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On Certain Results Involving Mock- Theta Functions

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Abstract: In this paper, making use of certain identities, we have established interesting results involving mock-theta functions, partial mock theta functions and complete mock theta functions.
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1. Notations and definitions:

Throughout this note, we shall adopt the following definitions and notations. The q- factorial is defined by,

$$[a;q]_0 = 1$$

$$[a;q]_n = (1-a)(1-aq)...(1-aq^{n-1}), \quad n = 1,2,3,...,$$

and

$$[a;q]_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

For product of q- shifted factorials, we use the short notation

$$[a_1, a_2, a_3, ..., a_r; q]_n = [a_1;q]_n[a_2;q]_n[a_3;q]_n...[a_r;q]_n,$$

where n is an integer or infinity.
Basic and bilateral basic hypergeometric series are defined by,

$${}_r\Phi_s\left[\begin{array}{l} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s \end{array}\right] = \sum_{n=0}^{\infty}\frac{[a_1, a_2, ..., a_r; q]_n z^n}{[q, b_1, b_2, ..., b_s; q]_n}\left[(-)^n q^{n(n-1)/2}\right]^{1+s-r}.$$

and

$${}_r\Psi_s\left[\begin{array}{l} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s \end{array}\right] = \sum_{n=-\infty}^{\infty}\frac{[a_1, a_2, ..., a_r; q]_n z^n}{[b_1, b_2, ..., b_s; q]_n}\left[(-)^n q^{n(n-1)/2}\right]^{s-r}.$$

respectively. We shall use the following known partial sums in our analysis.

$${}_2\Phi_1\left[\begin{array}{l} a, y; q; q \\ ayz \end{array}\right]_N = \sum_{r=0}^{N}\frac{[a, y; q]_r q^r}{[q, ayq; q]_r} = \frac{[aq, yq; q]_N}{[q, ayq; q]_N}. \tag{1.1}$$

Taking y=0 in (1.1) we get another summation,

$$ {}_1\Phi_0\left[\begin{array}{c} a;q;q \\ - \end{array}\right]_N = \sum_{r=0}^{N} \frac{[a;q]_r q^r}{[q;q]_r} = \frac{[aq;q]_N}{[q;q]_N}. \tag{1.2} $$

As $n \to \infty$, (1.1) and (1.2) give following summation formulae respectively,

$$ {}_2\Phi_1\left[\begin{array}{c} a,y;q;q \\ ayz \end{array}\right] = \sum_{r=0}^{\infty} \frac{[a,y;q]_n q^n}{[q,ayq;q]_n} = \frac{[aq,yq;q]_\infty}{[q,ayq;q]_\infty}. \tag{1.3} $$

$$ {}_1\Phi_0\left[\begin{array}{c} a;q;q \\ - \end{array}\right] = \sum_{r=0}^{\infty} \frac{[a;q]_r q^r}{[q;q]_r} = \frac{[aq;q]_\infty}{[q;q]_\infty}. \tag{1.4} $$

(a) The complete set of mock theta functions of order three are
Mock theta function of order three

$$ f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{[-q;q]_n^2}, \qquad \Phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{[-q^2;q^2]_n}, $$

$$ \Psi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{[q;q^2]_n}, \qquad \chi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{[-\omega q,-\omega^2 q;q]_n} $$

$$ \omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{[q;q^2]_{n+1}^2}, \qquad \rho(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{[\omega q,\omega^2 q;q^2]_{n+1}} $$

$$ \nu(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{[-q;q^2]_{n+1}} $$

Partial mock theta functions of order three

$$ f_N(q) = \sum_{n=0}^{N} \frac{q^{n^2}}{[-q;q]_n^2}, \qquad \Phi_N(q) = \sum_{n=0}^{N} \frac{q^{n^2}}{[-q^2;q^2]_n}, $$

$$ \Psi_N(q) = \sum_{n=0}^{N} \frac{q^{n^2}}{[q;q^2]_n}, \qquad \chi_N(q) = \sum_{n=0}^{N} \frac{q^{n^2}}{[-\omega q,-\omega^2 q;q]_n} $$

$$ \omega_N(q) = \sum_{n=0}^{N} \frac{q^{2n(n+1)}}{[q;q^2]_{n+1}^2}, \qquad \rho_N(q) = \sum_{n=0}^{N} \frac{q^{2n(n+1)}}{[\omega q,\omega^2 q;q^2]_{n+1}} $$

$$ \nu_N(q) = \sum_{n=0}^{N} \frac{q^{n(n+1)}}{[-q;q^2]_{n+1}} $$

Complete mock theta functions of order three

$$ f_c(q) = \sum_{n=-\infty}^{\infty} \frac{q^{n^2}}{[-q;q]_n^2}, \qquad \Phi_c(q) = \sum_{n=-\infty}^{\infty} \frac{q^{n^2}}{[-q^2;q^2]_n}, $$

$$\Psi_c(q)=\sum_{n=-\infty}^{\infty}\frac{q^{n^2}}{[q;q^2]_n},\qquad \chi_c(q)=\sum_{n=-\infty}^{\infty}\frac{q^{n^2}}{[-\omega q,-\omega^2 q;q]_n}$$

$$\omega_c(q)=\sum_{n=-\infty}^{\infty}\frac{q^{2n(n+1)}}{[q;q^2]_{n+1}^2},\qquad \rho_c(q)=\sum_{n=-\infty}^{\infty}\frac{q^{2n(n+1)}}{[\omega q,\omega^2 q;q^2]_{n+1}}$$

$$\nu_c(q)=\sum_{n=-\infty}^{\infty}\frac{q^{n(n+1)}}{[-q;q^2]_{n+1}}$$

(b) Mock theta functions of order five (first group)

$$f_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{[-q;q]_n},\qquad \Phi_0(q)=\sum_{n=0}^{\infty}q^{n^2}[-q;q^2]_n,$$

$$\Psi_0(q)=\sum_{n=0}^{\infty}[-q;q]_n q^{(n+1)(n+2)/2},\qquad F_0(q)=\sum_{n=0}^{\infty}\frac{q^{2n^2}}{[q;q^2]_n},$$

$$\chi_0(q)=\sum_{n=0}^{\infty}\frac{q^n[q;q]_n}{[q;q]_{2n}},$$

Partial mock theta functions of order five (first group)

$$f_{0,N}(q)=\sum_{n=0}^{N}\frac{q^{n^2}}{[-q;q]_n},\qquad \Phi_{0,N}(q)=\sum_{n=0}^{N}q^{n^2}[-q;q^2]_n,$$

$$\Psi_{0,N}(q)=\sum_{n=0}^{N}[-q;q]_n q^{(n+1)(n+2)/2},\qquad F_{0,N}(q)=\sum_{n=0}^{N}\frac{q^{2n^2}}{[q;q^2]_n},$$

$$\chi_{0,N}(q)=\sum_{n=0}^{N}\frac{q^n[q;q]_n}{[q;q]_{2n}},$$

Complete mock theta functions of order five (first group)

$$f_{0,c}(q)=\sum_{n=-\infty}^{\infty}\frac{q^{n^2}}{[-q;q]_n},\qquad \Phi_{0,c}(q)=\sum_{n=-\infty}^{\infty}q^{n^2}[-q;q^2]_n,$$

$$\Psi_{0,c}(q)=\sum_{n=-\infty}^{\infty}[-q;q]_n q^{(n+1)(n+2)/2},\qquad F_{0,c}(q)=\sum_{n=-\infty}^{\infty}\frac{q^{2n^2}}{[q;q^2]_n},$$

$$\chi_{0,c}(q)=\sum_{n=-\infty}^{\infty}\frac{q^n[q;q]_n}{[q;q]_{2n}},$$

(c) Mock theta functions of order five (second group)

$$f_1(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{[-q;q]_n},\qquad \Phi_1(q)=\sum_{n=0}^{\infty}q^{(n+1)^2}[-q;q^2]_n,$$

$$\Psi_1(q)=\sum_{n=0}^{\infty}[-q;q]_n q^{n(n+1)/2},\qquad F_1(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{[q;q^2]_{n+1}},$$

$$\chi_1(q) = \sum_{n=0}^{\infty} \frac{q^n[q;q]_n}{[q;q]_{2n+1}}$$

Partial mock theta functions of order five (second group)

$$f_{1,N}(q) = \sum_{n=0}^{N} \frac{q^{n(n+1)}}{[-q;q]_n}, \qquad \Phi_{1,N}(q) = \sum_{n=0}^{N} q^{(n+1)^2}[-q;q^2]_n,$$

$$\Psi_{1,N}(q) = \sum_{n=0}^{N}[-q;q]_n q^{n(n+1)/2}, \qquad F_{1,N}(q) = \sum_{n=0}^{N} \frac{q^{2n(n+1)}}{[q;q^2]_{n+1}},$$

$$\chi_{1,N}(q) = \sum_{n=0}^{N} \frac{q^n[q;q]_n}{[q;q]_{2n+1}}$$

Complete mock theta functions of order five (second group)

$$f_{1,c}(q) = \sum_{n=-\infty}^{\infty} \frac{q^{n(n+1)}}{[-q;q]_n}, \qquad \Phi_{1,c}(q) = \sum_{n=-\infty}^{\infty} q^{(n+1)^2}[-q;q^2]_n,$$

$$\Psi_{1,c}(q) = \sum_{n=-\infty}^{\infty} [-q;q]_n q^{n(n+1)/2}, \qquad F_{1,c}(q) = \sum_{n=-\infty}^{\infty} \frac{q^{2n(n+1)}}{[q;q^2]_{n+1}},$$

$$\chi_{1,c}(q) = \sum_{n=-\infty}^{\infty} \frac{q^n[q;q]_n}{[q;q]_{2n+1}}$$

(d) Mock theta functions of order seven

$$\Im_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}[q;q]_n}{[q;q]_{2n}}, \qquad \Im_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}[q;q]_n}{[q;q]_{2n+1}}$$

$$\Im_2(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}[q;q]_n}{[q;q]_{2n+1}}$$

Partial Mock theta functions of order seven

$$\Im_{0,N}(q) = \sum_{n=0}^{N} \frac{q^{n^2}[q;q]_n}{[q;q]_{2n}}, \qquad \Im_{1,N}(q) = \sum_{n=0}^{N} \frac{q^{(n+1)^2}[q;q]_n}{[q;q]_{2n+1}}$$

$$\Im_{2,N}(q) = \sum_{n=0}^{N} \frac{q^{n(n+1)}[q;q]_n}{[q;q]_{2n+1}}$$

Complete Mock theta functions of order seven

$$\Im_{0,c}(q) = \sum_{n=-\infty}^{\infty} \frac{q^{n^2}[q;q]_n}{[q;q]_{2n}}, \qquad \Im_{1,c}(q) = \sum_{n=-\infty}^{\infty} \frac{q^{(n+1)^2}[q;q]_n}{[q;q]_{2n+1}}$$

$$\Im_{2,c}(q) = \sum_{n=-\infty}^{\infty} \frac{q^{n(n+1)}[q;q]_n}{[q;q]_{2n+1}}$$

2. Identities

In this section we establish two identities which will be used in next section.
(a) Bailey in 1947 establish the following lemma which is simple but very useful.
If

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r} \tag{2.1}$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n} \tag{2.2}$$

Then under suitable conditions of convergence

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \tag{2.3}$$

provided $\alpha_r, \delta_r, u_r, v_r$ are functions of r alone and the infinite series defining γ_n is convergent.

If we take $u_r = v_r = 1$ in the above lemma than it takes the following form, If

$$\beta_n = \sum_{r=0}^{n} \alpha_r, \qquad \gamma_n = \sum_{r=n}^{\infty} \delta_r \tag{2.4}$$

Then under suitable conditions of convergence

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n. \tag{2.5}$$

Using (2.4) in (2.5) we have,

$$\sum_{n=0}^{\infty} \alpha_n \left\{ \sum_{r=0}^{\infty} \delta_r - \sum_{r=0}^{n} \delta_r + \delta_n \right\} = \sum_{n=0}^{\infty} \delta_n \sum_{r=0}^{n} \alpha_r$$

which on simplification gives the identity,

$$\sum_{n=0}^{\infty} \alpha_n \sum_{r=0}^{\infty} \delta_r = \sum_{n=0}^{\infty} \delta_n \sum_{r=0}^{n} \alpha_r + \sum_{n=0}^{\infty} \alpha_n \sum_{r=0}^{n} \delta_r - \sum_{n=0}^{\infty} \alpha_n \delta_n, \tag{2.6}$$

where α_n and δ_n are any two arbitrary sequences.
(b) The symmetric bilateral Bailey transform due to Andrews and Warnaar [2] is given by, If

$$\beta_n = \sum_{r=-n}^{n} \alpha_r u_{n-r} v_{n+r} \tag{2.7}$$

and

$$\gamma_n = \sum_{r=|n|}^{\infty} \delta_r u_{r-n} v_{r+n} \tag{2.8}$$

Then

$$\sum_{n=-\infty}^{\infty} \alpha_n\gamma_n = \sum_{n=0}^{\infty} \beta_n\delta_n, \tag{2.9}$$

subject to conditions on the four sequences $\alpha_n, \beta_n, \gamma_n$ and δ_n which make all the relevant infinite series absolutely convergent.
Taking $u_r = v_r = 1$ in (2.7) and (2.9) we have, If

$$\beta_n = \sum_{r=-n}^{n} \alpha_r, \qquad \gamma_n = \sum_{r=n}^{\infty} \delta_r$$

Then

$$\sum_{n=-\infty}^{\infty} \alpha_n\gamma_n = \sum_{n=0}^{\infty} \beta_n\delta_n. \tag{2.10}$$

provided all infinite series are convergent.
Now, we can write (2.10) as,

$$\sum_{n=-\infty}^{\infty} \alpha_n \sum_{r=n}^{\infty} \delta_r = \sum_{n=0}^{\infty} \delta_n \sum_{r=-n}^{n} \alpha_n$$

or

$$\sum_{n=-\infty}^{\infty} \alpha_n \left\{ \sum_{r=0}^{\infty} \delta_r - \sum_{r=0}^{n} \delta_r + \delta_n \right\} = \sum_{n=0}^{\infty} \delta_n \sum_{r=-n}^{n} \alpha_r$$

$$\sum_{n=-\infty}^{\infty} \alpha_n \sum_{r=0}^{\infty} \delta_r = \sum_{n=0}^{\infty} \delta_n \sum_{r=-n}^{n} \alpha_r + \sum_{n=-\infty}^{\infty} \alpha_n \sum_{r=0}^{n} \delta_r - \sum_{n=-\infty}^{\infty} \alpha_n\delta_n.$$

Thus we have the identity

$$\sum_{n=-\infty}^{\infty} \alpha_n \sum_{r=0}^{\infty} \delta_r = \sum_{n=0}^{\infty} \delta_n \sum_{r=-n}^{n} \alpha_r + \sum_{n=-\infty}^{\infty} \alpha_n \sum_{r=0}^{n} \delta_r - \sum_{n=-\infty}^{\infty} \alpha_n\delta_n, \tag{2.11}$$

where α_r and δ_r arbitrary sequences.

3. Main Results

In this section we shall make use of the identities (2.6) and (2.11) in order to establish certain results involving mock theta functions.

(i) Taking $\delta_r = \frac{[a, y; q]_r q^r}{[q, ayq; q]_r}$ in (2.6) we get,

$$\sum_{n=0}^{\infty} \alpha_n \frac{[aq, yq; q]_\infty}{[q, ayq; q]_\infty} = \sum_{n=0}^{\infty} \frac{[a, y; q]_n q^n}{[q, ayq; q]_n} \sum_{r=0}^{n} \alpha_r + \sum_{n=0}^{\infty} \alpha_n \frac{[aq, yq; q]_n}{[q, ayq; q]_n} - \sum_{n=0}^{\infty} \frac{[a, y; q]_n q^n}{[q, ayq; q]_n} \alpha_n,$$

which on simplification gives

$$\frac{[aq, yq; q]_\infty}{[q, ayq; q]_\infty} \sum_{n=0}^{\infty} \alpha_n = \sum_{n=0}^{\infty} \frac{[a, y; q]_n q^n}{[q, ayq; q]_n} \sum_{r=0}^{n} \alpha_r + \sum_{n=0}^{\infty} \frac{[aq, yq; q]_n}{[q, ayq; q]_n} \alpha_{n+1}. \tag{3.1}$$

We shall make use of (3.1) in order to give series representation of different mock theta functions.

(ii) Choosing $\alpha_n = \dfrac{q^{n^2}}{[-q;q]_n^2}$ in (3.1) we find,

$$\frac{[aq,yq;q]_\infty}{[q,ayq;q]_\infty} f(q) = \sum_{n=0}^{\infty} \frac{[a,y;q]_n q^n}{[q,ayq;q]_n} f_n(q) + \sum_{n=0}^{\infty} \frac{[aq,yq;q]_n q^{(n+1)^2}}{[q,ayq;q]_n [-q;q]_{n+1}^2}. \tag{3.2}$$

For $a = y = -1$ in (3.2) we have,

$$\frac{[-q;q]_\infty^2}{[q;q]_\infty^2} f(q) = \sum_{n=0}^{\infty} \frac{[-1;q]_n^2 q^n}{[q;q]_n^2} f_n(q) + \sum_{n=0}^{\infty} \frac{[-q;q]_n^2 q^{(n+1)^2}}{[q;q]_n^2 [-q;q]_{n+1}^2}, \tag{3.3}$$

Other mock theta functions of order three can be expressed similarly by proper choice of α_n in (3.1).

(iii) Choosing $\alpha_n = \dfrac{q^{n^2}}{[-q;q]_n}$ in (3.1) we get,

$$\frac{[aq,yq;q]_\infty}{[q,ayq;q]_\infty} f_0(q) = \sum_{n=0}^{\infty} \frac{[a,y;q]_n q^n}{[q,ayq;q]_n} f_{0,n}(q) + \sum_{n=0}^{\infty} \frac{[aq,yq;q]_n q^{(n+1)^2}}{[q,ayq;q]_n [-q;q]_{n+1}}, \tag{3.4}$$

where $f_0(q)$ is a mock theta function of order five belonging to the first group.
Taking $a = -1$ in (3.4) we have,

$$\frac{[-q,yq;q]_\infty}{[q,-yq;q]_\infty} f_0(q) = \sum_{n=0}^{\infty} \frac{[-1,y;q]_n q^n}{[q,-yq;q]_n} f_{0,n}(q) + \sum_{n=0}^{\infty} \frac{[yq;q]_n q^{(n+1)^2}}{[q,-yq;q]_n (1+q^{n+1})}. \tag{3.5}$$

Taking $y = -1$ in (3.5) we have,

$$\left\{\frac{[-q;q]_\infty}{[q;q]_\infty}\right\}^2 f_1(q) = \sum_{n=0}^{\infty} \frac{[-1;q]_n^2 q^n}{[q;q]_n^2} f_{0,n}(q) + \sum_{n=0}^{\infty} \frac{[-q;q]_n q^{(n+1)^2}}{[q;q]_n^2 (1+q^{n+1})}. \tag{3.6}$$

(iv) Choosing $\alpha_n = \dfrac{q^{n(n+1)}}{[-q;q]_n}$ in (3.1) we get,

$$\frac{[aq,yq;q]_\infty}{[q,ayq;q]_\infty} f_1(q) = \sum_{n=0}^{\infty} \frac{[a,y;q]_n q^n}{[q,ayq;q]_n} f_{1,n}(q) + \sum_{n=0}^{\infty} \frac{[aq,yq;q]_n q^{(n+1)(n+2)}}{[q,ayq;q]_n [-q;q]_{n+1}}, \tag{3.7}$$

Taking $a = -1, y = -1$ in (3.7) we get,

$$\frac{[-q;q]_\infty^2}{[q;q]_\infty^2} f_1(q) = \sum_{n=0}^{\infty} \frac{[-1;q]_n^2 q^n}{[q;q]_n^2} f_{1,n}(q) + \sum_{n=0}^{\infty} \frac{[-q;q]_n q^{(n+1)(n+2)}}{[q;q]_n^2 (1+q^{n+1})}. \tag{3.8}$$

(v) Choosing $\alpha_n = \dfrac{q^{n^2}[q;q]_n}{[q;q]_{2n}}$ in (3.1) we get,

$$\frac{[aq,yq;q]_\infty}{[q,ayq;q]_\infty} \Im_0(q) = \sum_{n=0}^{\infty} \frac{[a,y;q]_n q^n}{[q,ayq;q]_n} \Im_{0,n}(q) + \sum_{n=0}^{\infty} \frac{[aq,yq;q]_n q^{(n+1)^2} [q;q]_{n+1}}{[q,ayq;q]_n [q;q]_{2n+2}}, \tag{3.9}$$

for $y = 0$ it gives

$$\frac{[aq;q]_\infty}{[q;q]_\infty}\Im_0(q) = \sum_{n=0}^{\infty}\frac{[q;q]_n q^n}{[q;q]_n}\Im_{0,n}(q) + \sum_{n=0}^{\infty}\frac{[aq;q]_n q^{(n+1)^2}(1+q^{n+1})}{[q;q]_{2n+2}}. \tag{3.10}$$

taking $a = -1$ in (3.10) we have,

$$\frac{[-q;q]_\infty}{[q;q]_\infty}\Im_0(q) = \sum_{n=0}^{\infty}\frac{[-1;q]_n q^n}{[q;q]_n}\Im_{0,n}(q) + \sum_{n=0}^{\infty}\left(\frac{1-q^{n+1}}{1+q^{n+1}}\right)\frac{q^{(n+1)^2}}{[q;q]_{n+1}[q;q^2]_{n+1}}. \tag{3.11}$$

Similar representations for other mock theta functions of order seven can be established by proper choice of α_n in (3.1).

4. Product formulae for mock theta functions

In this section we shall establish product formulae for any two mock theta functions

(i) Choosing $\alpha_n = \dfrac{q^{n^2}}{[-q;q]_n^2}$ and $\delta_n = \dfrac{q^{n^2}}{[-q^2;q^2]_n}$ in (2.6) we get,

$$f(q)\Phi(q) = \sum_{n=0}^{\infty}\frac{q^{n^2}}{[-q^2;q^2]_n}f_n(q) + \sum_{n=0}^{\infty}\frac{q^{n^2}}{[-q;q]_n^2}\Phi_n(q) - \sum_{n=0}^{\infty}\frac{q^{2n^2}}{[-q;q]_n^2[-q^2;q^2]_n}. \tag{4.1}$$

This is a product formula for two mock theta functions of order three. Similar product formulae for other pairs of mock theta functions of order three can also be established.

(ii) Choosing $\alpha_n = \dfrac{q^{n^2}}{[-q;q]_n^2}$ and $\delta_n = \dfrac{q^{n(n+1)}}{[-q;q]_n}$ in (2.6) we get,

$$f(q)f_1(q) = \sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{[-q;q]_n}f_n(q) + \sum_{n=0}^{\infty}\frac{q^{n^2}}{[-q;q]_n^2}f_{1,n}(q) - \sum_{n=0}^{\infty}\frac{q^{n(2n+1)}}{[-q;q]_n^3}, \tag{4.2}$$

where $f(q)$ is a mock theta function of order three and $f_1(q)$ is a mock theta function of order five belonging to the second group.

(iii) Choosing $\alpha_n = \dfrac{q^{n^2}}{[-q;q]_n^2}$ and $\delta_n = \dfrac{q^{n^2}}{[-q;q]_n}$ in (2.6) we get,

$$f(q)f_0(q) = \sum_{n=0}^{\infty}\frac{q^{n^2}}{[-q;q]_n}f_n(q) + \sum_{n=0}^{\infty}\frac{q^{n^2}}{[-q;q]_n^2}f_{0,n}(q) - \sum_{n=0}^{\infty}\frac{q^{2n^2}}{[-q;q]_n^3}, \tag{4.3}$$

where $f_0(q)$ is a mock theta function of order five belonging to the first group. Thus similar results can be established of mock theta function of order three and mock theta function of order five.

(iv) Choosing $\alpha_n = \dfrac{q^{n^2}}{[-q;q]_n}$ and $\delta_n = \dfrac{q^{n(n+1)}}{[-q;q]_n}$ in (2.6) we get,

$$f_0(q)f_1(q) = \sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{[-q;q]_n}f_{0,n}(q) + \sum_{n=0}^{\infty}\frac{q^{n^2}}{[-q;q]_n}f_{1,n}(q) - \sum_{n=0}^{\infty}\frac{q^{n(2n+1)}}{[-q;q]_n^2}, \tag{4.4}$$

where $f_0(q)$ and $f_1(q)$ are mock theta function of order five one belonging to the first group and another belonging to the second group respectively. Similar results for any pair of mock theta function of order five can be established.

(v) Choosing $\alpha_n = \dfrac{q^{n^2}}{[-q;q]_n^2}$ and $\delta_n = \dfrac{q^{n^2}[q;q]_n}{[q;q]_{2n}}$ in (2.6) we get,

$$f(q)\Im_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}[q;q]_n}{[q;q]_{2n}} f_n(q) + \sum_{n=0}^{\infty} \frac{q^{n^2}}{[-q;q]_n^2} \Im_{0,n}(q) - \sum_{n=0}^{\infty} \frac{q^{2n^2}[q;q]_n}{[-q;q]_n^2[q;q]_{2n}}, \tag{4.5}$$

where $f(q)$ is a mock theta function of order three and $\Im_0(q)$ is a mock theta function of order seven. Similar results involving any mock theta function of order three and any mock theta function of order seven can be established.

(vi) Choosing $\alpha_n = \dfrac{q^{n^2}}{[-q;q]_n}$ and $\delta_n = \dfrac{q^{n^2}[q;q]_n}{[q;q]_{2n}}$ in (2.6) we get,

$$f(q)\Im_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}[q;q]_n}{[q;q]_{2n}} f_{0,n}(q) + \sum_{n=0}^{\infty} \frac{q^{n^2}}{[-q;q]_n} \Im_{0,n}(q) - \sum_{n=0}^{\infty} \frac{q^{2n^2}[q;q]_n}{[-q;q]_n[q;q]_{2n}}, \tag{4.6}$$

where $f_0(q)$ is a mock theta function of order five.

(vii) Choosing $\alpha_n = \dfrac{q^{n^2}[q;q]_n}{[q;q]_{2n}}$ and $\delta_n = \dfrac{[q;q]_n q^{n(n+1)^2}}{[q;q]_{2n+1}}$ in (2.6) we get,

$$\Im_0(q)\Im_1(q) = \sum_{n=0}^{\infty} \frac{[q;q]_n q^{n(n+1)^2}}{[q;q]_{2n+1}} \Im_{0,n}(q) + \sum_{n=0}^{\infty} \frac{[q;q]_n q^{n^2}}{[q;q]_{2n}} \Im_{1,n}(q) - \sum_{n=0}^{\infty} \frac{[q;q]_n^2 q^{2n(n+1)+1}}{[q;q]_{2n}[q;q]_{2n+1}}, \tag{4.7}$$

where $\Im_0(q)$ and $\Im_1(q)$ are mock theta functions of order seven. Similar other results can also be scored.

5. New Product Formulae

In this section we shall make use of (2.11) in order to establish product formulae for mock theta functions.

(i) Choosing $\alpha_n = \dfrac{q^{n^2}}{[-q;q]_n^2}$ and $\delta_n = \dfrac{q^{n^2}}{[-q^2;q^2]_n}$ in (2.11) we get,

$$f_c(q)\Phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{[-q^2;q^2]_n} f_{(-n,n)}(q) + \sum_{n=-\infty}^{\infty} \frac{q^{n^2}}{[-q;q]_n^2} \Phi_n(q) - \sum_{n=-\infty}^{\infty} \frac{q^{2n^2}}{[-q;q]_n^2[-q^2;q^2]_n}, \tag{5.1}$$

where $f_c(q)$ is complete mock theta function of order three and $\Phi(q)$ is another mock theta function of order three.

(ii) Choosing $\alpha_n = \dfrac{q^{n^2}}{[-q;q]_n^2}$ and $\delta_n = \dfrac{q^{n^2}}{[-q;q]_n}$ in (2.11) we get,

$$f_c(q)f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{[-q;q]_n} f_{(-n,n)}(q) + \sum_{n=-\infty}^{\infty} \frac{q^{n^2}}{[-q;q]_n^2} f_{0,n}(q) - \sum_{n=-\infty}^{\infty} \frac{q^{2n^2}}{[-q;q]_n^3}, \tag{5.2}$$

where $\Phi(q)$ is a mock theta function of order five belonging to the first group.

(iii) Choosing $\alpha_n = \dfrac{q^{n^2}}{[-q;q]_n^2}$ and $\delta_n = \dfrac{q^{n(n+1)}}{[-q;q]_n}$ in (2.11) we get,

$$f_c(q)f_1(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{[-q;q]_n} f_{(-n,n)}(q) + \sum_{n=-\infty}^{\infty} \frac{q^{n^2}}{[-q;q]_n^2} f_{1,n}(q) - \sum_{n=-\infty}^{\infty} \frac{q^{n(2n+1)}}{[-q;q]_n^3}, \tag{5.3}$$

where $f_1(q)$ is a mock theta function of order five belonging to the second group.

(iv) Choosing $\alpha_n = \dfrac{q^{n^2}}{[-q;q]_n^2}$ and $\delta_n = \dfrac{q^{n^2}[q;q]_n}{[q;q]_{2n}}$ in (2.11) we get,

$$f_c(q)\Im_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}[q;q]_n}{[q;q]_{2n}} f_{(-n,n)}(q) + \sum_{n=-\infty}^{\infty} \frac{q^{n^2}}{[-q;q]_n^2} \Im_{0,n}(q) - \sum_{n=-\infty}^{\infty} \frac{q^{2n^2}[q;q]_n}{[-q;q]_n^2[q;q]_{2n}}, \tag{5.4}$$

where $\Im_0(q)$ is a mock theta function of order seven and

$$f_{(-n,n)}(q) = \sum_{r=-n}^{n} \frac{q^{r^2}}{[-q;q]_r^2}.$$

Thus we can establish similar expression for the product of any two mock theta functions of same order or of different order.

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