a_r : c_{1,1}, ..., c_{1,r_1}; ...; c_{m,1}, ..., c_{m,r_m}; q, q_1, ..., q_m; z \\ b_1, b_2, ..., b_s : d_{1,1}, ..., d_{1,s_1}; ...; d_{m,1}, ..., d_{m,s_m} \end{matrix} \right]$$

$$= \sum_{n=0}^{\infty} \frac{[a_1, a_2, a_3, ..., a_r; q]_n}{[q, b_1, b_2, b_3, ..., b_s; q]_n} \left[(-)^n q^{n(n-1)/2}\right]^{1+s-r} z^n \times$$

$$\times \prod_{j=1}^{m} \frac{\left[c_{j,1}, a_{j,2}, ..., c_{j,r_j}; q_j\right]_n}{\left[d_{j,1}, d_{j,2}, ..., d_{j,s_j}; q_j\right]_n} \left[(-)^n q^{n(n-1)/2}\right]^{s_j - r_j} . \tag{1.2}$$

We shall make use of following known results.

$$\sum_{n=0}^{\infty} (-a)^n q^{n(n+1)/2} = \frac{1}{1+} \frac{aq}{1+} \frac{a(q^2-q)}{1+} \frac{aq^3}{1+} \frac{a(q^4-q^2)}{1+...}. \tag{1.3}$$

[Andrews & Berndt 1;(6.2.29) p. 152]

$$\sum_{n=0}^{\infty} a^n q^{3n(n+1)/2} = [q;q]_\infty \sum_{n=0}^{\infty} \frac{[-aq^{n+1};q]_n q^n}{[q;q]_n}. \tag{1.4}$$

[Andrews & Berndt 1;(9.3.11) p. 230]

If

$$P_n(a) = [q^2;q^2]_\infty [-aq;q^2]_n \sum_{j=0}^{n} \left[\begin{array}{c} n \\ j \end{array}\right]_{q^2} \frac{q^{2j}}{[-aq;q^2]_j}$$

then

$\lim_{n\to\infty} P_n(a) = \sum_{n=0}^{\infty} a^j q^{j^2}.$

So,

$$\sum_{j=0}^{\infty} a^j q^{j^2} = [q^2;q^2]_\infty [-aq;q^2]_\infty \sum_{j=0}^{\infty} \frac{q^{2j}}{[q^2;q^2]_j [-aq;q^2]_j}. \tag{1.5}$$

[Andrews & Berndt 1;(13.8.2) p. 298]

If we put m = 0 in [Gasper & Rahman 2; App.II (II.36)] we get

$$\sum_{k=0}^{n} \frac{(1-adp^k q^k)\left(1-\frac{bp^k}{dq^k}\right)[a,b;p]_k \left[c, \frac{ad^2}{bc}; q\right]_k q^k}{(1-ad)(1-b/d)[dq, adq/b;q]_k [adp/c, bcp/d;p]_k}$$

$$= \frac{(1-a)(1-b)(1-c)\left(1-\frac{ad^2}{bc}\right)[ap,bp;p]_n \left[cq, \frac{ad^2q}{bc}; q\right]_n}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)[dq, adq/b;q]_n [adp/c, bcp/d;p]_n}$$

$$-\frac{a^2d(1-c/ad)(1-d/bc)(1-1/d)(1-b/ad)}{(1-ad)(1-b/d)(1-c/d)(1-ad/bc)}. \tag{1.6}$$

The main aim of this paper is to establish transformation formulae for poly-basic hypergeometric series. In this section we establish following theorems which will be used in next section.

Theorem 1.

If

$$\beta_n = \sum_{r=0}^{n} \alpha_r, \tag{1.7}$$

Then multiplying both sides of (1.7) by $\Omega_n z^n$ and summing over n from 0 to ∞ we have,

$$\sum_{n=0}^{\infty} \Omega_n \beta_n z^n = \sum_{n=0}^{\infty} \Omega_n z^n \sum_{r=0}^{n} \alpha_r,$$

which by an appeal of the identity,

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n} A(n,r) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} A(n+r,r),$$

[Srivastava and Manocha 3; Lemma 1 (2) p. 100]

gives

$$\sum_{n=0}^{\infty} \Omega_n \beta_n z^n = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \Omega_{n+r} z^{n+r} \alpha_r. \tag{1.8}$$

Most summation formulae for poly-basic hypergeometric series arises because the series involved telescope. Thus using the terms of such series for α_n in (1.7)we can find β_n. Again, putting these values of α_n and β_n in (1.8) we find transformation formulae for poly-basic hypergeometric series. Proceeding in this way we shall establish the following theorem.

Theorem 2.

Choosing $\alpha_r = \dfrac{(1-adp^rq^r)\left(1-\dfrac{bp^r}{dq^r}\right)[a,b;p]_r\left[c,\dfrac{ad^2}{bc};q\right]_r q^r}{(1-ad)(1-b/d)[dq,adq/b;q]_r[adp/c,bcp/d;p]_r}$ in (1.7) and using (1.8) we get,

$$\beta_n = \frac{(1-a)(1-b)(1-c)\left(1-\dfrac{ad^2}{bc}\right)[ap,bp;p]_n\left[cq,\dfrac{ad^2q}{bc};q\right]_n}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)[dq,adq/b;q]_n[adp/c,bcp/d;p]_n}$$

$$-\frac{a^2d(1-c/ad)(1-d/bc)(1-1/d)(1-b/ad)}{(1-ad)(1-b/d)(1-c/d)(1-ad/bc)}.$$

Putting these values of α_n and β_n in (1.8) we have,

$$\frac{(1-a)(1-b)(1-c)\left(1-\frac{ad^2}{bc}\right)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)}\times$$

$$\times\sum_{n=0}^{\infty}\Omega_n z^n\frac{[ap,bp;p]_n\left[cq,\frac{ad^2q}{bc};q\right]_n}{[dq,adq/b;q]_n[adp/c,bcp/d;p]_n}$$

$$-\frac{a^2d(1-c/ad)(1-d/bc)(1-1/d)(1-b/ad)}{(1-ad)(1-b/d)(1-c/d)(1-ad/bc)}\sum_{n=0}^{\infty}\Omega_n z^n$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{n+r}z^{n+r}\frac{(1-adp^rq^r)\left(1-\frac{bp^r}{dq^r}\right)[a,b;p]_r\left[c,\frac{ad^2}{bc};q\right]_r q^r}{(1-ad)(1-b/d)[dq,adq/b;q]_r[adp/c,bcp/d;p]_r} \qquad (1.9)$$

2. Main Transformation formulae

In this section we shall make use of the theorems of previous section in order to establish the transformation formulae.

(i) Taking $\Omega_n = q_1^{n(n+1)/2}$ in (1.9) we get,

$$\frac{(1-a)(1-b)(1-c)\left(1-\frac{ad^2}{bc}\right)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)}\times$$

$$\times\sum_{n=0}^{\infty}\frac{[ap,bp;p]_n\left[cq,\frac{ad^2q}{bc};q\right]_n z^n q_1^{n(n+1)/2}}{[dq,adq/b;q]_n[adp/c,bcp/d;p]_n}$$

$$-\frac{a^2d(1-c/ad)(1-d/bc)(1-1/d)(1-b/ad)}{(1-ad)(1-b/d)(1-c/d)(1-ad/bc)}\sum_{n=0}^{\infty}q_1^{n(n+1)/2}z^n$$

$$=\sum_{r=0}^{\infty}\frac{(1-adp^rq^r)\left(1-\frac{bp^r}{dq^r}\right)[a,b;p]_r\left[c,\frac{ad^2}{bc};q\right]_r (zq)^r q_1^{r(r+1)/2}}{(1-ad)(1-b/d)[dq,adq/b;q]_r[adp/c,bcp/d;p]_r}\times$$

$$\times \sum_{n=0}^{\infty} (zq_1^r)^n q_1^{n(n+1)/2}. \tag{2.1}$$

(ii) Using (1.3) in (2.1) we get,

$$\frac{(1-a)(1-b)(1-c)\left(1-\frac{ad^2}{bc}\right)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)} \times$$

$$\times \sum_{n=0}^{\infty} \frac{[ap, bp; p]_n \left[cq, \frac{ad^2q}{bc}; q\right]_n z^n q_1^{n(n+1)/2}}{[dq, adq/b; q]_n [adp/c, bcp/d; p]_n}$$

$$= \sum_{r=0}^{\infty} \frac{(1-adp^r q^r)\left(1-\frac{bp^r}{dq^r}\right)[a,b;p]_r \left[c, \frac{ad^2}{bc}; q\right]_r (zq)^r q_1^{r(r+1)/2}}{(1-ad)(1-b/d)[dq, adq/b; q]_r [adp/c, bcp/d; p]_r} \times$$

$$\times \left\{ \frac{1}{1-} \frac{zq_1^{r+1}}{1+} \frac{zq_1^{r+1}(1-q_1)}{1-} \frac{zq_1^{r+3}}{1+} \frac{zq_1^{r+2}(1-q_1^2)}{1-\ldots} \right\}$$

$$+ \frac{a^2d(1-c/ad)(1-d/bc)(1-1/d)(1-b/ad)}{(1-ad)(1-b/d)(1-c/d)(1-ad/bc)} \times$$

$$\times \left\{ \frac{1}{1-} \frac{zq_1}{1+} \frac{zq_1(1-q_1)}{1-} \frac{zq_1^3}{1+} \frac{zq_1^2(1-q_1^2)}{1-\ldots} \right\}. \tag{2.2}$$

(iii) Taking $\Omega_n = q_1^{n^2}$ in (1.9) we get,

$$\frac{(1-a)(1-b)(1-c)\left(1-\frac{ad^2}{bc}\right)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)} \times$$

$$\times \sum_{n=0}^{\infty} \frac{[ap, bp; p]_n \left[cq, \frac{ad^2q}{bc}; q\right]_n z^n q_1^{n^2}}{[dq, adq/b; q]_n [adp/c, bcp/d; p]_n}$$

$$= \sum_{r=0}^{\infty} \frac{(1-adp^r q^r)\left(1-\frac{bp^r}{dq^r}\right)[a,b;p]_r \left[c, \frac{ad^2}{bc}; q\right]_r (zq)^r q_1^r}{(1-ad)(1-b/d)[dq, adq/b; q]_r [adp/c, bcp/d; p]_r} \sum_{n=0}^{\infty} (zq_1^{2r})^n q_1^{n^2}$$

$$+ \frac{a^2d(1-c/ad)(1-d/bc)(1-1/d)(1-b/ad)}{(1-ad)(1-b/d)(1-c/d)(1-ad/bc)} \sum_{n=0}^{\infty} z^n q_1^{n^2}. \tag{2.3}$$

(iv) Using (1.5) on the right hand side of (2.3) we find,

$$\frac{(1-a)(1-b)(1-c)\left(1-\frac{ad^2}{bc}\right)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)}\times$$

$$\times\sum_{n=0}^{\infty}\frac{[ap,bp;p]_n\left[cq,\frac{ad^2q}{bc};q\right]_n z^n q_1^{n^2}}{[dq,adq/b;q]_n[adp/c,bcp/d;p]_n}=[q_1^2;q_1^2]_\infty[-zq_1;q_1^2]_\infty\times$$

$$\times\sum_{r=0}^{\infty}\frac{(1-adp^rq^r)\left(1-\frac{bp^r}{dq^r}\right)[a,b;p]_r\left[c,\frac{ad^2}{bc};q\right]_r (zq)^r q_1^{r^2}}{(1-ad)(1-b/d)[dq,adq/b;q]_r[adp/c,bcp/d;p]_r}\,{}_2\Phi_1\left[\begin{matrix}0,0;q_1^2;q_1^2\\-zq_1^{2r+1}\end{matrix}\right]$$

$$+[q_1^2;q_1^2]_\infty[-zq_1;q_1^2]_\infty\frac{a^2d(1-c/ad)(1-d/bc)(1-1/d)(1-b/ad)}{(1-ad)(1-b/d)(1-c/d)(1-ad/bc)}\times$$

$$\times{}_2\Phi_1\left[\begin{matrix}0,0;q_1^2;q_1^2\\-zq_1\end{matrix}\right]. \tag{2.4}$$

(v) Replacing q_1 by q_1^3 in (2.1) we get,

$$\frac{(1-a)(1-b)(1-c)\left(1-\frac{ad^2}{bc}\right)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)}\times$$

$$\times\sum_{n=0}^{\infty}\frac{[ap,bp;p]_n\left[cq,\frac{ad^2q}{bc};q\right]_n z^n q_1^{3n(n+1)/2}}{[dq,adq/b;q]_n[adp/c,bcp/d;p]_n}$$

$$=\sum_{r=0}^{\infty}\frac{(1-adp^rq^r)\left(1-\frac{bp^r}{dq^r}\right)[a,b;p]_r\left[c,\frac{ad^2}{bc};q\right]_r (zq)^r q_1^{3r(r+1)/2}}{(1-ad)(1-b/d)[dq,adq/b;q]_r[adp/c,bcp/d;p]_r}\times$$

$$\times\sum_{n=0}^{\infty}(zq_1^{3r})^n q_1^{3n(n+1)/2}$$

$$+\frac{a^2d(1-c/ad)(1-d/bc)(1-1/d)(1-b/ad)}{(1-ad)(1-b/d)(1-c/d)(1-ad/bc)}\sum_{n=0}^{\infty}q_1^{3n(n+1)/2}z^n \tag{2.5}$$

(vi) Now, using (1.4) in (2.5) we get,

$$\frac{(1-a)(1-b)(1-c)\left(1-\frac{ad^2}{bc}\right)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)}\times$$

$$\times\sum_{n=0}^{\infty}\frac{[ap,bp;p]_n\left[cq,\frac{ad^2q}{bc};q\right]_n z^n q_1^{3n(n+1)/2}}{[dq,adq/b;q]_n[adp/c,bcp/d;p]_n}$$

$$=[q_1;q_1]_\infty\sum_{r=0}^{\infty}\frac{(1-adp^rq^r)\left(1-\frac{bp^r}{dq^r}\right)[a,b;p]_r\left[c,\frac{ad^2}{bc};q\right]_r(zq)^rq_1^{3r(r+1)/2}}{(1-ad)(1-b/d)[dq,adq/b;q]_r[adp/c,bcp/d;p]_r}\times$$

$$\times{}_4\Phi_3\left[\begin{array}{l}i\sqrt{zq_1^{3r+1}},-i\sqrt{zq_1^{3r+1}},iq_1\sqrt{zq_1^{3r}},-iq_1\sqrt{zq_1^{3r}};q;q\\0,0,-zq_1^{3r}\end{array}\right]$$

$$+[q_1;q_1]_\infty\frac{a^2d(1-c/ad)(1-d/bc)(1-1/d)(1-b/ad)}{(1-ad)(1-b/d)(1-c/d)(1-ad/bc)}\times$$

$$\times{}_4\Phi_3\left[\begin{array}{l}i\sqrt{zq_1},-i\sqrt{zq_1},iq_1\sqrt{z},-iq_1\sqrt{z};q;q\\0,0,-zq\end{array}\right]. \tag{2.6}$$

3. Special Cases

In this section we shall deduce certain interesting transformations from the results established in previous section.

(i) Putting d = 1 in (2.1) we get,

$$\sum_{n=0}^{\infty}\frac{[ap,bp;p]_n\left[cq,\frac{aq}{bc};q\right]_n z^nq_1^{n(n+1)/2}}{[q,aq/b;q]_n[ap/c,bcp;p]_n}$$

$$=\sum_{r=0}^{\infty}\frac{(1-ap^rq^r)\left(1-\frac{bp^r}{q^r}\right)[a,b;p]_r\left[c,\frac{a}{bc};q\right]_r(zq)^rq_1^{r(r+1)/2}}{(1-a)(1-b)[q,aq/b;q]_r[ap/c,bcp;p]_r}\times$$

$$\times\sum_{n=0}^{\infty}(zq_1^r)^nq_1^{n(n+1)/2}. \tag{3.1}$$

(ii) For d = 1, (2.2) yields,

$$\sum_{n=0}^{\infty}\frac{[ap,bp;p]_n\left[cq,\frac{aq}{bc};q\right]_n z^nq_1^{n(n+1)/2}}{[q,aq/b;q]_n[ap/c,bcp;p]_n}$$

$$= \sum_{r=0}^{\infty} \frac{(1-ap^r q^r)\left(1-\frac{bp^r}{q^r}\right)[a,b;p]_r\left[c,\frac{a}{bc};q\right]_r q_1^{r(r+1)/2}(zq)^r}{(1-a)(1-b)[q,aq/b;q]_r[ap/c,bcp;p]_r}\times$$

$$\times\left\{\frac{1}{1-}\frac{zq_1^{r+1}}{1+}\frac{zq_1^{r+1}(1-q_1)}{1-}\frac{zq_1^{r+3}}{1+}\frac{zq_1^{r+2}(1-q_1^2)}{1-...}\right\} \tag{3.2}$$

(iii) As $q_1 \to 1$ (3.1) gives,

$$(1-z)\sum_{n=0}^{\infty}\frac{[ap,bp;p]_n\left[cq,\frac{aq}{bc};q\right]_n z^n}{[q,aq/b;q]_n[ap/c,bcp;p]_n}$$

$$= \sum_{r=0}^{\infty} \frac{(1-ap^r q^r)\left(1-\frac{bp^r}{q^r}\right)[a,b;p]_r\left[c,\frac{a}{bc};q\right]_r (zq)^r}{(1-a)(1-b)[q,aq/b;q]_r[ap/c,bcp;p]_r}. \tag{3.3}$$

(iv) Taking c = q in (3.2) we have,

$$\sum_{n=0}^{\infty}\frac{[ap,bp;p]_n\left[\frac{a}{b};q\right]_n z^n q_1^{n(n+1)/2}}{[q,aq/b;q]_n[ap/q,bqp;p]_n}\left(\frac{1-q^{n+1}}{1-q}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(1-ap^r q^r)\left(1-\frac{bp^r}{q^r}\right)[a,b;p]_r\left[\frac{a}{bq};q\right]_r q_1^{r(r+1)/2}(zq)^r}{(1-a)(1-b)[aq/b;q]_r[ap/q,bqp;p]_r}\times$$

$$\times\left\{\frac{1}{1-}\frac{zq_1^{r+1}}{1+}\frac{zq_1^{r+1}(1-q_1)}{1-}\frac{zq_1^{r+3}}{1+}\frac{zq_1^{r+2}(1-q_1^2)}{1-...}\right\}. \tag{3.4}$$

(v) Taking b = 0 in (3.2) we get,

$$\sum_{n=0}^{\infty}\frac{[ap;p]_n\,[cq;q]_n\left(\frac{z}{c}\right)^n q_1^{n(n+1)/2}}{[q;q]_n[ap/c;p]_n}$$

$$= \sum_{r=0}^{\infty} \frac{(1-ap^r q^r)[a;p]_r\,[c;q]_r\,q_1^{r(r+1)/2}\left(\frac{zq}{c}\right)^r}{(1-a)[q;q]_r[ap/c;p]_r}\times$$

$$\times\left\{\frac{1}{1-}\frac{zq_1^{r+1}}{1+}\frac{zq_1^{r+1}(1-q_1)}{1-}\frac{zq_1^{r+3}}{1+}\frac{zq_1^{r+2}(1-q_1^2)}{1-...}\right\}. \tag{3.5}$$

(vi) Taking d = 1 in (2.4) we get,

$$\times \sum_{n=0}^{\infty} \frac{[ap, bp; p]_n \left[cq, \frac{aq}{bc}; q\right]_n z^n q_1^{n^2}}{[q, aq/b; q]_n [ap/c, bcp; p]_n} = [q_1^2; q_1^2]_\infty [-zq_1; q_1^2]_\infty \times$$

$$\times \sum_{r=0}^{\infty} \frac{(1 - ap^r q^r) \left(1 - \frac{bp^r}{q^r}\right) [a, b; p]_r \left[c, \frac{a}{bc}; q\right]_r (zq)^r q_1^{r^2}}{(1-a)(1-b)[q, aq/b; q]_r [ap/c, bcp; p]_r [-zq_1; q_1^2]_r} {}_2\Phi_1 \left[\begin{array}{l} 0, 0; q_1^2; q_1^2 \\ -zq_1^{2r+1} \end{array} \right]. \quad (3.6)$$

(vii) Taking d = 1 in (2.6) we have,

$$\sum_{n=0}^{\infty} \frac{[ap, bp; p]_n \left[cq, \frac{aq}{bc}; q\right]_n z^n q_1^{3n(n+1)/2}}{[q, aq/b; q]_n [ap/c, bcp; p]_n}$$

$$= [q_1; q_1]_\infty \sum_{r=0}^{\infty} \frac{(1 - ap^r q^r) \left(1 - \frac{bp^r}{q^r}\right) [a, b; p]_r \left[c, \frac{a}{bc}; q\right]_r (zq)^r q_1^{3r(r+1)/2}}{(1-a)(1-b)[q, aq/b; q]_r [ap/c, bcp; p]_r} \times$$

$$\times {}_4\Phi_3 \left[\begin{array}{l} i\sqrt{zq_1^{3r+1}}, -i\sqrt{zq_1^{3r+1}}, iq_1\sqrt{zq_1^{3r}}, -iq_1\sqrt{zq_1^{3r}}; q; q \\ 0, 0, -zq_1^{3r} \end{array} \right] \quad (3.7)$$

(viii) Taking $p = q = q_1$ in (3.2) we have,

$${}_4\Phi_4 \left[\begin{array}{l} aq, bq, cq, aq/bc; q; -zq \\ aq/b, aq/c, bcq, 0 \end{array} \right]$$

$$= \sum_{r=0}^{\infty} \frac{(1 - aq^{2r})[a, b, c, a/bc; q]_r q^{r(r+1)/2} (zq)^r}{(1-a)[aq/b, aq/c, bcq; q]_r [q; q]_r} \times$$

$$\times \left\{ \frac{1}{1-} \frac{zq^{r+1}}{1+} \frac{zq^{r+1}(1-q)}{1-} \frac{zq^{r+3}}{1+} \frac{zq^{r+2}(1-q^2)}{1-\ldots} \right\}. \quad (3.8)$$

As $a \to 1$, (3.8) yields,

$${}_4\Phi_4 \left[\begin{array}{l} q, bq, cq, q/bc; q; -zq \\ q/b, q/c, bcq, 0 \end{array} \right]$$

$$= \left\{ \frac{1}{1-} \frac{zq}{1+} \frac{zq(1-q)}{1-} \frac{zq^3}{1+} \frac{zq^2(1-q^2)}{1-\ldots} \right\}$$

$$+\sum_{r=1}^{\infty}(1+q^r)\frac{[b,c,1/bc;q]_r(zq)^r q^{r(r+1)/2}}{[q/b,q/c,bcq;q]_r}\times$$

$$\times\left\{\frac{1}{1-}\frac{zq^{r+1}}{1+}\frac{zq^{r+1}(1-q)}{1-}\frac{zq^{r+3}}{1+}\frac{zq^{r+2}(1-q^2)}{1-...}\right\} \tag{3.9}$$

For b = 1, (3.9) yields,

$$\sum_{n=0}^{\infty}z^n q^{n(n+1)/2}=\left\{\frac{1}{1-}\frac{zq}{1+}\frac{zq(1-q)}{1-}\frac{zq^3}{1+}\frac{zq^2(1-q^2)}{1-...}\right\} \tag{3.10}$$

which is a known result (1.3).
Taking $b, c \to \infty$ in (3.9) we find,

$$\sum_{r=0}^{\infty}(-z)^r q^{r(r+1)}=\frac{1}{1-}\frac{zq}{1+}\frac{zq(1-q)}{1-}\frac{zq^3}{1+}\frac{zq^2(1-q^2)}{1-...}$$

$$+\sum_{r=1}^{\infty}(1+q^r)(-z)^r q^{r^2}\left\{\frac{1}{1-}\frac{zq^{r+1}}{1+}\frac{zq^{r+1}(1-q)}{1-}\frac{zq^{r+3}}{1+}\frac{zq^{r+2}(1-q^2)}{1-...}\right\}. \tag{3.11}$$

Using (1.3) on the left hand side of (3.11) we obtain,

$$\frac{1}{1+}\frac{zq^2}{1-}\frac{zq^2(1-q^2)}{1+}\frac{zq^6}{1-}\frac{zq^4(1-q^4)}{1+...}$$

$$=\frac{1}{1-}\frac{zq}{1+}\frac{zq(1-q)}{1-}\frac{zq^3}{1+}\frac{zq^2(1-q^2)}{1-...}$$

$$+\sum_{r=1}^{\infty}(1+q^r)(-z)^r q^{r^2}\left\{\frac{1}{1-}\frac{zq^{r+1}}{1+}\frac{zq^{r+1}(1-q)}{1-}\frac{zq^{r+3}}{1+}\frac{zq^{r+2}(1-q^2)}{1-...}\right\}. \tag{3.12}$$

Similar results can also be scored proceeding as above by taking theorems

Acknowledgement

The authors are thankful to Department of Science and Technology, Government of India, New Delhi, for support under a major research project No. SR/ S4/ MS : 561 / 08 entitled "Investigations of Ramanujan's work in the field of basic hypergeometric series".

References

[1] Andrews, G. E. and Berndt, B. C., Ramanujan's Lost Notebook Part I, Springer 2005.

[2] Gasper, G. and Rahman, M., Basic Hypergeometric Series, Cambridge University Press 1991.

[3] Srivasatva, H. M. and Manocha, H. L., A trestise on generating functions, Ellishorwood limited, Halsted Press; a division of John Wiley and Sons,New York 1984.