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LUCKY LABELING ON SHELL FAMILY OF GRAPHS

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Abstract: Let $f : V(G) \to N$ be a labeling of the vertices of a graph G. Let $S(v)$ denote the sum of labels of the neighbours of the vertex v in G. If v is an isolated vertex of G, then $S(v) = 0$. A labeling f is lucky if $S(u) \neq S(v)$ for every pair of adjacent vertices u and v. The lucky number of a graph G, denoted by $\eta(G)$, is the least positive integer k such that G has a lucky labeling with $\{1, 2, ..., k\}$ as the set of labels. In this paper we prove that shell graph, bow graph and wheel graph admits Lucky labeling.

Keywords and Phrases: Lucky Labeling, Shell Graph, Bow Graph, Wheel Graph, Lucky number.

2020 Mathematics Subject Classification: 05C78.

1. Introduction

The first paper in Graph Theory was written by Euler in 1736, when he solved the Konigsberg Bridge problem. Graph theory is a dynamic mathematical discipline that has many applications in wide variety of subjects such as Physics, Chemistry, Operations Research and so on. Graph labeling serves as a frontier between Number Theory and structure of graphs. Labeling the vertices or edges or both, subject to certain conditions is called as Graph Labeling. Most Graph labeling methods trace their origin to one introduced by Rosa [6] in 1967. It was further developed by Graham and Sloane [5] in 1980. Labeled graph serves as useful models for many applications such as x-ray, cryptography, radar, astronomy, circuit design, communication network addressing and data base management, secret sharing schemes, models for constraint programming over finite domains. For more applications refer to Gallian survey [4].

Lucky labeling is assigning the vertices arbitrarily such that the sum of labels of all adjacent vertices of a vertex is not equal to the sum of labels of all adjacent vertices of any vertex which is adjacent to it. The lucky number of a graph G, denoted by $\eta(G)$, is the least positive integer k such that G has a lucky labeling with $\{1, 2, \ldots, k\}$ as the set of labels. Lucky labeling of graphs was studied in recent times by Ahai *et al* [1] and Akbari *et al* [2]. Many results exist on Lucky Labeling. Chiranjilal Kujur *et al* [3] have proved that Bloom graph admits Lucky labeling. Lucky labeling is applied in real life situations such as transportation network, where pairwise connections are given some numerical values and each weight could represent the stations or city with certain expenses or costs. Lucky labeling is also applicable in computational biology to model protein structures.

2. Main Results

In this paper we prove that shell graph, bow graph and wheel graph admits Lucky labeling and we compute the lucky number $\eta(G)$.

Definition 2.1. *Let $f : V(G) \to N$ be a labeling of the vertices of a graph G. Let $S(v)$ denote the sum of labels of the neighbours of the vertex v in G. If v is an isolated vertex of G, then $S(v) = 0$. A labeling f is* ***lucky*** *if $S(u) \neq S(v)$ for every pair of adjacent vertices u and v. The lucky number of a graph G, denoted by $\eta(G)$, is the least positive integer k such that G has a* ***lucky labeling*** *with $\{1, 2, \ldots, k\}$ as the set of labels.*

Definition 2.2. *A* ***Shell graph*** *S_n is the graph obtained by taking $(n–3)$ concurrent chords in cycle C_n. The vertex at which all the chords are concurrent is called apex vertex.*

Definition 2.3. *A* ***Bow graph*** *$B_{m,n}$ is defined to be a double shell in which each shell has any order. A bow graph in which each shell has the same order is called uniform bow graph. The vertex at which all the chords are concurrent is called apex vertex.*

Definition 2.4. *A* ***Wheel graph*** *W_n is a graph with n vertices $(n \geq 4)$ formed by connecting a single vertex to all vertices of an $(n–1)$ cycle.*

Theorem 1. *The Shell graph S_n admits Lucky labeling.*

Proof. Let S_n be the shell graph where n denotes the number of vertices in the shell graph.

The total number of vertices and edges in the shell graph is given by

$|V(Sn)| = |n|$ and $|E(Sn)| = |(2n–3)|$
The vertices of the shell graph are denoted as $\{v_1, v_2, ..., v_n\}$ in anticlockwise direction where v_1 is the apex vertex as shown in the Figure 1.

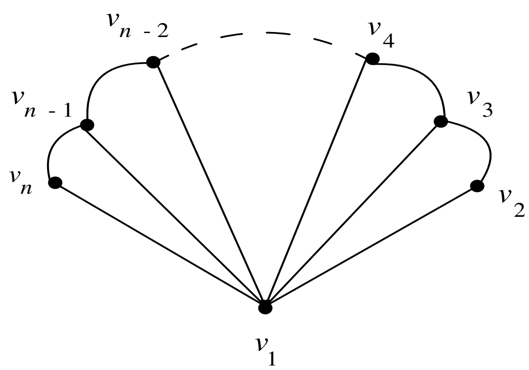

Figure 1: Shell graph S_n

Case 1: When n is even
We define the vertex labels as follows
$f : V(S_n) \to N$

$$f(v_i) = \begin{cases} 1 & \text{if} \quad i = \text{odd} \\ 2 & \text{if} \quad i = \text{even} \end{cases} \tag{2.1}$$

where $1 \leq i \leq n, n \geq 6$.
Let $S(v)$ denote the sum of the labels of the neighbours of the vertex v in S_n.
The vertex v_1 will have the neighbourhood sum $S(v_1) = ((n - 1) + n/2)$ where $n \geq 6$.
The vertices v_2 and v_n will have the neighbourhood sum $S(v_2) = S(v_n) = 2$.
The vertices $\{v_3, ..., v_{n-1}\}$ will have the neighbourhood sum

$$S(v_j) = \begin{cases} 5 & \text{if} \quad j = \text{odd} \\ 3 & \text{if} \quad j = \text{even} \end{cases} \tag{2.2}$$

where $3 \leq j \leq n–1$.
It is evident that no two adjacent vertices have equal neighbourhood sums.
Hence it satisfies the condition $S(u) \neq S(v)$ for every pair of adjacent vertices u and v.
Therefore shell graph admits Lucky labeling when n is even.
An illustration is shown in Figure 2

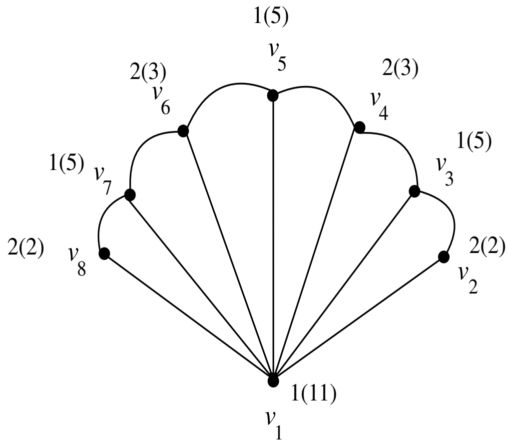

Figure 2: Lucky labeling of S_8

Case 2: When n is odd

We define the vertex labels as follows

$f : V(S_n) \to N$

$$f(v_i) = \begin{cases} 1 & \text{if} \quad i = \text{odd} \\ 2 & \text{if} \quad i = \text{even} \end{cases} \tag{2.3}$$

where $1 \leq i \leq n-1, n \geq 5$.

Without the loss of generality $f(v_n)$ is labeled as 2.

Let $S(v)$ denote the sum of the labels of the neighbours of the vertex v in S_n .

The vertex v_1 will have the neighbourhood sum $S(v_1) = ((n+1)/2)+(n-1)$ where $n \geq 5$.

The vertex v_2 will have the neighbourhood sum $S(v_2) = 2$.

The vertex v_n will have the neighbourhood sum $S(v_n) = 3$.

The vertex v_{n-1} will have the neighbourhood sum $S(v_{n-1}) = 4$.

The vertices $\{v_3, ..., v_{n-2}\}$ will have the neighbourhood sum

$$S(v_j) = \begin{cases} 5 & \text{if} \quad j = \text{odd} \\ 3 & \text{if} \quad j = \text{even} \end{cases} \tag{2.4}$$

where $3 \leq j \leq n–2$.

It is evident that no two adjacent vertices have equal neighbourhood sums.

Hence it satisfies the condition $S(u) \neq S(v)$ for every pair of adjacent vertices u and v.

Therefore shell graph admits Lucky labeling when n is odd.

An illustration is shown in Figure 3.

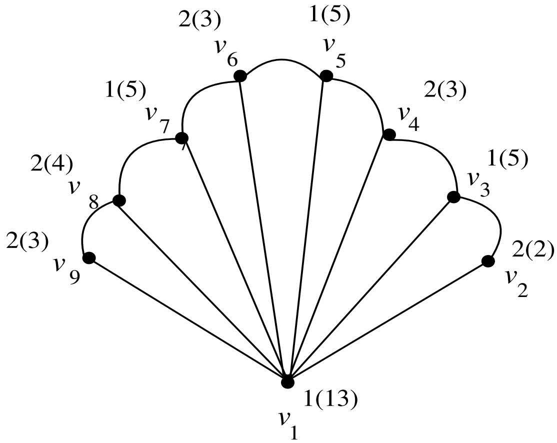

Figure 3: Lucky labeling of S_9

Thus shell graph admits Lucky labeling with $\eta(S_n) = 2$.

Theorem 2. *The Bow graph $B_{m,n}$ admits Lucky labeling where $m \leq n$.*
Proof. Let $B_{m,n}$ be the bow graph with shells of order m and n excluding the apex. The shell that is in the left side of the apex is called as the left shell and the shell that is in the right side of the apex is called the right shell. Let m be the order of the right shell in $B_{m,n}$ and n be the order of the left shell. Let the apex of $B_{m,n}$ be denoted as v_0. The total number of vertices and edges in the bow graph is given by $|V(B_{m,n})| = |m + n + 1|$ and $|E(B_{m,n})| = |2(m + n + 1)–4|$

The vertices in the right shell of the bow graph from top to bottom is denoted as $\{v_1, v_2, ..., v_m\}$ and the vertices in the left shell of the bow graph from top to bottom is denoted as $\{u_1, u_2, ..., u_n\}$ as shown in the Figure 4.

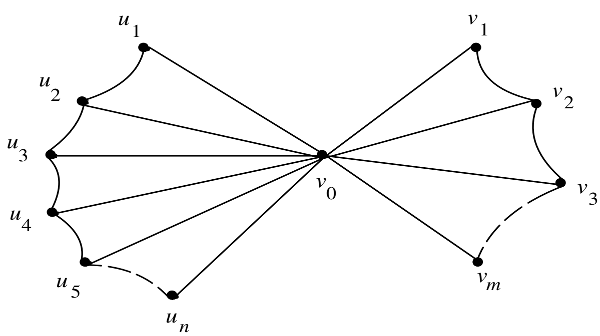

Figure 4: Bow graph B_m, n

Case 1: When m is even and n is odd
We define the vertex labels as follows
$f : V(B_{m,n}) \to N$
The apex vertex v_0 is labeled as $f(v_0) = 1$

$$f(v_i) = \begin{cases} 2 & \text{if} i = \text{odd} \\ 1 & \text{if} i = \text{even} \end{cases} \tag{2.5}$$

in the right shell where $1 \leq i \leq m–1, m \geq 4$.
Without the loss of generality $f(v_m)$ is labeled as 2

$$f(u_i) = \begin{cases} 2 & \text{if} \quad i = \text{odd} \\ 1 & \text{if} \quad i = \text{even} \end{cases} \tag{2.6}$$

in the left shell where $1 \leq i \leq n, n \geq 5$.
Let $S(v)$ denote the sum of the labels of the neighbours of the vertex v in $B_{m,n}$.
The vertex v_0 will have the neighbourhood sum $S(v_0) = 3n - 3(((n - m) - 1)/2)$.
The vertices v_1, u_1 and u_n will have the neighbourhood sum $S(v_1) = S(u_1) = S(u_n) = 2$.
The vertex v_m will have the neighbourhood sum $S(v_m) = 3$.
The vertex v_{m-1} will have the neighbourhood sum $S(v_{m-1}) = 4$.
The vertices $\{v_2, ..., v_{m-2}\}$ will have the neighbourhood sum

$$S(v_j) = \begin{cases} 5 & \text{if} \quad j = \text{even} \\ 3 & \text{if} \quad j = \text{odd} \end{cases} \tag{2.7}$$

where $2 \leq j \leq m–2$.
The vertices $\{u_2, ..., u_{n-1}\}$ will have the neighbourhood sum

$$S(u_j) = \begin{cases} 5 & \text{if} \quad j = \text{even} \\ 3 & \text{if} \quad j = \text{odd} \end{cases} \tag{2.8}$$

where $2 \leq j \leq n–1$.
It is evident that no two adjacent vertices have equal neighbourhood sums.
Hence it satisfies the condition $S(u) \neq S(v)$ for every pair of adjacent vertices u and v.
Therefore bow graph admits Lucky labeling when m is even and n is odd.
An illustration is shown in Figure 5.

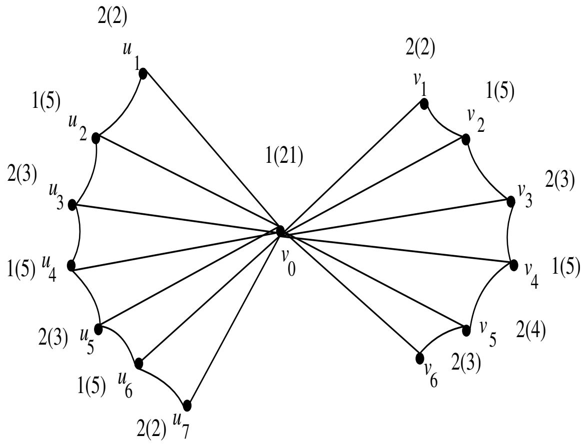

Figure 5: Lucky labeling of $B_{6,7}$

Case 2: When m is odd and n is even

We define the vertex labels as follows

$f : V(B_{m,n}) \to N$.

The apex vertex v_0 is labeled as $f(v_0) = 1$

$$f(v_i) = \begin{cases} 2 & \text{if} \quad i = \text{odd} \\ 1 & \text{if} \quad i = \text{even} \end{cases} \tag{2.9}$$

in the right shell where $1 \leq i \leq m, m \geq 5$.

$$f(u_i) = \begin{cases} 2 & \text{if} \quad i = \text{odd} \\ 1 & \text{if} \quad i = \text{even} \end{cases} \tag{2.10}$$

in the left shell where $1 \leq i \leq n–1, n \geq 6$.

Without the loss of generality $f(u_n)$ is labeled as 2.

Let $S(v)$ denote the sum of the labels of the neighbours of the vertex v in $B_{m,n}$.

The vertex v_0 will have the neighbourhood sum $S(v_0) = 3n - 3(((n - m) - 1)/2)$.

The vertices v_1, u_1 and vm will have the neighbourhood sum $S(v_1) = S(u_1) = S(v_m) = 2$.

The vertex u_n will have the neighbourhood sum $S(u_n) = 3$.

The vertex u_{n-1} will have the neighbourhood sum $S(u_{n-1}) = 4$.

The vertices $\{v_2, ..., v_{m-1}\}$ will have the neighbourhood sum

$$S(v_j) = \begin{cases} 5 & \text{if} \quad j = \text{even} \\ 3 & \text{if} \quad j = \text{odd} \end{cases} \tag{2.11}$$

where $2 \leq j \leq m–1$.
The vertices $\{u_2, ..., u_{n-2}\}$ will have the neighbourhood sum

$$S(u_j) = \begin{cases} 5 & \text{if} \quad j = \text{even} \\ 3 & \text{if} \quad j = \text{odd} \end{cases} \tag{2.12}$$

where $2 \leq j \leq n–2$.
It is evident that no two adjacent vertices have equal neighbourhood sums.
Hence it satisfies the condition $S(u) \neq S(v)$ for every pair of adjacent vertices u and v.
Therefore bow graph admits Lucky labeling when m is odd and n is even.
An illustration is shown in Figure 6.

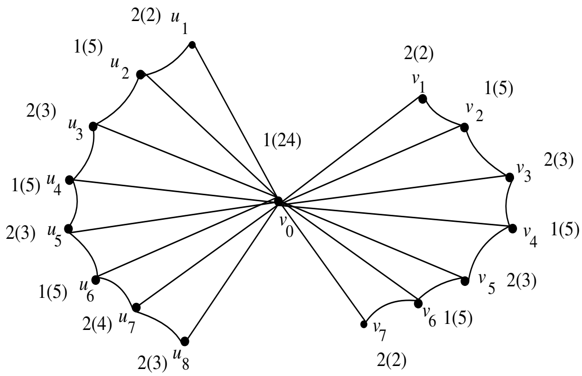

Figure 6: Lucky labeling of $B_{7,8}$

Case 3: When m and n is even but $m \neq n$
We define the vertex labels as follows
$f : V(B_{m,n}) \to N$ The apex vertex v_0 is labeled as $f(v_0) = 1$.

$$f(v_i) = \begin{cases} 2 & \text{if} \quad i = \text{odd} \\ 1 & \text{if} \quad i = \text{even} \end{cases} \tag{2.13}$$

in the right shell where $1 \leq i \leq m–1, m \geq 4$.
Without the loss of generality $f(v_m)$ is labeled as 2.

$$f(u_i) = \begin{cases} 2 & \text{if} \quad i = \text{odd} \\ 1 & \text{if} \quad i = \text{even} \end{cases} \tag{2.14}$$

in the left shell where $1 \leq i \leq n$–$1, n \geq 4$.
Without the loss of generality $f(u_n)$ is labeled as 2.
Let $S(v)$ denote the sum of the labels of the neighbours of the vertex v in $B_{m,n}$.
The vertex v_0 will have the neighbourhood sum $S(v_0) = 3n-(1+3((n-m)-2)/2)$.
The vertices v_1, u_1 will have the neighbourhood sum $S(v_1) = S(u_1) = 2$.
The vertices v_m, u_n will have the neighbourhood sum $S(v_m) = S(u_n) = 3$.
The vertices v_{m-1}, u_{n-1} will have the neighbourhood sum $S(v_{m-1}) = S(u_{n-1}) = 4$.
The vertices $\{v2, ..., v_{m-2}\}$ will have the neighbourhood sum.

$$S(v_j) = \begin{cases} 5 & \text{if} \quad j = \text{even} \\ 3 & \text{if} \quad j = \text{odd} \end{cases} \tag{2.15}$$

where $2 \leq j \leq m$–2.
The vertices $\{u_2, ..., u_{n-2}\}$ will have the neighbourhood sum.

$$S(u_j) = \begin{cases} 5 & \text{if} \quad j = \text{even} \\ 3 & \text{if} \quad j = \text{odd} \end{cases} \tag{2.16}$$

where $2 \leq j \leq n$–2.
It is evident that no two adjacent vertices have equal neighbourhood sums.
Hence it satisfies the condition $S(u) \neq S(v)$ for every pair of adjacent vertices u and v.
Therefore bow graph admits Lucky labeling when m and n is even but $m \neq n$.
An illustration is shown in Figure 7

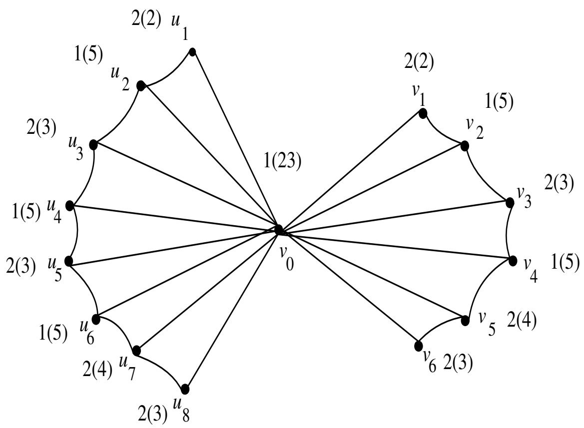

Figure 7: Lucky labeling of $B_{6,8}$

Case 4: When m and n is odd but $m \neq n$
We define the vertex labels as follows
$f : V(Bm, n) \rightarrow N$
The apex vertex v_0 is labeled as $f(v_0) = 1$.

$$f(v_i) = \begin{cases} 2 & \text{if} \quad i = \text{odd} \\ 1 & \text{if} \quad i = \text{even} \end{cases} \tag{2.17}$$

in the right shell where $1 \leq i \leq m–1, m \geq 5$.

$$f(u_i) = \begin{cases} 2 & \text{if} \quad i = \text{odd} \\ 1 & \text{if} \quad i = \text{even} \end{cases} \tag{2.18}$$

in the left shell where $1 \leq i \leq n–1, n \geq 5$.
Let $S(v)$ denote the sum of the labels of the neighbours of the vertex v in $B_{m,n}$.
The vertex v_0 will have the neighbourhood sum $S(v_0) = (3n-(2+3((n-m)-2)/2)$.
The vertices v_1, v_m, u_1, u_n will have the neighbourhood sum
$S(v_1) = S(v_m) = S(u_1) = S(u_n) = 2$.
The vertices $\{v_2, \ldots, v_{m–1}\}$ will have the neighbourhood sum

$$S(v_j) = \begin{cases} 5 & \text{if} j = \text{even} \\ 3 & \text{if} j = \text{odd} \end{cases} \tag{2.19}$$

where $2 \leq j \leq m–1$.
The vertices $\{u_2, ..., u_{n-1}\}$ will have the neighbourhood sum

$$S(u_j) = \begin{cases} 5 & \text{if} \quad j = \text{even} \\ 3 & \text{if} \quad j = \text{odd} \end{cases} \tag{2.20}$$

where $2 \leq j \leq n–1$.
It is evident that no two adjacent vertices have equal neighbourhood sums.
Hence it satisfies the condition $S(u) \neq S(v)$ for every pair of adjacent vertices u and v.
Therefore bow graph admits Lucky labeling when m and n is odd but $m \neq n$.
An illustration is shown in Figure 8
Thus bow graph admits Lucky labeling with $\eta(B_{m,n}) = 2$.

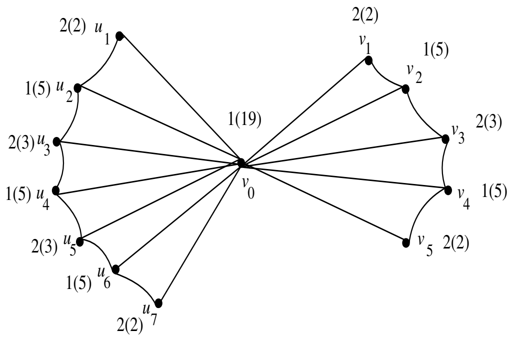

Figure 8: Lucky labeling of $B_{5,7}$

Theorem 3. *The Wheel graph W_n admits Lucky labeling.*
Proof. Let W_n be the wheel graph where n denotes the number of vertices in the wheel graph. The total number of vertices and edges in the wheel graph is given by $|V(Wn)| = |n|$ and $|E(Wn)| = |2(n–1)|$
The vertices of the wheel graph are denoted as $\{v_1, v_2, ..., v_n\}$ in anticlockwise direction where v_1 is the single vertex as shown in the Figure 9.

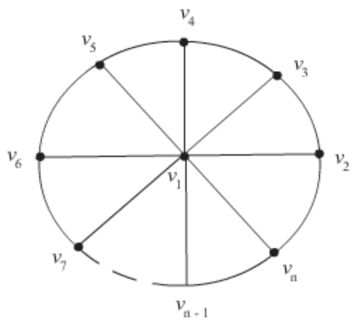

Figure 9: Wheel graph W_n

Case 1: When n is even
We define the vertex labels as follows
$f : V(W_n) \to N$

$$f(v_i) = \begin{cases} 1 & \text{if} \quad i \equiv 1 \pmod 4 \\ 2 & \text{if} \quad i \equiv 2 \pmod 4 \\ 3 & \text{if} \quad i \equiv 3 \pmod 4 \\ 4 & \text{if} \quad i \equiv 0 \pmod 4 \end{cases} \tag{2.21}$$

where $1 \leq i \leq n, n \geq 4$
Let $S(v)$ denote the sum of the labels of the neighbours of the vertex v in W_n .
For W_n where $n = 4i$, $1 \leq i \leq n$.
The vertex v_1 will have the neighbourhood sum $S(v_1) = 10i - 1$.
The vertex v_2 will have the neighbourhood sum $S(v_2) = 8$.
The vertex v_n will have the neighbourhood sum $S(v_n) = 6$.
For W_n where $n = 4i + 2$, $1 \leq i \leq n$.
The vertex v_1 will have the neighbourhood sum $S(v_1) = 10i + 2$.
The vertex v_2 will have the neighbourhood sum $S(v_2) = 6$.
The vertex v_n will have the neighbourhood sum $S(v_n) = 4$.
For W_n when n is even.
The vertices $\{v_3, ..., v_{n-1}\}$ will have the neighbourhood sum.

$$S(v_j) = \begin{cases} 7 & \text{if} \quad j = \text{odd} \\ 5 & \text{if} \quad j = \text{even} \end{cases} \tag{2.22}$$

where $3 \leq j \leq n$–1.
It is evident that no two adjacent vertices have equal neighbourhood sums.
Hence it satisfies the condition $S(u) \neq S(v)$ for every pair of adjacent vertices u and v.
Therefore wheel graph admits Lucky labeling when n is even.
An illustration is shown in Figure 10

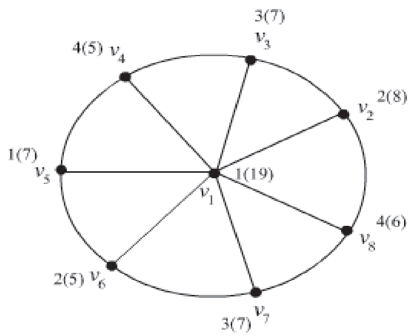

Figure 10: Lucky labeling of W_8

Case 2: When n is odd
We define the vertex labels as follows

$f : V(S_n) \to N$. For W_n where $n = 4i + 1$, $1 \le i \le n$

$$f(v_i) = \begin{cases} 1 & \text{if} \quad i \equiv 1 \pmod 4 \\ 2 & \text{if} \quad i \equiv 2 \pmod 4 \\ 3 & \text{if} \quad i \equiv 3 \pmod 4 \\ 4 & \text{if} \quad i \equiv 0 \pmod 4 \end{cases} \tag{2.23}$$

where $1 \le i \le n, n \ge 5$.
For W_n where $n = 4i + 3$, $1 \le i \le n$

$$f(v_i) = \begin{cases} 1 & \text{if} \quad i \equiv 1 \pmod 4 \\ 2 & \text{if} \quad i \equiv 2 \pmod 4 \\ 3 & \text{if} \quad i \equiv 3 \pmod 4 \\ 4 & \text{if} \quad i \equiv 0 \pmod 4 \end{cases} \tag{2.24}$$

$f(v_n) = 2$, where $1 \le i \le n, n \ge 7$.
Let $S(v)$ denote the sum of the labels of the neighbours of the vertex v in W_n .
For W_n where $n = 4i + 1$, $1 \le i \le n$.
The vertex v_1 will have the neighbourhood sum $S(v_1) = 10i$.
The vertices $\{v_2, ..., v_n\}$ will have the neighbourhood sum

$$S(v_j) = \begin{cases} 7 & \text{if} \quad j = \text{odd} \\ 5 & \text{if} \quad j = \text{even} \end{cases} \tag{2.25}$$

where $2 \le j \le n$.
For W_n where $n = 4i + 3$, $1 \le i \le n$.
The vertex v_1 will have the neighbourhood sum $S(v_1) = 10i + 4$.
The vertex v_2 will have the neighbourhood sum $S(v_2) = 6$.
The vertex v_n will have the neighbourhood sum $S(v_n) = 4$.
The vertices $\{v_3, ..., v_{n-1}\}$ will have the neighbourhood sum

$$S(v_j) = \begin{cases} 7 & \text{if} \quad j = \text{odd} \\ 5 & \text{if} \quad j = \text{even} \end{cases} \tag{2.26}$$

where $3 \le j \le n$–1.
It is evident that no two adjacent vertices have equal neighbourhood sums.
Hence it satisfies the condition $S(u) \ne S(v)$ for every pair of adjacent vertices u and v. Therefore wheel graph admits Lucky labeling when n is odd.
An illustration is shown in Figure 11.

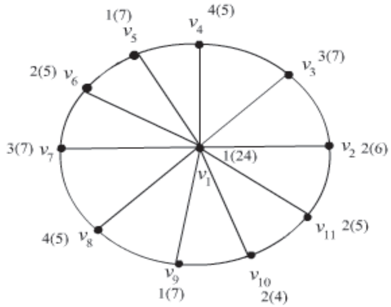

Figure 11: Lucky labeling of W_{11}

Thus wheel graph admits Lucky labeling with Lucky number $\eta(W_n) = 4$.

3. Conclusion

In this paper we have proved that shell graph, bow graph and wheel graph admits Lucky labeling and we have computed the lucky number $\eta(G)$. Further we intend to prove the family of shell graphs admits Proper Lucky labeling.

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