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A FOUR-COMPARTMENT MODEL TO ESTIMATE OXYGEN AND CARBON DIOXIDE EXCHANGE CONCENTRATIONS VIA BLOOD USING EIGENVALUE APPROACH

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Abstract: A mathematical model of oxygen and carbon dioxide transport via blood in the human body has been formulated. The model is represented by four compartments: alveolar tissue, arterial blood, tissue and venous blood. The aim of this study is to estimate the concentration profiles of oxygen and carbon dioxide over alveolar tissue, arterial blood, tissue and venous blood compartments. The formulation is based on the transport of oxygen from atmospheric air to alveolar tissue and subsequently to capillary bed through inspiration and back flow of carbon dioxide through expiration. Ordinary differential equations and balance law have been employed to formulate compartment-wise transport phenomenon of both oxygen and carbon dioxide in the respiratory tract via blood. The solution of the model has been obtained using eigenvalue approach. The model provide the information regarding absorption rate of oxygen and release rate of carbon dioxide at the respective compartments. The results obtained in this study may help clinical and bio-medical sciences to deal with respiratory ailments faced by the people living at high altitudes. The results are in agreement with those arrived at by N. S. Cherniack et al. (1968). In addition, these results may have utility in biomedical engineering and physiological research problems.

Keywords and Phrases: Oxygen tension, Carbon dioxide tension, Blood; Compartment model, Balance law, Transfer rate, Eigenvalue method.

2020 Mathematics Subject Classification: 92-10, 92BXX, 92CXX.

1. Introduction

The respiratory system in human body plays a key role in transport and exchange of oxygen (O_2) and carbon dioxide (CO_2) between the surrounding atmosphere and the tissues in the body. Oxygen is essential for life and it has been verified by Nunn [9, 15, 16] that a human being consumes about 260 ml/min under normal conditions. Oxygen is transported from atmospheric air into the lungs via respiratory tract and is then carried in blood as dissolved in plasma and bound to hemoglobin in RBC. It is transported to tissue via plasma (not directly from Hb-O_2). Hemoglobin acts as an O_2 storage to increase the oxygen carrying capacity of blood. Carbon dioxide is carried by the blood as a waste product of oxidative metabolism in opposite direction as that of oxygen transport, from tissues via blood into the lungs, where it is removed by ventilation. The elimination rate of CO_2 is approximately 160 ml/min under normal conditions as estimated by Nunn [9, 15, 16]. In addition to other functions the acid-base balance in the blood is maintained by the exchange of O_2 and CO_2 in the body.

The respiration starts from the outside air towards lungs by inspiration. Air enters the lungs connecting the atmospheric air with the alveoli - the air filled sacs. From the alveoli O_2 diffuses across the wall of pulmonary capillaries into the blood and transported in association with hemoglobin. The distribution of gases throughout the body is accomplished by the blood circulation and it is important to mention here that gas diffuses independently from the area where its partial pressure is higher to the area where its partial pressure is lower. The larger partial pressure differences accelerate the rates of gas diffusion. The oxygen leaves the blood stream by diffusion and it enters the cells of the target tissue, where it is used in respiratory metabolism. The oxygen transported through blood is utilized by the cells in the metabolism that drives growth and other activities of the body. The metabolic processes produce CO_2 that is transported into pulmonary capillaries and finally diffuses across the lung membrane into alveoli. From the alveoli, it is transported through the respiratory tract to the atmosphere.

Oxygen and carbon dioxide transport is known to play an important role in cellular respiration operating in tissues. Oxygen is available free of cost in the environment and therefore, humans need to be more conscious about the utility of this valuable gas. Human tissues can survive for many days without food but not without transport and exchange of oxygen and carbon dioxide. The first simple mathematical model for oxygen transport to tissue was formulated by Krogh [13]. The model is based on the concept that oxygen from a capillary diffuses only into a tissue cylinder concentric with the capillary, so that the O_2 flux at the external

surface of the cylinder vanishes. This model was further extended and modified by Blum [1] by adding capillary wall resistance; Salathe et al. [19] introduced and revised the model by adding flow conditions in the capillary; Reneau et al. [18] refined the model in light of intra-capillary diffusion and convective effect of blood and axial diffusion in the tissue. The mathematical study of O_2 uptake in spherical tissues has also been studied by many researchers like Lin [14], Simpson and Ellery [21] and Clark et al. [6]. Lin [14] studied diffusion of oxygen in a spherical cell with non-linear uptake. Simpson and Ellery [21] studied steady state model of oxygen transport in spherical tissue through Maclaurin's series. This model was originally reported by Lin [14] to represent the distribution of oxygen inside a cell and has since been studied extensively by both the numerical analysis and formal analysis communities. They extended these previous studies by deriving an analytic solution to a generalized reaction-diffusion equation that encompasses Lin's model as a particular case. Clark et al. [6] developed a mathematical model of oxygen diffusion across the bovine and murine cumulus-oocyte complex. They simulated their results by MATLAB software and finally concluded that the model equations cannot be solved analytically. However, Khanday and Najar [11] solved the model equations analytically and the process of solution was based on the Maclaurin's series method. Khanday and Najar [11, 12] computed the amount of oxygen uptake in a spherical tissue through the capillary bed using analytical approach as well as finite element approach.

Oxygen and carbon dioxide tensions of blood are important chemical determinants of the level of ventilation. To a large extent, these tensions are determined by the amount and distribution of oxygen and carbon dioxide stored in the body. A mathematical model of carbon dioxide stores, reported by Cherniack et al. [5], which closely predicted the changing carbon dioxide tensions during hyperventilation and apnea at constant arterial oxygen tensions. In that model, carbon dioxide was considered to be stored in multiple compartments representing several different organs. Each compartment was assigned its own blood flow rate, metabolic rate, and dissociation curve for carbon dioxide. In this paper, we formulate a mathematical model of oxygen and carbon dioxide transport via blood in the human body; and estimate the concentration profiles of O_2 and CO_2 over alveolar tissue, arterial blood, tissue and venous blood compartments. The model is represented by four compartments: alveolar tissue, arterial blood, tissue and venous blood. The transport and exchange mechanism of O_2 and CO_2 in the human body has partial pressure as a main driving force. The concentration profiles of oxygen and carbon dioxide at alveolar tissue, arterial blood, tissue and venous blood compartments have been estimated with respect to time, initial O_2 and CO_2 concentration and

transfer rate from one compartment to another. The results obtained in this study are in agreement with those arrived at by Cherniack et al. [5].

2. Mathematical Model and Solution

2.1. Postulated Conditions

Before setting up the O_2 and CO_2 transport model equations, following assumptions were made:

(i) Blood flowing through the capillaries is treated as a homogeneous mixture of erythrocytes and plasma.

(ii) The interaction of O_2 and CO_2 in the blood is ignored.

(iii) The source and sink for both O_2 and CO_2 are taken constants.

(iv) The mass exchange between the compartments is represented by concentration gradient (diffusion).

(v) The production of carbon dioxide in the tissue is taken to be proportional to the consumption of oxygen.

2.2. Nomenclature

The parameters used in the formulation of O_2 and CO_2 transport model equations via alveolar tissue, arterial blood, tissue and venous blood are given in Table 1.

Table 1: Nomenclature of different parameters.

Quantity	Symbol
Initial concentration of oxygen in alveolar tissue, mol/cm^3	X_0
Initial concentration of carbon dioxide in tissue, mol/cm^3	Y_0
Concentration of total oxygen (bounded and dissolved) in the i^{th}-compartment, mol/cm^3	$X_i(t)$
Concentration of total carbon dioxide (bounded and dissolved) in the i^{th}-compartment, mol/cm^3	$Y_i(t)$
Partial pressure of oxygen, $mmHg$	PO_2
Partial pressure of carbon dioxide, $mmHg$	PCO_2
Solubility coefficient of oxygen in alveolar tissue, $mol/(cm^3\ mmHg)$	α
Solubility coefficient of carbon dioxide in tissue, $mol/(cm^3\ mmHg)$	β

2.3. Compartment Modelling

The compartments considered in the model can be generally represented as shown in Figure 1. If the concentration of gas (O_2 or CO_2) in the i^{th}-compartment at time $t \geq 0$ is Z_i, then

$$\begin{aligned} \frac{dZ_i}{dt} &= \text{Input Rate} - \text{Output Rate.} \\ &= k_{i-1}Z_{i-1}(t) - k_iZ_i(t) \end{aligned} \tag{1}$$

This is know as balance law [7]. The parameters k_{i-1} and k_i denotes the transfer rates of gases (O_2 or CO_2) from $(i-1)^{th}$-compartment to i^{th}-compartment and from i^{th}-compartment to $(i+1)^{th}$-compartment, respectively.

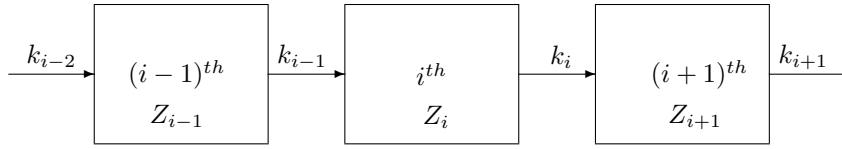


Figure 1: Schematic representation of the compartment model.

Consider a four-compartment model consisting of alveolar tissue, arterial blood and tissue with concentration of oxygen $X_L(t)$, $X_B(t)$ and $X_T(t)$, respectively; and tissue, venous blood and alveolar tissue with concentration of carbon dioxide $Y_T(t)$, $Y_B(t)$ and $Y_L(t)$, respectively. The parameters a_1 and a_2 denotes, respectively, the transfer rates of O_2 from alveolar tissue to arterial blood and from arterial blood to tissue. The transfer rate of CO_2 from tissue to venous blood is denoted by b_1 and from venous blood to alveolar tissue by b_2. A schematic overview of the four-compartment model of oxygen and carbon dioxide transport via blood in the human body is shown in Figure 2.

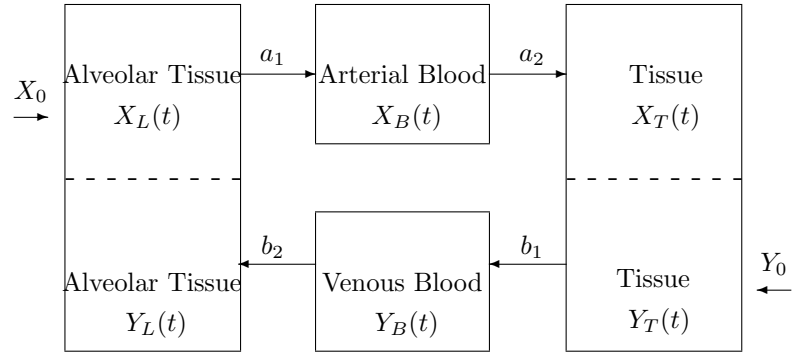


Figure 2: A four-compartment model of oxygen and carbon dioxide transport via blood in the human body with initial oxygen concentration X_0 and carbon dioxide concentration Y_0.

The oxygen and carbon dioxide are carried by the blood to and from the tissues. The O_2 (or CO_2) that is carried by blood leaves one compartment and enters into another at the rate proportional to the concentration of O_2 (or CO_2) present in the first compartment and so on. If the concentration of O_2 in the i^{th}-compartment at time $t \geq 0$ is $X_i(t)$ $(i = L,\ B,\ T)$, then according to Balance Law [2]:

$$\frac{dX_i}{dt} = (\text{Amount of } O_2 \text{ diffusion from } (i-1)^{th}\text{-compartment to } i^{th}\text{-compartment}) - (\text{Amount of } O_2 \text{ diffusion from } i^{th}\text{-compartment to } (i+1)^{th}\text{-compartment}) \quad (2)$$

Also, if the concentration of CO_2 in the i^{th}-compartment at time $t \geq 0$ is $Y_i(t)$ $(i = L,\ B,\ T)$, then again by Balance Law [2]:

$$\frac{dY_i}{dt} = (\text{Amount of } CO_2 \text{ diffusion from } (i-1)^{th}\text{-compartment to } i^{th}\text{-compartment}) - (\text{Amount of } CO_2 \text{ diffusion from } i^{th}\text{-compartment to } (i+1)^{th}\text{-compartment}) \quad (3)$$

The partial pressure of a gas (O_2 or CO_2), P, is related to the concentration, C, through the coefficient of solubility, σ, according to Henry's Law [9, 16]:

$$C = \sigma P \quad (4)$$

2.4. Oxygen Transport

Oxygen is continually being absorbed from the alveoli into the blood of the lungs, and new oxygen is continually being breathed into the alveoli from the atmosphere. The more rapidly oxygen is absorbed, the lower its concentration in the alveoli becomes; conversely, the more rapidly new oxygen is breathed into the alveoli from the atmosphere, the higher its concentration becomes. Therefore, oxygen concentration in the alveoli, as well as its partial pressure, is controlled by the rate of absorption of oxygen into the blood. The oxygen transport through arterial blood admits the pattern shown in Figure 2.

The mathematical form of the compartment model describing oxygen transport is given by the following set of ordinary differential equations:

$$\left.\begin{aligned} \frac{dX_L(t)}{dt} &= -a_1 X_L(t) + S; & X_L(0) &= X_0 \\ \frac{dX_B(t)}{dt} &= a_1 X_L(t) - a_2 X_B(t); & X_B(0) &= \sigma \\ \frac{dX_T(t)}{dt} &= a_2 X_B(t) - S; & X_T(0) &= \delta \end{aligned}\right\} \quad (5)$$

where σ and δ are constant oxygen concentrations in the blood and tissue compartments at $t = 0$, respectively; $a_1,\ a_2\ (> 0)$ are transfer rates that determines how O_2 flux in the respective compartments (alveolar tissue/arterial blood/tissues) decreases and the parameter S represents the source term or sink term for oxygen. It represents source term with plus sign i.e., $+S$ and sink term with minus sign i.e., $-S$.

The solution to any initial value problem can be obtained using standard methods such as separation of variables, Laplace transform, Fourier transform etc. [3, 7, 17]. We use eigenvalue method to solve the model Equations (5) and (14). The model Equations (5) and (14) have a solution on $[0, \infty)$ that is well behaved (i.e., bounded at any finite time and continuous for $t > 0$) and unique [3, 7, 17]. We write Equations (5) in matrix form:

$$U'(t) = \frac{dU(t)}{dt} = A(t)U(t) + b(t), \qquad U(0) = U_0 \tag{6}$$

$$\text{where} \quad A(t) = \begin{bmatrix} -a_1 & 0 & 0 \\ a_1 & -a_2 & 0 \\ 0 & a_2 & 0 \end{bmatrix},$$

$$U(t) = \begin{bmatrix} X_L(t) \\ X_B(t) \\ X_T(t) \end{bmatrix}, \quad b(t) = \begin{bmatrix} S \\ 0 \\ -S \end{bmatrix} \quad \text{and} \quad U_0 = \begin{bmatrix} X_0 \\ \sigma \\ \delta \end{bmatrix}$$

The eigenvectors u_1, u_2, u_3 corresponding to the eigenvalues $-a_1$, $-a_2$, 0 of the matrix A are:

$$u_1 = \begin{bmatrix} \frac{a_1 - a_2}{a_2} \\ -\frac{a_1}{a_2} \\ 1 \end{bmatrix}, \ u_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \ u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{7}$$

Therefore, the general solution of the system of Equations (5) is:

$$U(t) = c_1 u_1 e^{-a_1 t} + c_2 u_2 e^{-a_2 t} + c_3 u_3 + P.I \tag{8}$$

where $P.I = \begin{bmatrix} S/a_1 \\ S/a_2 \\ -S(a_1 + a_2)/a_1 a_2 \end{bmatrix}$ is the particular integral and $c_1,\ c_2,\ c_3$ are arbitrary constants which can be determined from the initial condition. By using the initial condition, we obtain $c_1,\ c_2,\ c_3$:

$$c_1 = \frac{a_2}{a_1 - a_2}\Big(X_0 - \frac{S}{a_1}\Big), \ c_2 = -\frac{a_1}{a_1 - a_2}\Big(X_0 - \frac{S}{a_1}\Big) + \frac{S}{a_2} - \sigma, \ c_3 = X_0 + \delta \tag{9}$$

Hence, the solution of the system of Equations (5) is:

$$X_L(t) = \Big(X_0 - \frac{S}{a_1}\Big)\exp(-a_1 t) + \frac{S}{a_1} \tag{10}$$

$$X_B(t) = \frac{a_1}{a_2 - a_1}\Big(X_0 - \frac{S}{a_1}\Big)\Big\{\exp(-a_1 t) - \exp(-a_2 t)\Big\} - \big(\sigma + \frac{S}{a_2}\big)\exp(-a_2 t) + \frac{S}{a_2} \tag{11}$$

$$X_T(t) = \Big(X_0 - \frac{S}{a_1}\Big)\Big\{1 + \frac{a_1\exp(-a_2 t) - a_2\exp(-a_1 t)}{(a_2 - a_1)}\Big\} + \big(\frac{S}{a_2} - \sigma\big)\exp(-a_2 t) - \frac{S}{a_2} + \delta \tag{12}$$

Equations (10), (11) and (12) represent, respectively, the compartment-wise concentration of oxygen in the alveolar tissue, arterial blood and tissues.

Flowing cases arise:

(A) If $a_1 > a_2$, then from Equation (11) it follows that $X_B(t) > 0$.

(B) If $a_1 < a_2$, we have $X_B(t) < 0$, which is not possible.

(C) If $a_1 = a_2$, then there will be no pressure gradient and therefore, no flux of O_2.

Hence, $a_1 > a_2$ admits the flux between the compartments.

2.5. Carbon Dioxide Transport

The respiratory system is responsible for gas transfer between the tissues and the outside air. The glucose oxidative metabolism reaction in the tissue cells is given by [9]:

$$C_6H_{12}O_6 + 6O_2 \rightarrow 6CO_2 + 6H_2O + ATP \tag{13}$$

The carbon dioxide produced as a waste product by the metabolism in the tissues must be moved by the venous blood to the outside air. Because an excessive amount of CO_2 produces acidity that can be toxic to cells, excess CO_2 must be eliminated quickly and efficiently. The carbon dioxide transport through venous blood admits the reverse pattern shown in Figure 2.

The mathematical form of the compartment model describing carbon dioxide transport is given by the following set of ordinary differential equations:

$$\left.\begin{aligned} \frac{dY_T(t)}{dt} &= -b_1 Y_T(t) + M; & Y_T(0) &= Y_0 \\ \frac{dY_B(t)}{dt} &= b_1 Y_T(t) - b_2 Y_B(t); & Y_B(0) &= \sigma^* \\ \frac{dY_L(t)}{dt} &= b_2 Y_B(t) - M; & Y_L(0) &= \delta^* \end{aligned}\right\} \tag{14}$$

where σ^* and δ^* are constant carbon dioxide concentrations in the blood and lung compartments at $t = 0$, respectively; $b_1,\ b_2(> 0)$ are CO_2 transfer rates, and the terms $+M$ and $-M$ represents, respectively, the source and sink terms for carbon dioxide.

The matrix formulation of the system of Equations (14) is:

$$V'(t) = \frac{dV(t)}{dt} = B(t)V(t) + b^*(t), \qquad V(0) = V_0 \tag{15}$$

$$\text{where} \quad B(t) = \begin{bmatrix} -b_1 & 0 & 0 \\ b_1 & -b_2 & 0 \\ 0 & b_2 & 0 \end{bmatrix},$$

$$V(t) = \begin{bmatrix} Y_T(t) \\ Y_B(t) \\ Y_L(t) \end{bmatrix}, \quad b^*(t) = \begin{bmatrix} M \\ 0 \\ -M \end{bmatrix}, \quad \text{and} \quad V_0 = \begin{bmatrix} Y_0 \\ \sigma^* \\ \delta^* \end{bmatrix}$$

By similar procedure as we did above in case of oxygen transport, we obtain the general solution of the system of Equations (14):

$$V(t) = d_1 v_1 e^{-b_1 t} + d_2 v_2 e^{-b_2 t} + d_3 v_3 + P.I \tag{16}$$

where v_1, v_2, v_3 are the eigenvectors corresponding to the eigenvalues $-b_1$, $-b_2$, 0 of the matrix B; d_1, d_2, d_3 are arbitrary constants which can be determined from the initial condition; and $P.I$ is the particular integral and are given in (17).

$$\left.\begin{aligned} v_1 &= \begin{bmatrix} \frac{b_1 - b_2}{b_2} \\ -\frac{b_1}{b_2} \\ 1 \end{bmatrix};\ v_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix};\ v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \\ d_1 &= \frac{b_2}{b_1 - b_2}\Big(Y_0 - \frac{M}{b_1}\Big);\ d_2 = -\frac{b_1}{b_1 - b_2}\Big(Y_0 - \frac{M}{b_1}\Big) + \frac{M}{b_2} - \sigma^*;\ d_3 = Y_0 + \delta^*; \\ P.I &= \begin{bmatrix} M/b_1 \\ M/b_2 \\ -M(b_1 + b_2)/b_1 b_2 \end{bmatrix} \end{aligned}\right\} \tag{17}$$

Hence, the solution of the system of Equations (14) is:

$$Y_L(t) = \Big(Y_0 - \frac{M}{b_1}\Big)\exp(-b_1 t) + \frac{M}{b_1} \tag{18}$$

$$Y_B(t) = \frac{b_1}{b_2 - b_1}\Big(Y_0 - \frac{M}{b_1}\Big)\Big\{\exp(-b_1 t) - \exp(-b_2 t)\Big\} - \big(\sigma^* + \frac{M}{b_2}\big)\exp(-b_2 t) + \frac{M}{b_2} \tag{19}$$

$$Y_T(t) = \Big(Y_0 - \frac{M}{b_1}\Big)\Big\{1 + \frac{b_1\exp(-b_2 t) - b_2\exp(-b_1 t)}{(b_2 - b_1)}\Big\} + \big(\frac{M}{b_2} - \sigma^*\big)\exp(-b_2 t) - \frac{M}{b_2} + \delta^* \tag{20}$$

Equations (18), (19) and (20) represent, respectively, the compartment-wise concentration of carbon dioxide in the tissues, venous blood and alveolar tissue.

Flowing cases arise:

(I) From Equation (19), we have $Y_B(t) > 0$ if $b_1 > b_2$, and $Y_B(t) < 0$ if $b_1 < b_2$, which is not possible.

(II) If $b_1 = b_2$, then there will be no pressure gradient and therefore, no flux of CO_2.

Hence, we have $b_1 > b_2$.

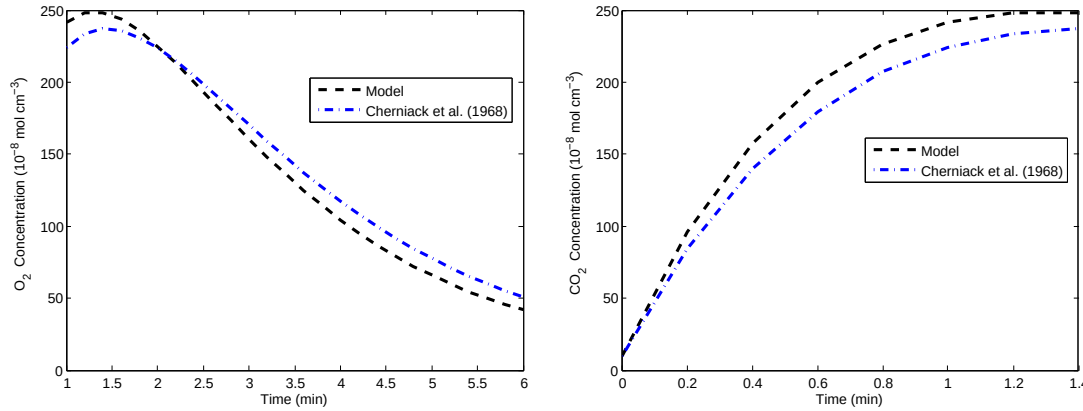
3. Results and Discussion

A mathematical model based on ordinary differential equations and balance law has been formulated to estimate the concentration profiles of oxygen and carbon dioxide at alveolar tissue, arterial blood, tissue and venous blood compartments using four-compartment model scheme. We validate our model to induce respiratory changes such as apnea using the transfer rates a_1, a_2, b_1, b_2 by comparing computed results with the reference results [4], as depicted in Figure 3. Apnea denotes the cessation of breading and ensuring hypoxia. Apnea can affect people of all ages and the cause depends on the type of apnea we have. Apnea usually occurs while we are sleeping. For this reason, it is often called sleep apnea.

Table 2: Numerical values of different parameters.

Parameter	Numerical Value
α	$5.18 \times 10^{-8}\ mol\ cm^{-3}\ mmHg^{-1}$ [10]
β	$2.59 \times 10^{-8}\ mol\ cm^{-3}\ mmHg^{-1}$ [20]

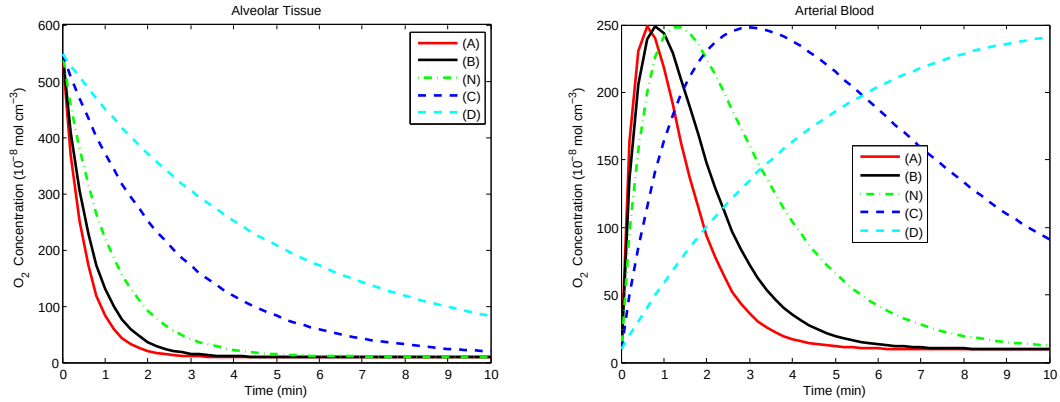
The transfer rate $a_2 = 0.625/sec$ determines how oxygen flux in the tissues decreases for $PO_2 > 0$ [8]. By cases-(A), (B), (C), we have $a_1 > a_2$ and we take $a_1 = 0.9375/sec$. Thus, we have O_2 transfer rates $N := \{a_1,\ a_2 : a_1 = 0.9375,\ a_2 = 0.625\ (/sec)\}$. By assumption-$(v)$, we have $a_i = b_i,\ i = 1, 2$ and by cases-(I), (II), we have CO_2 transfer rates $N^* := \{b_1,\ b_2 : b_1 = 0.9375,\ b_2 = 0.625\ (/sec)\}$. The numerical value of the parameter $S\ (= M)$ is taken to be $3.72\times10^{-8}\ mol\ cm^{-3}\ sec^{-1}$ [19].



(a) (b)

Figure 3: Comparison of model result and the result reported by Cherniack et al. (1968) [4] during apnea: (a) change in oxygen concentration in arterial blood, and (b) change in carbon dioxide concentration in venous blood.

The partial pressure of oxygen in alveolar air is assumed to be 104 $mmHg$, it decreases gradually due to flux once it enters inside alveolar tissue (lungs). The initial oxygen concentration in alveolar tissue is taken to be $X_0 = 104\ mmHg \times \alpha$ and based on this input, the oxygen transport patterns through alveolar tissue, arterial blood and tissues are presented in Figures 4. It has been observed from the graphs that O_2 has a rapid decay in lungs, a hyperbolic behaviour in blood and a sigmoid characteristic in tissue. The dash-dotted $(-.-.)$ curves shown in Figure 4 show efficient transport of O_2 in human body. These dash-dotted curves are drawn against the O_2 transfer rates N for alveolar tissue, arterial blood and tissue. The neighbourhood values of these O_2 transfer rates N represent normal rates of respiration taking place in the respective compartments. The results, however, reflect that this neighbourhood of O_2 transfer rates N is very small defined



(a) (b)

(c)

Figure 4: Temporal variation of oxygen concentration in the respective compartments (alveolar tissue/arterial blood/tissue) with $X_0 = \alpha \times PO_2$ and at different transfer rates $/sec$: $(A) = \{a_1 = 1.98,\ a_2 = 1.32\}$, $(B) = \{a_1 = 1.5,\ a_2 = 1.0\}$, $(N) = \{a_1 = 0.9375,\ a_2 = 0.625\}$, $(C) = \{a_1 = 0.4,\ a_2 = 0.27\}$ and $(D) = \{a_1 = 0.1,\ a_2 = 0.07\}$.

by $\{a : a_i - 0.25 < a < a_i + 0.25,\ i = 1, 2\}$. Any increase or decrease in values of O_2 transfer rates outside the neighbourhood values of N affects respiration.

From our results, it follows that more the value of transfer rates (greater than the neighbourhood values of N), greater will be the decrease in O_2 concentration as depicted in Figure 4(a) by solid $(-)$ curves and hence, faster will be the absorption of O_2 within the alveolar tissue. The cases in the point stem from different situations in daily life like running, heavy exercise, etc. For values of O_2 transfer

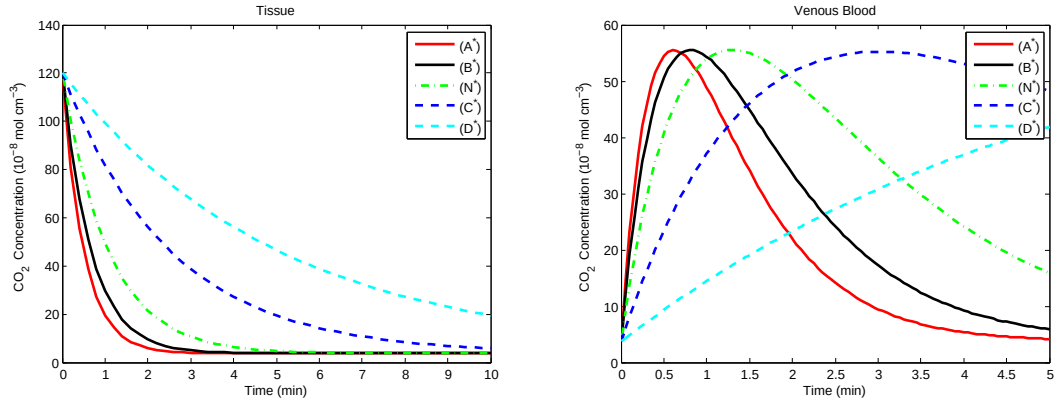
rates lesser than the neighbourhood values of N, absorption of O_2 will be slow as depicted in Figure 4(a) by dashed $(---)$ curves. As a result less molecules of O_2 are available for oxy-hemoglobin formation in the blood compartment and hence, respiratory ailments like hypoxia happen to occur.

The plots in Figure 4(b) shows O_2 concentration increases rapidly inside the blood compartment up to a certain (peak) value, following which it abruptly falls down and plateaus thenceforth. The initial increasing trend can be attributed to increase in influx of oxygen from alveolar tissue to arterial blood wherein it combines with available hemoglobin molecules to yield oxy-hemoglobin, carrier of oxygen to tissues. The oxy-hemoglobin molecules unload the oxygen in the tissues, decreasing the concentration of O_2 in the blood as depicted by the decreasing trend in the graph. The plateau part of the graph indicates a steady state in the formation of oxy-hemoglobin in the arterial blood (determined by a_1) and oxygen unloading into tissues (determined by a_2). The plots also show that the decrease in the value of transfer rates (a_1 and a_2) is associated with the dampening of peaks of concentration plots; the higher values of transfer rates result in higher peaks of oxygen concentration in blood. The graph is in confirmation with the results arrived at by Cherniack et al. (1968) [4], attesting to the credibility of our results.

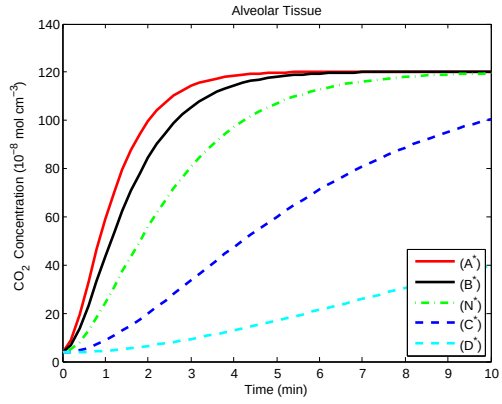
The curves in Figure 4(c) shows logistic patterns of the O_2 concentration in the tissues with increase in the value of transfer rates and almost linear pattern with decrease in the value of transfer rates. The O_2 concentration increases to a certain level and then, after a definite time period, the consumption level reaches saturation point resulting in the horizontal curve. This may be attributed to the fact that inside the tissue, oxygen consumption in the respiratory metabolism keeps the gradient in favour of oxygen diffusion into the tissues.

The carbon dioxide flow in backward direction, from tissues via blood into the alveolar tissue, with respect to $PCO_2 = 45\ mmHg$ assumed to be produced in the tissue is depicted in Figures 5. The graphs with different transfer rates shows CO_2 behaviour in tissues, venous blood and alveolar tissue compartments. The dash-dotted $(-.-.)$ curves shown in Figures 5 drawn against the CO_2 transfer rates N^* show efficient expulsion of CO_2 from human body. The neighbourhood values of these CO_2 transfer rates N^* show normal elimination of CO_2, without occurrence of any CO_2 related respiratory problems. This neighbourhood of CO_2 transfer rates N^* is similarly defined by $\{b : b_i - 0.25 < b < b_i + 0.25,\ i = 1, 2\}$.

The increase in CO_2 transfer rates above the neighbourhood values of N^* results in faster elimination of CO_2 from the body and hence, results in steep decrease in CO_2 concentration as depicted in Figure 5(a) by solid $(-)$ curves. This rapid expulsion of CO_2 from the body results in less availability of CO_2 for carbonic acid



(a) (b)



(c)

Figure 5: Temporal variation of carbon dioxide concentration in the respective compartments (tissue/venous blood/alveolar tissue) with $Y_0 = \beta \times PCO_2$ and at different transfer rates $/sec$: $(A^*) = \{b_1 = 1.98,\ b_2 = 1.32\}$, $(B^*) = \{b_1 = 1.5,\ b_2 = 1.0\}$, $(N^*) = \{b_1 = 0.9375,\ b_2 = 0.625\}$, $(C^*) = \{b_1 = 0.4,\ b_2 = 0.27\}$ and $(D^*) = \{b_1 = 0.1,\ b_2 = 0.07\}$.

formation in the blood compartment and consequently increases blood pH above normal level. The increase in pH of blood is attributed to respiratory alkalosis in human body. Conversely, if the CO_2 transfer rates decreases below the neighbourhood values of N^*, the expulsion of CO_2 will be slow and hence, it would lead to increase in CO_2 concentration as shown in Figure 5(a) by dashed $(- - -)$ curves. This increase in CO_2 content in the blood increases carbonic acid formation and decreases blood pH below normal level, resulting respiratory acidosis in human

body.

The graphs in Figure 5(b) represent change in concentration of CO_2 in the venous blood compartment in relation to time. The graph is in agreement with the results arrived at by Cherniack et al. (1968) [4]. The plots reflect that the CO_2 concentration increases initially to a peak value followed by steep decrease to a certain value before it gets plateaued. The trend is similar to that shown by O_2 concentration in the blood except that the path followed by CO_2 is opposite to that of O_2: oxygen flows from alveolar tissue to tissues via arterial blood, while as CO_2 flows back from tissues to alveolar tissue via venous blood. The plot of CO_2 concentration versus time shows similar trend in alveolar tissue as that of O_2 concentration in the tissue, shown in Figure 5(c).

4. Conclusion

A mathematical model of O_2 and CO_2 transport via blood between the lung and the tissue in the human body represented by four compartments: alveolar tissue, arterial blood, tissue and venous blood was formulated. The aim of this study was to estimate the concentration profiles of O_2 and CO_2 over alveolar tissue, arterial blood, tissue and venous blood compartments. From the above discussion, it can be concluded that our model provides useful information regarding absorption rate of O_2 at alveolar tissue, arterial blood and tissue compartments and release rate of CO_2 at tissue, venous blood and alveolar tissue compartments. The results obtained in this study have applications in medical sciences in general and in the field of biomedical engineering in particular to deal with respiratory ailments faced by the people living at high altitudes. Hypoxia, pH level increase or decrease and other respiratory ailment conditions can be handled by using suitable dataset in the present model. This work can be further extended by incorporating hypoxia, the interaction of O_2 and CO_2 in the blood and other environmental issues as parameters in the model.

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