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ON THE INTEGRAL SOLUTIONS OF BINARY QUADRATIC DIOPHANTINE EQUATION $ax^2 - bx = cy^2$

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Abstract: In this paper, we show that the Diophantine equation $ax^2 - bx = cy^2$ in positive integers x, y, a, b, c has infinitely many solutions where ac is not a square. We transform the above equation into a Pellian equation to find its infinitely many integer solutions only when ac is not a square. Finally, we present some recurrence relations for (x, y).

Keywords and Phrases: Diophantine Equation, Quadratic Equation, Integral Solutions, Pell's Equation, hyperbola.

2020 Mathematics Subject Classification: 11DXX, 11D09.

1. Introduction

The aim of this paper is to find the general solution of non-homogenous Diophantine equation of the form

$$ax^2 - bx = cy^2.$$

This equation is considered as a more general form of the equation introduced in [19]. Moreover, the equation is a special form of the Diophantine equation

$X^2 - DY^2 = N$ which is a generalized Pell's equations. Its fundamental form is $X^2 - DY^2 = 1$ which was first studied by the Indian Mathematician and Astronomer, Brahmagupta in 7th Century. Subsequently in 12th Century, it was also studied by another Indian Mathematician and Astronomer, Bhaskara II in an extensive manner. However, its complete information was given by Lagrange in theoretical form. Recently many authors have involved themselves to study the Diophantine equations of infinitely many integer solutions with various orders. As per the objective of the proposed work, here, we have made a sample survey about the study of Diophantine equations related to its order and the number of variables involved in it.

Primarily, the survey has been made about the study of Diophantine Equations with two variables for their integer solutions. In 1978 [22], Stanley Rabinowitz-solved non-homogenous cubic Diophantine Equation of the form $3y^2 \pm 2^n\gamma = x^3$ with $\gamma = \pm 1$. In 2000, Bremner and Silverman [8] considered the problem of finding the integral points form an arithmetic progression on the elliptic curve $y^2 = x(x^2 - n^2)$. In 2002, Arifet. al [6] proved that the Diophantine Equation of the form $x^2 + q^{2k+1} = y^n$ has exactly two families of solutions to the certain conditions for k, n and q in order to find its integer solutions. The Diophantine equation $px^2 + q^{2m} = y^p$ and its similar forms were considered by Fadwa S. Abu Muriefah in 2008 [11], Wang Yongxing and Wang Tingting in 2011 [24], Florian Luca and GokhanSoydanc in 2012 [13], Fadwa S. Abu Muriefah and Amal Al-Rashed in 2012 [12], Eva G. Goedhart and Helen G. Grundman [10] in 2014, Musa Demirci [17] in 2017 for the analysis of existence and non-existence of the non-zero integer solutions. LajosHajdu and Istvan Pink [14] made an attempt to find the complete solutions of the Diophantine Equation $1 + 2^a + x^b = y^n$ for the odd positive integers of $x < 50$ in the year 2014.

Finally, we have made the survey about the study of Quadratic Diophantine Equations and its integer solutions. Mollinet. al. in 2002 [15] provided criteria in terms of continued fraction expansion of $\sqrt{a^2b}$ for solving the Diophantine Equation $aX^2 - bY^2 = c$. In 2005, Chung-Gang Ji [9] gave sufficient conditions in terms of certain graph for the solvability of the Diophantine Equations $x^2 - Dy^2 = -1, \pm 2$. Richard A. Mollan in 2006 [21] provided the continued fraction solutions in term of central norm for the Diophantine Equations $x^2 - Dy^2 = c^n$. In 2010, RefikKeskin [20] made an investigation about the existence and non-existence of positive integer solutions of the Diophantine Equations $x^2 - kxy + y^2 \mp x = 0$ and $x^2 - kxy + y^2 \mp y = 0$ for the certain limits of t involved in it. In 2011, G. Arag On-Gonzalez et. al. [5] approached the Quadratic Diophantine Equations $a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2 = a_1x_{n+1}^2$ and its more generalized form

$a_1x_1^2 + a_2x_2^2 + \cdots + a_px_p^2 = a_{p+1}x_{p+1}^2 + \cdots + a_1x_{n+1}^2$ for the further improvement to find its all integer solutions. In the same year, Pingzhi Yuan and Yongzhong Hu [18] proved that the Diophantine Equation $x^2 - kxy + y^2 + lx = 0$ has infinitely many positive integer solutions for some specific values of k and l. Subsequently they also proved in the same work that the equation has infinite number integer solutions iff$k \neq 0, \pm 1$. Ahmet Tekcanet. al in 2010 and 2014 [1, 2], Amara Chandoul [3, 4] and Arzu Ozkoc et. al. [7] in 2011, considered the Diophantine equations of similar formats such as $x^2 - (t^2 + t)\,y^2 - (4t + 2)x + (4t^2 + 4t)\,y = 0$, $8x^2 - y^2 - 8x(1+t) + (2t+1)^2 = 0$, $X^2 - (P^2 - P)\,Y^2 - (4P^2 - 2)\,X + (4P^2 - 4P)\,Y = 0$, $x^2 - (t^2 - t)\,y^2 - (16t - 4)x + (16t^2 - 16t)\,y = 0$, and $x^2 + (t - t^2)y^2 + (4 - 8t)x + (8t^2 - 8t)y + 3 = 0$ for finding its integer solutions over Z and F_p for the certain limits of the prime p respectively. Moreover, they derived some algebraic identities related for its integer solutions including recurrence relation and continued fractions. In 2013, the formulae for the number of integer solutions of the Diophantine Equation $n^2 = x^2 + by^2 + cz^2$ for some specific values of b and c were stated and also proved by Shaun Cooper and Heung Yeung Lam [23].

The survey stimulates us to study about the solvability of the Diophantine Equation considered under this study. The direct motivation to write this paper is the question whether there are infinitely many twins of integers satisfying the equation $ax^2 - bx = cy^2$. We observe that the question concerning the existence of integer solutions is answered through converting the given indeterminate Diophantine equation $ax^2 - bx = cy^2$ as a Pell's equation. In this way we obtain its general solution based on the general solution of Pell's equations where the product of the constants a and c is a non-square integer and the constant b is a multiple of other constant a. Furthermore, some recurrence relations in the form of identities may be arrived for the integer solutions of the proposed non-homogenous quadratic Diophantine equation at the end of this work. The method of analysis for solving the equation taken in this study is as follows:

2. Method of Analysis

The non-homogenous quadratic Diophantine equation representing the hyperbola to be solved for its non-zero distinct integral solution is

$$ax^2 - bx = cy^2 \tag{1}$$

The equation (1) can be written as

$$(2ax - b)^2 - 4acy^2 = b^2 \tag{2}$$

Now the above equation is in the form of a generalized Pellian equation

$$X^2 - Dy^2 = N$$

where $X = 2ax - b$, $D = 4ac$, $N = b^2$, $b = ma$, $m \in Z_+$.

Since D is a square free number, ac is not a square number.

Using the general solution of generalized Pell's Equation [16], we obtain the general solution (x_n, y_n) of equation (2) which is given as follows:

$$x_n = \frac{\frac{1}{2}\left[\left(X'_0 + y'_0\sqrt{4ac}\right)^n\left(X_0 + \sqrt{4ac}y_0\right) + \left(X'_0 - y'_0\sqrt{4ac}\right)^n\left(X_0 - \sqrt{4ac}y_0\right)\right] + b}{2a}$$

$$y_n = \frac{1}{2\sqrt{4ac}}\left[\left(X'_0 + y'_0\sqrt{4ac}\right)^n\left(X_0 + \sqrt{4ac}y_0\right) - \left(X'_0 - y'_0\sqrt{4ac}\right)^n\left(X_0 - \sqrt{4ac}y_0\right)\right] \tag{3}$$

where $X'_0 + y'_0\sqrt{4ac}$ is the fundamental solution of $(2ax-b)^2 - 4acy^2 = 1$ and $X_0 + \sqrt{4ac}y_0$ is the fundamental solution of $(2ax-b)^2 - 4acy^2 = b^2$. In order to generate the sequence of integer solutions of the equation under this study based on the nature of constants a and c, we take the multiplication of the constants a and c as the non-square integer valued second order polynomial $P(\alpha)$ for all $\alpha \in Z$ which is reducible over the integer field.

2.1. Illustration 1

Case I $a = \alpha + 2$, $c = \alpha + 3$.

In this case, the equation (2) becomes,

$$(2(\alpha+2)x - b)^2 - 4(\alpha^2 + 5\alpha + 6)y^2 = b^2 \tag{4}$$

where $b = m(\alpha + 2)$, $m \in Z_+$.

The above equation may be rewritten as

$$X^2 - Dy^2 = N$$

where $X = 2(\alpha+2)x - b$, $N = b^2$ and $D = 4(\alpha^2 + 5\alpha + 6)$.

Using (3), we obtain the general solution (x_n, y_n) of equation (3) which is given as follows:

$$\begin{aligned} x_n &= \frac{\frac{1}{2}\begin{bmatrix}\left(X'_0 + y'_0\sqrt{4(\alpha^2+5\alpha+6)}\right)^n\left(X_0 + \sqrt{4(\alpha^2+5\alpha+6)}y_0\right) \\ + \left(X'_0 - y'_0\sqrt{4(\alpha^2+5\alpha+6)}\right)^n\left(X_0 - \sqrt{4(\alpha^2+5\alpha+6)}y_0\right)\end{bmatrix} + b}{2(\alpha+2)} \\ y_n &= \frac{\begin{bmatrix}\left(X'_0 + y'_0\sqrt{4(\alpha^2+5\alpha+6)}\right)^n\left(X_0 + \sqrt{4(\alpha^2+5\alpha+6)}y_0\right) \\ - \left(X'_0 - y'_0\sqrt{4(\alpha^2+5\alpha+6)}\right)^n\left(X_0 - \sqrt{4(\alpha^2+5\alpha+6)}y_0\right)\end{bmatrix}}{2\sqrt{4(\alpha^2+5\alpha+6)}} \end{aligned} \tag{5}$$

where
$X'_0 + y'_0\sqrt{4(\alpha^2+5\alpha+6)}$ is the fundamental solution of $(2(\alpha+2)x-b)^2 - 4(\alpha^2+5\alpha+6)y^2 = 1$ and $X_0 + y_0\sqrt{4(\alpha^2+5\alpha+6)}$ is the fundamental solution of $(2(\alpha+2)x-b)^2 - 4(\alpha^2+5\alpha+6)y^2 = b^2$.

For the sake of simplicity, solutions of (4) are presented for $\alpha = 1$ as follows:

Substitute $\alpha = 1$ in equation (4), we get.

$$(6x-3m)^2 - 48y^2 = 9m^2 \tag{6}$$

Using (5), we obtain the general solution (x_n, y_n) of equation (6) which is given as follows:

$$x_n = \frac{\frac{1}{2}\left[\left(X'_0 + y'_0\sqrt{48}\right)^n\left(X_0+\sqrt{48}y_0\right) + \left(X'_0 - y'_0\sqrt{48}\right)^n\left(X_0-\sqrt{48}y_0\right)\right] + 3m}{6}$$

$$y_n = \frac{1}{2\sqrt{48}}\left[\left(X'_0 + y'_0\sqrt{48}\right)^n\left(X_0+\sqrt{48}y_0\right) - \left(X'_0 - y'_0\sqrt{48}\right)^n\left(X_0-\sqrt{48}y_0\right)\right]$$

where $X'_0 + y'_0\sqrt{48}$ is the fundamental solution of $(6x-3m)^2 - 48y^2 = 1$ and $X_0 + y_0\sqrt{48}$ is the fundamental solution of $(6x-3m)^2 - 48y^2 = 9m^2$.

It is the general solution of the specific form of the proposed equation $3x^2 - 3mx = 4y^2$. Few numerical solutions of this equation for certain values of m are listed in Table 1.

Further the solutions satisfy the following recurrence relations:

(a) Recurrence relations for the solution $(x_{(n,m)}, y_{(n,m)})$ among the different values of n.

   (i) $x_{(n,k)} - 14x_{(n+1,k)} + x_{(n+2,k)} + 2b = 0$ where $n \geq 0$ and $k = 0, 1, 2, \ldots,$

   (ii) $y_{(n,k)} - 14y_{(n+1,k)} + y_{(n+2,k)} = 0$ where $n \geq 0$ and $k = 0, 1, 2, \ldots$

(b) Recurrence relations for the solution $(x_{(n,m)}, y_{(n,m)})$ among the different values of b.

   (i) $x_{(n,2k+1)} - 2x_{(n,2k+2)} + x_{(n,2k+3)} = 0$ where $k = 0, 1, 2, \ldots,$

   (ii) $x_{(n,1)} - x_{(n,2k+2)} + x_{(n,2k+1)} = 0,$ where $k = 1, 2, \ldots$

(c) Recurrence relations for the solution $(x_{i(n,m)}, y_{i(n,m)})_{i=1,2}$.

   (i) $4x_{1(n,4k)} - 2x_{2(n,4k)} + 4y_{1(n,4k)} = m$ where $k = 1, 2, \ldots$

Table 1: Solutions $(x_{(n,m)}, y_{(n,m)})$ of $3x^2 - 3mx = 4y^2$ for small values of n, m

n	$(x_{(n,m)}, y_{(n,m)})$ for $(m, b) = (1, 3)$	$(x_{(n,m)}, y_{(n,m)})$ for $(m, b) = (2, 6)$
0	(1, 0)	(2, 0)
1	(4, 3)	(8, 6)
2	(49, 42)	(98, 84)
3	(676, 585)	(1352, 1170)
4	(9409, 8148)	(18818, 16296)
5	(131044, 113487)	(262088, 226974)

n	$(x_{(n,m)}, y_{(n,m)})$ for $(m, b) = (3, 9)$	$(x_{i(n,m)}, y_{i(n,m)})_{i=1,2}$ for $(m, b) = (4, 12)$	
0	(3, 0)	(4, 0)	(6, 3)
1	(12, 9)	(16, 12)	(54, 45)
2	(147, 126)	(196, 168)	(726, 627)
3	(2028, 1755)	(2704, 2340)	(10086, 8733)
4	(28227, 24444)	(37636, 32592)	(140454, 121635)
5	(393132, 340461)	(524176, 453948)	(1956246, 1694157)

n	$(x_{(n,m)}, y_{(n,m)})$ for $(m, b) = (5, 15)$	$(x_{(n,m)}, y_{(n,m)})$ for $(m, b) = (6, 18)$
0	(5, 0)	(6, 0)
1	(20, 15)	(24, 18)
2	(245, 210)	(294, 252)
3	(3380, 2925)	(4056, 3510)
4	(47045, 40740)	(56454, 48888)
5	(655220, 567435)	(786264, 680922)

n	$(x_{(n,m)}, y_{(n,m)})$ for $(m, b) = (7, 21)$	$(x_{i(n,m)}, y_{i(n,m)})_{i=1,2}$ for $(m, b) = (8, 24)$	
0	(7, 0)	(8, 0)	(12, 6)
1	(28, 21)	(32, 24)	(108, 90)
2	(343, 294)	(392, 336)	(1452, 1254)
3	(4732, 4095)	(5408, 4680)	(20172, 17466)
4	(65863, 57036)	(75272, 65184)	(280908, 243270)
5	(917308, 794409)	(1048352, 907896)	(3912492, 3388314)

Remark: More than one class of solutions are obtained for the Diophantine Equation $3x^2 - 3mx = 4y^2$ when $m = 4, 8, \ldots$

Proof of the recurrence relation $x_{(n,k)} - 14x_{(n+1,k)} + x_{(n+2,k)} + 2b = 0$ where $n \geq 0$ and $k = 0, 1, 2, \ldots$:

The above recurrence relation can be written as

$$x_n - 14x_{n+1} + x_{n+2} + 2b = 0$$

The solution of the equation $3x^2 - 3x = 4y^2$ is given by

$$x_n = \frac{\frac{1}{2}\left[\left(7+\sqrt{48}\right)^n\left(21+3\sqrt{48}\right)+\left(7-\sqrt{48}\right)^n\left(21-3\sqrt{48}\right)\right]+3}{6}$$

$$y_n = \frac{1}{2\sqrt{48}}\left[\left(7+\sqrt{48}\right)^n\left(21+3\sqrt{48}\right)-\left(7-\sqrt{48}\right)^n\left(21-3\sqrt{48}\right)\right]$$

where $b = 3$.

The above equations can be rewritten as

$$12x_n - 6 = \left(7+\sqrt{48}\right)^n\left(21+3\sqrt{48}\right)+\left(7-\sqrt{48}\right)^n\left(21-3\sqrt{48}\right) \qquad (*)$$

$$2\sqrt{48}y_n = \left(7+\sqrt{48}\right)^n\left(21+3\sqrt{48}\right)-\left(7-\sqrt{48}\right)^n\left(21-3\sqrt{48}\right) \qquad (**)$$

Now

$$\begin{aligned}
12x_n - 6 &= \left(7+\sqrt{48}\right)^n\left(21+3\sqrt{48}\right)+\left(7-\sqrt{48}\right)^n\left(21-3\sqrt{48}\right)\\
&= 21\left(7+\sqrt{48}\right)^n + 3\sqrt{48}\left(7+\sqrt{48}\right)^n + 21\left(7-\sqrt{48}\right)^n - 3\sqrt{48}\left(7-\sqrt{48}\right)^n\\
&= 21\left[\left(7+\sqrt{48}\right)^n+\left(7-\sqrt{48}\right)^n\right] + 3\sqrt{48}\left[\left(7+\sqrt{48}\right)^n-\left(7-\sqrt{48}\right)^n\right]\\
&= 21E_1 + 3E_2
\end{aligned}$$

where $E_1 = \left(7+\sqrt{48}\right)^n + \left(7-\sqrt{48}\right)^n$ and $E_2 = \left(7+\sqrt{48}\right)^n - \left(7-\sqrt{48}\right)^n$.

Substitute $n = n+1$ in (*), we get

$$\begin{aligned}
12x_{n+1} - 6 &= \left(7+\sqrt{48}\right)^{n+1}\left(21+3\sqrt{48}\right)+\left(7-\sqrt{48}\right)^{n+1}\left(21-3\sqrt{48}\right)\\
&= \left(7+\sqrt{48}\right)^n\left(7+\sqrt{48}\right)\left(21+3\sqrt{48}\right)+\left(7-\sqrt{48}\right)^n\left(7-\sqrt{48}\right)\left(21-3\sqrt{48}\right)\\
&= \left(7+\sqrt{48}\right)^n\left(291+42\sqrt{48}\right)+\left(7-\sqrt{48}\right)^n\left(291-42\sqrt{48}\right)\\
&= 291\left[\left(7+\sqrt{48}\right)^n+\left(7-\sqrt{48}\right)^n\right]+42\sqrt{48}\left[\left(7+\sqrt{48}\right)^n-\left(7-\sqrt{48}\right)^n\right]\\
&= 291E_1 + 42E_2
\end{aligned}$$

Substitute $n = n + 2$ in (*), we get

$$12x_{n+2} - 6 = \left(7+\sqrt{48}\right)^{n+2}\left(21+3\sqrt{48}\right) + \left(7-\sqrt{48}\right)^{n+2}\left(21-3\sqrt{48}\right)$$
$$= \left(7+\sqrt{48}\right)^{n}\left(7+\sqrt{48}\right)^{2}\left(21+3\sqrt{48}\right) + \left(7-\sqrt{48}\right)^{n}\left(7-\sqrt{48}\right)^{2}\left(21-3\sqrt{48}\right)$$
$$= \left(7+\sqrt{48}\right)^{n}\left(4053+585\sqrt{48}\right) + \left(7-\sqrt{48}\right)^{n}\left(4053-585\sqrt{48}\right)$$
$$= 4053\left[\left(7+\sqrt{48}\right)^{n} + \left(7-\sqrt{48}\right)^{n}\right] + 585\sqrt{48}\left[\left(7+\sqrt{48}\right)^{n} - \left(7-\sqrt{48}\right)^{n}\right]$$
$$= 4053E_1 + 585E_2$$

Now

$$\begin{aligned}(12x_{n+2} - 6) - 14(12x_{n+1} - 6) &= (4053E_1 + 585E_2) - 14(291E_1 + 42E_2)\\ &= -(21E_1 + 3E_2)\\ &= -(12x_n - 6)\end{aligned}$$

(i.e.) $(12x_n - 6) - 14(12x_{n+1} - 6) + (12x_{n+2} - 6) = 0.$

$$\begin{aligned}&\Rightarrow x_n - 14x_{n+1} + x_{n+2} + 6 = 0\\ &\Rightarrow x_n - 14x_{n+1} + x_{n+2} + (2 \times 3) = 0\\ &\Rightarrow x_n - 14x_{n+1} + x_{n+2} + 2b = 0 \quad \text{where } b = 3\end{aligned}$$

This is the required recurrence relation.

Remark: The other recurrence relations can be proved with the suitable modification of the methodology involved.

Case II: $a = \alpha + 3$, $c = \alpha + 2$. In this case, the equation (2) becomes,

$$(2(\alpha+3)x - b)^2 - 4(\alpha^2 + 5\alpha + 6)y^2 = b^2 \tag{7}$$

where $b = m(\alpha + 3)$, $m \in Z_+$.

The above equation may be rewritten as

$$X^2 - Dy^2 = N$$

where, $X = 2(\alpha + 3)x - b$, $N = b^2$ and $D = 4(\alpha^2 + 5\alpha + 6)$.

Using (3), we obtain the general solution (x_n, y_n) of equation (7) which is given as follows:

$$x_n = \frac{\frac{1}{2}\left[\begin{array}{l}\left(X'_0 + y'_0\sqrt{4(\alpha^2+5\alpha+6)}\right)^n \left(X_0 + \sqrt{4(\alpha^2+5\alpha+6)}y_0\right) \\ + \left(X'_0 - y'_0\sqrt{4(\alpha^2+5\alpha+6)}\right)^n \left(X_0 - \sqrt{4(\alpha^2+5\alpha+6)}y_0\right)\end{array}\right] + b}{2(\alpha+3)}$$

$$y_n = \frac{\left[\begin{array}{l}\left(X'_0 + y'_0\sqrt{4(\alpha^2+5\alpha+6)}\right)^n \left(X_0 + \sqrt{4(\alpha^2+5\alpha+6)}y_0\right) \\ - \left(X'_0 - y'_0\sqrt{4(\alpha^2+5\alpha+6)}\right)^n \left(X_0 - \sqrt{4(\alpha^2+5\alpha+6)}y_0\right)\end{array}\right]}{2\sqrt{4(\alpha^2+5\alpha+6)}} \tag{8}$$

where,
$X'_0 + y'_0\sqrt{4(\alpha^2+5\alpha+6)}$ is the fundamental solution of $(2(\alpha+3)x - b)^2 - 4(\alpha^2 + 5\alpha+6)y^2 = 1$ and $X_0 + y_0\sqrt{4(\alpha^2+5\alpha+6)}$ is the fundamental solution of $(2(\alpha + 3)x - b)^2 - 4(\alpha^2+5\alpha+6)y^2 = b^2$.

For the sake of simplicity, solutions of (7) are presented for $\alpha = 1$ as follows:

Substitute $\alpha = 1$ in equation (7), we get

$$(8x - 4m)^2 - 48y^2 = 16m^2 \tag{9}$$

Using (8), we obtain the general solution (x_n, y_n) of equation (9) which is given as follows:

$$x_n = \frac{\frac{1}{2}\left[\left(X'_0 + y'_0\sqrt{48}\right)^n \left(X_0 + \sqrt{48}y_0\right) + \left(X'_0 - y'_0\sqrt{48}\right)^n \left(X_0 - \sqrt{48}y_0\right)\right] + 4m}{8}$$

$$y_n = \frac{1}{2\sqrt{48}}\left[\left(X'_0 + y'_0\sqrt{48}\right)^n \left(X_0 + \sqrt{48}y_0\right) - \left(X'_0 - y'_0\sqrt{48}\right)^n \left(X_0 - \sqrt{48}y_0\right)\right]$$

where $X'_0 + y'_0\sqrt{48}$ is the fundamental solution of $(8x - 4m)^2 - 48y^2 = 1$ and $X_0 + y_0\sqrt{48}$ is fundamental solutions of $(8x - 4m)^2 - 48y^2 = 16m^2$.

It is the general solution of the specific form of the proposed equation $4x^2 - 4mx = 3y^2$. Few numerical solutions of this equation for certain values of m are listed in Table 2.

Further the solutions satisfy the following recurrence relations:

(a) Recurrence relations for the solutions $(x_{(n,m)}, y_{(n,m)})$ and $(x_{i(n,m)}, y_{i(n,m)})_{i=1,2}$ among the different values of b.

Table 2: Solutions $(x_{(n,m)}, y_{(n,m)})$ of $4x^2 - 4mx = 3y^2$ for small values of n, m

n	$(x_{(n,m)}, y_{(n,m)})$ for $(m,b) = (1,4)$	$(x_{i(n,m)}, y_{i(n,m)})_{i=1,2}$ for $(m,b) = (2,8)$	
0	(1, 0)	(2, 0)	(3, 2)
1	(4, 4)	(8, 8)	(27, 30)
2	(49, 56)	(98, 112)	(363, 418)
3	(676, 780)	(1352, 1560)	(5043, 5822)
4	(9409, 10864)	(18818, 21728)	(70227, 81090)
5	(131044, 151316)	(262088, 302632)	(978123, 1129438)
n	$(x_{(n,m)}, y_{(n,m)})$ for $(m,b) = (3,12)$	$(x_{i(n,m)}, y_{i(n,m)})_{i=1,2}$ for $(m,b) = (4,16)$	
0	(3, 0)	(4, 0)	(6, 4)
1	(12, 12)	(16, 16)	(54, 60)
2	(147, 168)	(196, 224)	(726, 836)
3	(2028, 2340)	(2704, 3120)	(10086, 11644)
4	(28227, 32592)	(37636, 43456)	(140454, 162180)
5	(393132, 453948)	(524176, 605264)	(1956246,2258876)
n	$(x_{(n,m)}, y_{(n,m)})$ for $(m,b) = (5,20)$	$(x_{i(n,m)}, y_{i(n,m)})_{i=1,2}$ for (m, b) = (6, 24)	
0	(5, 0)	(6, 0)	(9, 6)
1	(20, 20)	(24, 24)	(81, 90)
2	(245, 280)	(294, 336)	(1089, 1254)
3	(3380, 3900)	(4056, 4680)	(15129, 17466)
4	(47045, 54320)	(56454, 65184)	(210681, 243270)
5	(655220, 756580)	(786264, 907896)	(2934369, 3388314)
n	$(x_{(n,m)}, y_{(n,m)})$ for $(m,b) = (7,28)$	$(x_{i(n,m)}, y_{i(n,m)})_{i=1,2}$ for (m, b) = (8, 24)	
0	(7, 0)	(8, 0)	(12, 8)
1	(28, 28)	(32, 32)	(108, 120)
2	(343, 392)	(392, 448)	(1452, 1672)
3	(4732, 5460)	(5408, 6240)	(20172, 23288)
4	(65863, 76048)	(75272, 86912)	(280908, 324360)
5	(917308, 1059212)	(1048352, 1210528)	(3912492, 4517752)
Remark: More than one class of solutions are obtained for the Diophantine Equation $4x^2 - 4mx = 3y^2$ when m =2, 4, 6, 8, ...			

(i) $x_{(n-1,2k+1)} - 14x_{(n,2k+1)} + x_{(n+1,2k+1)} + 2b = 2m$, where $n > 0$ and $k = 0, 1, 2, \dots,$

(ii) $x_{1(n-1,2k+2)} - 14x_{(n,2k+2)} + x_{(n+1,2k+2)} + 2b = 0$, where $n > 0$ and $k = 0, 1, 2, \dots$

(b) Recurrence relations for solution $(x_{(n,m)}, y_{(n,m)})$ and $(x_{i(n,m)}, y_{i(n,m)})_{i=1,2}$.

(i) $x_{(n,2k+1)} - 2x_{1(n,2k+2)} + x_{(n,2k+3)} = 0$, where $k = 0, 1, 2, \dots,$

(ii) $x_{(n,k)} - x_{1(n,2k+2)} + x_{(n,2k+1)} = 0$, where $k = 1, 2, \dots$

(c) Recurrence relations for solution $(x_{i(n,m)}, y_{i(n,m)})_{i=1,2}$

(i) $2x_{1(n,2k)} + 2y_{1(n,2k)} - y_{2(n,2k)} = m$ where $k = 1, 2, \dots$

2.2. Illustration 2

Case I: $a = \alpha + 2$, $c = \alpha + 5$.

In this case, the equation (2) becomes,

$$(2(\alpha+2)x - b)^2 - 4(\alpha^2 + 7\alpha + 10)y^2 = b^2 \tag{10}$$

where $b = m(\alpha + 2)$, $m \in Z_+$.

The above equation may be rewritten as

$$X^2 - Dy^2 = N$$

where $X = 2(\alpha + 2)x - b$, $N = b^2$ and $D = 4(\alpha^2 + 7\alpha + 10)$.

Using (3), we obtain the general solution (x_n, y_n) of equation (10) which is given by

$$\begin{aligned}
x_n &= \frac{\frac{1}{2}\left[\begin{array}{l}\left(X'_0 + y'_0\sqrt{4(\alpha^2+7\alpha+10)}\right)^n \left(X_0 + \sqrt{4(\alpha^2+7\alpha+10)}y_0\right) \\ + \left(X'_0 - y'_0\sqrt{4(\alpha^2+7\alpha+10)}\right)^n \left(X_0 - \sqrt{4(\alpha^2+7\alpha+10)}y_0\right)\end{array}\right] + b}{2(\alpha+2)} \\
y_n &= \frac{\left[\begin{array}{l}\left(X'_0 + y'_0\sqrt{4(\alpha^2+7\alpha+10)}\right)^n \left(X_0 + \sqrt{4(\alpha^2+7\alpha+10)}y_0\right) \\ - \left(X'_0 - y'_0\sqrt{4(\alpha^2+7\alpha+10)}\right)^n \left(X_0 - \sqrt{4(\alpha^2+7\alpha+10)}y_0\right)\end{array}\right]}{2\sqrt{4(\alpha^2+7\alpha+10)}}
\end{aligned} \tag{11}$$

where $X'_0 + y'_0\sqrt{4(\alpha^2+7\alpha+10)}$ is the fundamental solution of $(2(\alpha+2)x-b)^2 - 4(\alpha^2+7\alpha+10)y^2 = 1$ and $X_0 + y_0\sqrt{4(\alpha^2+7\alpha+10)}$ is the fundamental solution of $(2(\alpha+2)x-b)^2 - 4(\alpha^2+7\alpha+10)y^2 = b^2$.

For the sake of simplicity, solutions of (10) are presented for $\alpha = 1$ as follows:

Substitute $\alpha = 1$ in equation (10), we get,

$$(6x-3m)^2 - 72y^2 = 9m^2 \tag{12}$$

Using (11), we obtain the general solution (x_n, y_n) of equation (12) which is given as follows:

$$x_n = \frac{\frac{1}{2}\left[\left(X'_0 + y'_0\sqrt{72}\right)^n\left(X_0+\sqrt{72}y_0\right) + \left(X'_0 - y'_0\sqrt{72}\right)^n\left(X_0-\sqrt{72}y_0\right)\right] + 3m}{6}$$

$$y_n = \frac{1}{2\sqrt{72}}\left[\left(X'_0 + y'_0\sqrt{72}\right)^n\left(X_0+\sqrt{72}y_0\right) - \left(X'_0 - y'_0\sqrt{72}\right)^n\left(X_0-\sqrt{72}y_0\right)\right]$$

where $X'_0 + y'_0\sqrt{72}$ is the fundamental solution of $(6x-3m)^2 - 72y^2 = 1$ and $X_0 + y_0\sqrt{72}$ is the fundamental solution of $(6x-3m)^2 - 72y^2 = 9m^2$.

It is the general solution of the specific form of the proposed equation $3x^2 - 3mx = 6y^2$. Few numerical solutions of this equation for certain values of m are listed in Table 3.

Further the solutions satisfy the following recurrence relations:

(a) Recurrence relations for solution $(x_{(n,m)}, y_{(n,m)})$

  (i) $x_{(n-1,k)} - 34x_{(n,k)} + x_{(n+1,k)} + 5b + m = 0$ where $n > 0$ and $k = 1, 2, \ldots,$

  (ii) $y_{(n-1,k)} - 34y_{(n,k)} + y_{(n+1,k)} = 0$ where $n > 0$ and $k = 1, 2, \ldots$

(b) Recurrence relations for solution $(x_{(n,m)}, y_{(n,m)})$among the different values of b.

  (i) $x_{(n,k)} + x_{(n,k+1)} - x_{(n,k+2)} - x_{(n,k-1)} = 0$, where $k = 2, 3, \ldots,$ and $k \neq 7, 14, 21, \ldots$

  (ii) $y_{(n,k)} + y_{(n,k+1)} - y_{(n,k+2)} - y_{(n,k-1)} = 0$, where $k = 2, 3, \ldots,$ and $k \neq 7, 14, 21, \ldots$

(c) Recurrence relations for solution $(x_{i(n,m)}, y_{i(n,m)})_{i=1,2}$

  (i) $x_{1(n,k)} - x_{2(n,k)} + y_{1(n,k)} + y_{2(n,k)} = 0$ where $k = 2, 3, \ldots,$ and $k \neq 7, 14, 21, \ldots$

Table 3: Solutions $(x_{(n,m)}, y_{(n,m)})$ of $3x^2 - 3mx = 6y^2$ for small values of n, m

n	$(x_{i(n,m)}, y_{i(n,m)})_{i=1,2}$ for $(m,b) = (1,3)$		$(x_{i(n,m)}, y_{i(n,m)})_{i=1,2}$ for $(m,b) = (2,6)$	
0	(1, 0)	(2, 1)	(2, 0)	(4, 2)
1	(9, 6)	(50, 35)	(18, 12)	(100, 70)
2	(289, 204)	(1682, 1189)	(578, 408)	(3364, 2378)
3	(9801, 6930)	(57122, 40391)	(19602, 13860)	(114244, 80782)
4	(332929, 235416)	(1940450, 1372105)	(665858, 470832)	(3880900, 2744210)
5	(11309769,7997214)	(65918162,46611179)	(22619538,15994428)	(131836324, 93222358)
n	$(x_{i(n,m)}, y_{i(n,m)})_{i=1,2}$ for $(m,b) = (3,9)$		$(x_{i(n,m)}, y_{i(n,m)})_{i=1,2}$ for $(m,b) = (4,12)$	
0	(3, 0)	(6, 3)	(4, 0)	(8, 4)
1	(27, 18)	(150, 105)	(36, 24)	(200, 140)
2	(867, 612)	(5046, 3567)	(1156, 816)	(6728, 4756)
3	(29403, 20790)	(171366, 121173)	(39204, 27720)	(228488, 161564)
4	(998787, 706248)	(5821350, 4116315)	(1331716, 941664)	(7761800, 5488420)
5	(33929307,23991642)	(197754486,139833537)	(45239076,31988856)	(263672648,186444716)
n	$(x_{i(n,m)}, y_{i(n,m)})_{i=1,2}$ for $(m,b) = (5,15)$		$(x_{i(n,m)}, y_{i(n,m)})_{i=1,2}$ for $(m,b) = (6,18)$	
0	(5, 0)	(10, 5)	(6, 0)	(12, 6)
1	(45, 30)	(250, 175)	(54, 36)	(300, 210)
2	(1445, 1020)	(8410, 5945)	(1734, 1224)	(10092, 7134)
3	(49005, 34650)	(285610, 201955)	(58806, 41580)	(342732, 242346)
4	(1664645, 1177080)	(9702250, 6860525)	(1997574, 1412496)	(11642700, 8232630)
5	(56548845,39986070)	(329590810,233055895)	(67858614,47983284)	(395508972,279667074)
n	$(x_{i(n,m)}, y_{i(n,m)})_{i=1,2,3,4,5,6}$ for $(m,b) = (7,21)$			
0	(7, 0)	(8, 2)	(9, 3)	(14, 7)
1	(63, 42)	(128,88)	(169,117)	(350,245)
2	(2023, 1428)	(4232, 2990)	(5625, 3975)	(11774, 8323)
3	(68607, 48510)	(143648, 101572)	(190969, 135033)	(399854, 282737)
4	(2330503, 1647912)	(4879688,3450458)	(6487209, 4587147)	(13583150, 9604735)
5	(79168383, 55980498)	(165765632, 117214000)	(220374025, 155827965)	(461427134, 326278253)
0	(25, 15)		(32, 20)	
1	(729,513)		(968,682)	
2	(24649, 17427)		(32768, 23168)	
3	(837225, 592005)		(1113032,787030)	
4	(28440889,20110743)		(37810208, 26735852)	
5	(966152889,683173257)		(1284433928,908231938)	
n	$(x_{i(n,m)}, y_{i(n,m)})_{i=1,2}$ for $(m,b) = (8,24)$			
0	(8, 0)		(16, 8)	
1	72, 48)		400,280)	
2	2312, 1632)		13456,9512)	
3	78408, 55440)		456976, 323128)	
4	2663432, 1883328)		15523600,10976840)	
5	90478152, 63977712)		527345296,372889432)	
Remark: More than one class of solutions are obtained for the Diophantine Equation $3x^2 - 3mx = 6y^2$ for all $m \in Z_+$				

Case II: $a = \alpha + 5$, $c = \alpha + 2$.

In this case, the equation (2) becomes,

$$(2(\alpha+5)x - b)^2 - 4(\alpha^2 + 7\alpha + 10)y^2 = b^2 \tag{13}$$

where $b = m(\alpha + 5)$, $m \in Z_+$.

The above equation may be rewritten as

$$X^2 - Dy^2 = N$$

where $X = 2(\alpha + 5)x - b$, $N = b^2$ and $D = 4(\alpha^2 + 7\alpha + 10)$.

Using (3), we obtain the general solution (x_n, y_n) of equation (13) which is given as follows:

$$\begin{aligned} x_n &= \frac{\frac{1}{2}\left[\begin{array}{l}\left(X'_0 + y'_0\sqrt{4(\alpha^2+7\alpha+10)}\right)^n \left(X_0 + \sqrt{4(\alpha^2+7\alpha+10)}y_0\right) \\ + \left(X'_0 - y'_0\sqrt{4(\alpha^2+7\alpha+10)}\right)^n \left(X_0 - \sqrt{4(\alpha^2+7\alpha+10)}y_0\right)\end{array}\right] + b}{2(\alpha+5)} \\ y_n &= \frac{\left[\begin{array}{l}\left(X'_0 + y'_0\sqrt{4(\alpha^2+7\alpha+10)}\right)^n \left(X_0 + \sqrt{4(\alpha^2+7\alpha+10)}y_0\right) \\ - \left(X'_0 - y'_0\sqrt{4(\alpha^2+7\alpha+10)}\right)^n \left(X_0 - \sqrt{4(\alpha^2+7\alpha+10)}y_0\right)\end{array}\right]}{2\sqrt{4(\alpha^2+7\alpha+10)}} \end{aligned} \tag{14}$$

where $X'_0 + y'_0\sqrt{4(\alpha^2+7\alpha+10)}$ is the fundamental solution of $(2(\alpha+5)x - b)^2 - 4(\alpha^2+7\alpha+10)y^2 = 1$ and $X_0 + y_0\sqrt{4(\alpha^2+7\alpha+10)}$ is the fundamental solution of $(2(\alpha+5)x - b)^2 - 4(\alpha^2+7\alpha+10)y^2 = b^2$.

For the sake of simplicity, solutions of (13) are presented for $\alpha = 1$ as follows:

Substitute $\alpha = 1$ in equation (13), we get.

$$(12x - 6m)^2 - 72y^2 = 36m^2 \tag{15}$$

Using (14), we obtain the general solution (x_n, y_n) of equation (15) which is given as follows:

$$\begin{aligned} x_n &= \frac{\frac{1}{2}\left[\left(X'_0 + y'_0\sqrt{72}\right)^n \left(X_0 + \sqrt{72}y_0\right) + \left(X'_0 - y'_0\sqrt{72}\right)^n \left(X_0 - \sqrt{72}y_0\right)\right] + 6m}{12} \\ y_n &= \frac{1}{2\sqrt{72}}\left[\left(X'_0 + y'_0\sqrt{72}\right)^n \left(X_0 + \sqrt{72}y_0\right) - \left(X'_0 - y'_0\sqrt{72}\right)^n \left(X_0 - \sqrt{72}y_0\right)\right] \end{aligned}$$

where $X'_0 + y'_0\sqrt{72}$ is the fundamental solution of $(12x - 6m)^2 - 72y^2 = 1$ and

Table 4: Solutions $(x_{(n,m)}, y_{(n,m)})$ of $6x^2 - 6mx = 3y^2$ for small values of n, m

n	$(x_{i(n,m)}, y_{i(n,m)})_{i=1,2}$ for $(m,b) = (1,6)$		$(x_{i(n,m)}, y_{i(n,m)})_{i=1,2}$ for $(m,b) = (2,12)$	
0	(1, 0)	(2, 2)	(2, 0)	(4,4)
1	(9,12)	(50,70)	(18,24)	(100,140)
2	(289, 408)	(1682, 2378)	(578, 816)	(3364, 4756)
3	(9801, 13860)	(57122, 80782)	(19602, 27720)	(114244,161564)
4	(332929, 470832)	(1940450, 2744210)	(665858, 941664)	(3880900, 5488420)
5	(11309769,15994428)	(65918162, 93222358)	(22619538, 31988856)	(131836324, 186444716)
n	$(x_{i(n,m)}, y_{i(n,m)})_{i=1,2}$ for $(m,b) = (3,18)$		$(x_{i(n,m)}, y_{i(n,m)})_{i=1,2}$ for $(m,b) = (4,24)$	
0	(3, 0)	(6, 6)	(4, 0)	(8, 8)
1	(27,36)	(150,210)	(36,48)	(200,280)
2	(867, 1224)	(5046, 7134)	(1156, 1632)	(6728, 9512)
3	(29403, 41580)	(171366, 242346)	(39204, 55440)	(228488, 323128)
4	(998787, 1412496)	(5821350, 8232630)	(1331716, 1883328)	(7761800, 10976840)
5	(33929307, 47983284)	(197754486, 279667074)	(45239076, 63977712)	(263672648, 372889432)
n	$(x_{i(n,m)}, y_{i(n,m)})_{i=1,2}$ for $(m,b) = (5,30)$		$(x_{i(n,m)}, y_{i(n,m)})_{i=1,2}$ for $(m,b) = (6,36)$	
0	(5, 0)	(10,10)	(6, 0)	(12,12)
1	(45,60)	(250,350)	(54,72)	(300,420)
2	(1445, 2040)	(8410, 11890)	(1734, 2448)	(10092, 14268)
3	(49005, 69300)	(285610, 403910)	(58806, 83160)	(342732, 484692)
4	(1664645, 2354160)	(9702250, 13721050)	(1997574, 2824992)	(11642700, 16465260)
5	(56548845, 79972140)	(329590810, 466111790)	(67858614, 95966568)	(395508972, 559334148)
n	$(x_{i(n,m)}, y_{i(n,m)})_{i=1,2,3,4,5,6}$ for $(m,b) = (7,42)$			
0	(7, 0)	(8, 4)	(9, 6)	(14, 14)
1	(63,84)	(128,176)	(169,234)	(350,490)
2	(2023, 2856)	(4232,5980)	(5625, 7950)	(11774, 16646)
3	(68607, 97020)	(143648, 203144)	(190969, 270066)	(399854, 565474)
4	(2330503, 3295824)	(4879688, 6900916)	(6487209, 9174294)	(13583150, 19209470)
5	(79168383, 111960996)	(165765632,234428000)	(220374025, 311655930)	(461427134, 652556506)
0	(25, 30)		(32, 40)	
1	(729,1026)		(968,1364)	
2	(24649,34854)		(32768, 46336)	
3	(837225, 1184010)		(1113032, 1574060)	
4	(28440889, 40221486)		(37810208, 53471704)	
5	(966152889,1366346514)		(1284433928, 1816463876)	
n	$(x_{i(n,m)}, y_{i(n,m)})_{i=1,2}$ for $(m,b) = (8,48)$			
0	(8, 0)		(16, 16)	
1	(72,96)		(400,560)	
2	(2312, 3264)		(13456, 19024)	
3	(78408, 110880)		(456976, 646256)	
4	(2663432, 3766656)		(15523600, 21953680)	
5	(90478152, 127955424)		(527345296, 745778864)	
Remark: More than one class of solutions are obtained for the Diophantine Equation $6x^2 - 6mx = 3y^2$ for all $m \in Z_+$				

$X_0 + y_0\sqrt{72}$ is fundamental solutions of $(12x - 6m)^2 - 72y^2 = 36m^2$.

It is the general solution of the specific form of the proposed equation $6x^2 - 6mx = 3y^2$. Few numerical solutions of this equation for certain values of m are listed in Table 4.

Further the solutions satisfy the following recurrence relation:

(a) Recurrence relations for solution $(x_{(n,m)}, y_{(n,m)})$among the different values of n.

  (i) $x_{(n-1,k)} - 34x_{(n,k)} + x_{(n+1,k)} + 3b = 2m$ where $n > 0$ and $k = 1, 2, \ldots$

  (ii) $y_{(n-1,k)} - 34y_{(n,k)} + y_{(n+1,k)} = 0$ where $n > 0$ and $k = 1, 2, \ldots$

(b) Recurrence relations for solution $(x_{(n,m)}, y_{(n,m)})$ among the different values of b.

  (i) $x_{(n,k)} + x_{(n,k+1)} - x_{(n,k+2)} - x_{(n,k-1)} = 0$, where $k = 1, 2, \ldots$, if $k \neq 7, 14, 21, \ldots$

  (ii) $y_{(n,k)} + y_{(n,k+1)} - y_{(n,k+2)} - y_{(n,k-1)} = 0$ where $k = 1, 2, \ldots$ if $k \neq 7, 14, 21, \ldots$

(c) Recurrence relations for solution $(x_{i(n,m)}, y_{i(n,m)})_{i=1,2}$

  (i) $x_{1(n,k)} + x_{2(n,k)} + y_{1(n,k)} - y_{2(n,k)} = m$ where $k = 1, 2, \ldots$ and $k \neq 7, 14, 21, \ldots$

3. Conclusion

In this work, an effort has been made to obtain infinitely many integer solutions of the proposed binary quadratic Diophantine equation $ax^2 - bx = cy^2$ of hyperbolic shape. During this effort, some difficulties were identified in generating the integer solutions continuously after the conversion of the equation as a Pellian Equation. It was resolved when considering one of the constant a as a multiple of the other constant b through the deep study and the survey of past related works. Finally, few relational identities were fetched between the integer solutions of the equation for the specific values of its constants.

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