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STAR COLOURING IN FEW CLASSES OF GRAPHS

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Abstract: A proper vertex colouring of a graph G is called a star colouring if every path of G on four vertices is not 2-coloured. The star chromatic number is the minimum number of colours required to star colour G and it is denoted by $\chi_s(G)$. The Star Chromatic Number of the Middle Graphs of path (P_n); Shadow Graphs of path (P_n) and Tadpole graphs $(T_{3,n})$; m- fold Triangular Snake graphs $(S(C_3, m, n))$ have been discussed in this paper.

Keywords and Phrases: Star Colouring, Star Chromatic number, Middle graph, Shadow graph, Tadpole graph, m-fold Triangular Snake graphs.

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1. Introduction and Preliminaries

Let us consider the graph $G = (V, E)$ to be finite, simple and undirected. *Vertex Colouring* on a graph G is an assignment of colours to the vertices of a graph so that no two adjacent vertices get the same colour. A vertex colouring of a graph is said to be *proper* if no two vertices sharing the same edge have the same colour. The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colours required to colour G [2]. A proper vertex colouring of a graph G is called *star colouring*, if every path of G on four vertices is not 2 - coloured. The *star chromatic number* is the minimum number of colours required to star colour G and it is denoted by

$\chi_s(G)$. In 1973, Branko Grünbaum introduced the concept of star coloring and also he introduce the notion of star chromatic number [4].

The *Middle graph* of G is denoted by $M(G)$, the vertex set of $M(G)$ is $V(G) \cup E(G)$ and any two vertices x, y in $M(G)$ are adjacent in $M(G)$, if x, y are in $E(G)$ and x, y are adjacent in G or x is in $V(G)$, y is in $E(G)$ and x, y are adjacent in G.The *Shadow graph* denoted by shad(G) is that graph with the vertex set $V(G) \cup \{u_1, \ldots, u_n\}$, where u_i is called the shadow vertex of v_i and u_i is adjacent to u_j if v_i is adjacent to v_j and u_i is adjacent to v_j if v_i is adjacent to v_j for $1 \leq i$, $j \leq n$. The (m, n)-*Tadpole graph* is a graph is obtained by joining a cycle graph C_m to a path graph P_n with a bridge, is denoted by $T_{m,n}$. A *m-fold triangular snake graph* [1] denoted by $S(C_3, m, n)$, is of length n and is obtained from a path $v_1, v_2 \ldots v_n, v_{n+1}$ by joining v_i and v_{i+1} to new m vertices $w_{i1}, \ldots, w_{im}$, $i = 1, 2, \ldots n$ giving edges $(v_i w_{ij})$ and $(w_{ij} v_{i+1})$, $j = 1, 2, \ldots m$ and $i = 1, 2 \ldots n$.
To prove the main result we need the following theorems:

Theorem 1.1. [3] *Let P_n be a path graph on n vertices, then*

$$\chi_s(P_n) = 3, \forall\ n \geq 4$$

Theorem 1.2. [3] *If K_n is the complete graph on n vertices, then*

$$\chi_s(K_n) = n, \forall\ n \geq 3$$

Theorem 1.3. [3] *Let C_n be a cycle, $n \geq 3$.Then*

$$\chi_s(C_n) = \begin{cases} 4 & \text{when } n = 5 \\ 3 & \text{otherwise} \end{cases}$$

2. Main Results

Theorem 2.1. *For a path P_n where $n \geq 3$, the star chromatic number of middle graph of path is*

$$\chi_s(M(P_n)) = \begin{cases} 3 & n = 3, 4 \\ 4 & n \geq 5 \end{cases}$$

Proof. Let $V(P_n) = \{v_i : 1 \leq i \leq n\}$ and $E(P_n) = \{e_j = u_j : 1 \leq j \leq n-1\}$. Consider $M(P_n)$, the vertex set of the middle graph of P_n is $V(M(P_n)) = \{v_i \cup u_j : 1 \leq i \leq n; 1 \leq j \leq n-1\}$ and denote by S, the set of colours required to star colour the graph $M(P_n)$. We now star colour the graph.
Case 1: $n = 3, 4$. Clearly the vertices $\{v_2 \cup u_1 \cup u_2\}$ induces a clique of order 3.

Hence $|S| \geq 3$. We claim that $\chi_s(M(P_n)) = 3$. Let $f : V(M(P_n)) \to S$ such that

$$f(v_i) = 1 \ (1 \leq i \leq n)$$
$$f(u_{2i-1}) = 2 \ (1 \leq i \leq 2)$$
$$f(u_2) = 3$$

Thus there is no possibility for any path on four vertices to be bicoloured. Hence $\chi_s(M(P_n)) = 3$.

Case 2: $n \geq 5$. Let $f : V(M(P_n)) \to S$ such that

$$f(v_i) = 1 \ (1 \leq i \leq n)$$
$$f(u_{2j-1}) = 2 \ (j \geq 1) \ \textit{where} \ 2j - 1 \leq 2n - 7$$
$$f(u_2) = 3 \ \textit{for} \ n = 5, 6$$
$$\text{and} \quad f(u_{4j-2}) = 3$$

where j varies from k to $k+t$; $k \in N$ (the set of natural numbers) and $t = 1, 2, \ldots$ in succession and $(4k - 1 \leq n \leq 4k + 2)$, $k = 2, 3, \ldots$. If we assign the colours from $\{1, 2, 3\}$ to the vertex u_4, we arrive at a contradiction. Hence $|S| \geq 4$. We claim that $\chi_s(M(P_n)) = 4$. For $j = k$; $k = 1, 2, \ldots$ in succession, we have

$$f(u_{4j}) = 4$$

where $4k + 1 \leq n \leq 4k + 4$ $(k = 1, 2, \ldots)$. We observe that there is no possibility for any path on four vertices to be bicoloured. Hence $\chi_s(M(P_n)) = 4$.

Theorem 2.2. *For $m, n \geq 2$, the star chromatic number of m-fold Triangular Snake graph $S(C_3, m, n)$ is*

$$\chi_s(S(C_3, m, n)) = \begin{cases} 3 & n = 2 \\ 4 & \textit{otherwise} \end{cases}$$

Proof. Let $V(S(C_3, m, n)) = \{v_i \cup u_j : 1 \leq i \leq n + 1, 1 \leq j \leq mn\}$ and $E(S(C_3, m, n)) = \{e_i : 1 \leq i \leq 2n + 1\}$ and denote by S, the set of colours required to star colour the graph $S(C_3, m, n)$. We now star colour the graph.

Case 1: $n = 2$. Label the vertices of P_3 as v_1, v_2, v_3 and the m- vertices as $u_1, u_2, u_3, \ldots, u_{mn}$ respectively. We note that the vertices $\{v_1 \cup u_1 \cup v_2\}$ form a cycle of order 3. Hence $|S| \geq 3$. We claim that $\chi_s(S(C_3, m, 2)) = 3$. Let $f : V(S(C_3, m, 2)) \to S$ such that

$$f(u_j) = 1 \ (1 \leq j \leq mn)$$
$$f(v_{2i-1}) = 2 \ (1 \leq i \leq 2)$$
$$f(v_2) = 3$$

Thus there is no possibility for any path on four vertices to be bicoloured. Hence $\chi_s(S(C_3, m, 2)) = 3$.

Case 2: $n > 2$. Let $f : V(S(C_3, m, n)) \to S$ such that

$$f(u_j) = 1 \ (1 \le j \le mn)$$
$$f(v_{2i-1}) = 2 \ (1 \le i \le n-1)$$

For $i = k$; $k \in N$ (the set of natural numbers), i varies from k to $k+t$, $t = 1, 2, \ldots$ in succession, we have

$$f(v_{4i-2}) = 3$$

where $4k+1 \le n \le 4k+4$ ($k = 1, 2, \ldots$). If we assign the colours from $\{1, 2, 3\}$ to the vertex v_4, we arrive at a contradiction. Hence $|S| \ge 4$. We claim that $\chi_s(S(C_3, m, n)) = 4$. For $i = k$; $k = 1, 2, \ldots$ in succession, we have

$$f(u_{4i}) = 4$$

where $4k+1 \le n \le 4k+4$ ($k = 1, 2, \ldots$). We observe that there is no possibility for any path on four vertices to be bicoloured. Hence $\chi_s(S(C_3, m, n)) = 4$.

Theorem 2.3. *For $n \ge 2$, the star chromatic number of the shadow graph of a path is given by*

$$\chi_s(shad(P_n)) = \begin{cases} 3 & n = 2, 3 \\ 5 & otherwise \end{cases}$$

Proof. Let $V(P_n) = \{v_i : 1 \le i \le n\}$ and $E(P_n) = \{e_i : 1 \le i \le n-1\}$. Consider $shad(P_n)$, the vertex and edge sets are $V(shad(P_n)) = \{v_i \cup u_j : 1 \le i \le n, 1 \le j \le n\}$ and $E(shad(P_n)) = \{e_j : 1 \le j \le 4n-4\}$ respectively and denote by S, the set of colours required to star colour the graph $shad(P_n)$. We now star colour the graph.

Case 1: $n = 2, 3$. In $shad(P_3)$, we note that $v_1v_2u_1u_2$ is a path on four vertices. Hence $|S| \ge 3$. We claim that $\chi_s(shad(P_3)) = 3$. Let $f : V(shad(P_n)) \to S$ such that

$$f(v_{2i+1}) = 1 \ (0 \le i \le 1)$$
$$f(u_{2j+1}) = 1 \ (0 \le j \le 1)$$
$$f(v_2) = 2$$
$$f(u_2) = 3$$

Thus there is no possibility for any path on four vertices to be bicoloured. Hence $\chi_s(shad(P_n)) = 3$.

Case 2: $n \geq 4$. Let $f : V(shad(P_n)) \to S$ such that

$$f(v_{2i-1}) = 1 \ (i \geq 1) \ where \ 2i-1 \leq n$$
$$f(v_{4i-2}) = 2 \ (i \geq 1) \ where \ 4i-2 \leq n$$
$$f(v_{4i}) = 3 \ (i \geq 1) \ where \ 4i \leq n$$
$$f(u_{2j-1}) = 1 \ (j \geq 1) \ where \ 2j-1 \leq n$$

If we assign the colours from $\{1, 2, 3\}$ to the vertex u_2, we arrive at a contradiction. Hence $|S| \geq 4$. So'

$$f(u_{4j-2}) = 4 \ (j \geq 1) \ where \ 4j-2 \leq n$$

If we assign the colours from $\{1, 2, 3, 4\}$ to the vertex u_4, we arrive at a contradiction. Hence $|S| \geq 5$. We claim that $\chi_s(shad(P_n)) = 5$. If

$$f(u_{4j}) = 5 \ (j \geq 1) \ where \ 4j \leq n$$

then there is no possibility for any path on four vertices to be bicoloured. Hence $\chi_s(shad(P_n)) = 5$.

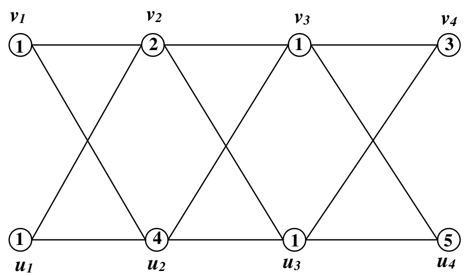

Figure 1: Star Colouring of $shad(P_4)$

Theorem 2.4. *For $n \geq 1$, the star chromatic number of the shadow graph of $T_{3,n}$ is given by*

$$\chi_s(shad(T_{3,n})) = 5$$

Proof. Let $V(T_{3,n}) = \{v_i : 1 \leq i \leq 3+n\}$ and $E(T_{3,n}) = \{e_i : 1 \leq i \leq 3+n\}$. Consider $shad(T_{3,n})$, the vertex and edge sets are $V(shad(T_{3,n})) = \{v_i \cup u_j : 1 \leq i \leq 3+n, 1 \leq j \leq 3+n\}$ and $E(shad(T_{3,n})) = \{e_j : 1 \leq j \leq 4n+8\}$ respectively and denote by S, the set of colours required to star colour the graph $shad(T_{3,n})$.

We now star colour the graph. Label the vertices of C_3 as v_1, v_2, v_3 and u_1, u_2, u_3 respectively. Hence $|S| \geq 3$. Assume that $\chi_s(shad(T_{3,n})) = 3$. If we assign the colours from $\{1, 2, 3\}$ to the vertex u_2 we get a contradiction. Hence $|S| \geq 4$. Assign the colour 4 to u_2. Assume that $\chi_s(shad(T_{3,n})) = 4$. If we assign the colours from $\{1, 2, 3, 4\}$ to the vertex u_3 we get a contradiction. Hence $|S| \geq 5$. Assign the colour 5 to u_3. We claim that $\chi_s(shad(T_{3,n})) = 5$.
Let $f : V(shad(T_{3,n})) \rightarrow S$ such that

$$\begin{aligned} f(v_{2i}) &= 1 \ (i \geq 2) \ where \ 2i \leq n+2 \\ f(u_{2j}) &= 1 \ (j \geq 2) \ where \ 2j \leq n+2 \\ f(v_{4i+1}) &= 2 \ (i \geq 1) \ where \ 4i+1 \leq n+2 \\ f(u_{4j+1}) &= 4 \ (j \geq 1) \ where \ 4j+1 \leq n+2 \\ f(v_{4i+3}) &= 3 \ (i \geq 1) \ where \ 4i+3 \leq n+3 \\ f(u_{4j+3}) &= 5 \ (j \geq 1) \ where \ 4j+3 \leq n+3 \end{aligned}$$

Thus there is no possibility for any path on four vertices to be bicoloured. Hence $\chi_s(shad(T_{3,n})) = 5$.

3. Conclusion

The Star Chromatic Number of the Middle Graphs of path (P_n); Shadow Graphs of path (P_n) and Tadpole graphs ($T_{3,n}$); m- fold Triangular Snake graphs ($S(C_3, m, n)$) are found in this paper. The Star Colouring of Honeycomb Network ($HC(n)$), the Middle and Total Graphs of cycle (C_n), complete graph (K_n), Closed Helm Graphs and m- fold Triangular Snake Graphs are under investigation.

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