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INTEGER CORDIAL LABELING OF SOME STAR AND BISTAR RELATED GRAPHS

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Abstract: An integer cordial labeling of a graph $G^*(p,q)$ is an injective map $g : V \to \left[\frac{-p}{2}, \dots, \frac{p}{2}\right]^*$ or $\left[-\left\lfloor\frac{p}{2}\right\rfloor, \dots, \left\lfloor\frac{p}{2}\right\rfloor\right]$ as p is even or odd, which induces an edge labeling $g : E \to \{0, 1\}$ defined by

$$g(uv) = \begin{cases} 1, & g(u) + g(v) \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

such that the number of edges labeled 1 and the number of edges labeled 0 differ by at most 1. If a graph has integer cordial labeling (I.C.L.), then it is called integer cordial graph (I.C.G.). In this paper, we investigate the existence of integer cordial Labeling of Star and Bistar related graphs.

Keywords and Phrases: Integer Cordial Labeling, Integer Cordial Graph, Shadow graph, Splitting of a Graph, Degree Splitting of a Graph.

2020 Mathematics Subject Classification: 05C78.

1. Introduction

Now in these days, Graph Theory and Graph Labeling act as essential tool in Data Science and Computer Engineering. It is very useful to assign networks communication, flow of computation and used to represent data organization. Here, we have investigated some important results on Integer cordial labeling which can be used as a tool in the mentioned fields. In [6], Nicholas et al. introduced the concept of integer cordial labeling of graphs and proved that some standard graphs such as Path P_n, Star graph $K_{1,n}$, Wheel graph W_n; $n \geq 3$, Cycle C_n, Helm graph H_n and Closed helm graph CH_n are integer cordial. K_n is not integer cordial whereas, $K_{n,n}$ is integer cordial iff n is even and $K_{n,n} \backslash M$ is integer cordial for any n, where M is perfect matching of $K_{n,n}$.

2. Definitions

Definition 2.1. *A graph $G^*(p,q)$ is said to have an integer cordial labeling if there exists an injective map g from V to $[\frac{-p}{2}, \dots, \frac{p}{2}]^*$ or $[-\lfloor \frac{p}{2} \rfloor, \dots, \lfloor \frac{p}{2} \rfloor]$ as p is even or odd, which induces an edge labeling, $g : E \to \{0, 1\}$ defined by*

$$g(uv) = \begin{cases} 1, & g(u) + g(v) \geq 0 \\ 0, & otherwise \end{cases}$$

such that the number of edges labeled 1 and the number of edges labeled 0 differ by at most 1. If a graph G^* admits integer cordial labeling, then the graph is called integer cordial graph [6]. *In general, $[-x, \dots, x] = \{y/(y\ )is\ an\ integer\ and\ |y| \leq x\}$ and $[-x, x]^* = \{y/(y\ )\ is\ an\ integer\ and\ |y| \leq x - \{0\}\}$* [6].

Definition 2.2. *For a graph G^* the splitting graph $S'(G^*)$ of a graph G^* is obtained by adding a new vertex v' corresponding to each vertex v of G^* such that $N(v) = N(v')$.*

Definition 2.3. *Let $G^* = (V(G^*), E(G^*))$ be a graph with $V = S_1 \cup S_2 \cup S_3 \cup \dots S_i \cup T$ where each S_i is a set of vertices having at least two vertices of the same degree and $T = V \backslash \cup S_i$. The degree splitting graph of G^* denoted by $DS(G^*)$ is obtained from G^* by adding vertices w_1, w_2, w_3,..., w_t and joining to each vertex of S_i for $1 \leq i \leq t$.*

Definition 2.4. *The shadow graph $D_2(G^*)$ of a connected graph G^* is constructed by taking two copies of G^* say G' and G''. Join each vertex u' in G' to the neighbours of the corresponding vertex v' in G''.*

Definition 2.5. *For a simple connected graph G^* the square of graph G^* is denoted by G^2 and defined as the graph with the same vertex set as of G^* and two vertices are adjacent in G^2 if they are at a distance 1 or 2 apart in G^*.*

3. Main Results

Theorem 3.1. *$D_2(K_{1,n})$ is an integer cordial graph.*
Proof. Consider two copies of $K_{1,n}$. Let $u, u_1, u_2, \dots, u_n$ be the vertices of the first copy of $K_{1,n}$ and $v, v_1, v_2, \dots, v_n$ be the vertices of the second copy of $K_{1,n}$ where

u and v are the apex vertices respectively.
Let G^* be $D_2(K_{1,n})$.
We define $g : V \to [-\lfloor \frac{p}{2} \rfloor, \ldots, \lfloor \frac{p}{2} \rfloor]$ as follows:
$g(u) = 1$,
$g(u_i) = i + 1$, $1 \leq i \leq n$.
$g(v) = -1$,
$g(v_i) = -(i + 1)$, $1 \leq i \leq n$.
Therefore $|e_g(0) - e_g(1)| \leq 1$.
Hence, $D_2(K_{1,n})$ is an integer cordial graph.

Example 3.1. Integer cordial labeling of $D_2(K_{1,4})$ is shown in **Figure 1**.

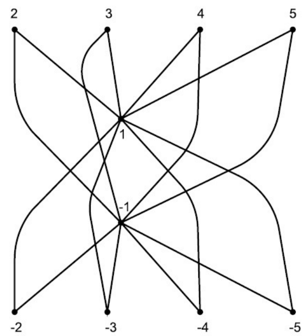

Figure 1

Theorem 3.2. *$D_2(B_{n,n})$ is an integer cordial graph.*
Proof. Consider two copies of $B_{n,n}$. Let $\{u, v, u_i, v_i, 1 \leq i \leq n\}$ and $\{u', v', u'_i, v'_i, 1 \leq i \leq n\}$ be the corresponding vertex sets of each copy of $B_{n,n}$. Let G^* be the graph $D_2(B_{n,n})$. Then $|V(G^*)| = 4n + 4$ and $|E(G^*)| = 8n + 4$.
We define $g : V \to [-\lfloor \frac{p}{2} \rfloor, \ldots, \lfloor \frac{p}{2} \rfloor]$ as follows:
$g(u) = 2$,
$g(u_i) = n + i + 2$, $1 \leq i \leq n$.
$g(u') = 1$,
$g(u'_i) = i + 2$, $1 \leq i \leq n$.
$g(v) = -1$,
$g(v_i) = -(n + i + 2)$, $1 \leq i \leq n$.
$g(v') = -3$,
$g(v'_1) = -2$,
$g(v'_i) = -(i + 2)$, $2 \leq i \leq n$.

Therefore $|e_g(0) - e_g(1)| \leq 1$.
Hence, $D_2(B_{n,n})$ is an integer cordial graph.

Example 3.2. Integer cordial labeling of $D_2(B_{4,4})$ is shown in **Figure 2**.

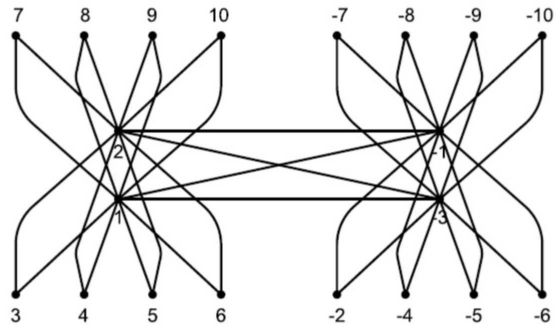

Figure 2

Theorem 3.3. *$DS(K_{1,n})$ is an integer cordial graph.*
Proof. Let $u, u_1, u_2, u_3, \ldots, u_n$ be the vertices of $K_{1,n}$. Introduce a new vertex v and join to the vertices of star graph $K_{1,n}$ of degree one. Then the resultant graph is $DS(K_{1,n})$ whose vertex set is $V = \{u, u_i/1 \leq i \leq n\} \cup \{v\}$ and edge set is $E = \{uu_i/1 \leq i \leq n\} \cup \{u_iv/1 \leq i \leq n\}$. Then $|V(DS(K_{1,n}))| = n + 2$ and $|E(DS(K_{1,n}))| = 2n$.
We define $g : V \to [-\lfloor \frac{p}{2} \rfloor, \ldots, \lfloor \frac{p}{2} \rfloor]$ as follows:
Case 1: when n is even
g(u)=1,
g(v) = -1,

$$g(u_i) = \begin{cases} (\frac{i+3}{2}) & \text{i is odd; } 1 \leq i \leq n \\ -i & \text{i is even; } 1 \leq i \leq n. \end{cases}$$

Case 2: when n is odd
$g(u) = \lceil \frac{n}{2} + 1 \rceil$,
$g(v) = -(\lceil \frac{n}{2} \rceil + 1)$,

$$g(u_i) = \begin{cases} (\frac{i+1}{2}) & \text{i is odd; } 1 \leq i \leq n \\ -(\frac{i}{2}) & \text{i is even; } 1 \leq i \leq n. \end{cases}$$

Therefore, $|e_g(0) - e_g(1)| \leq 1$.
Hence $DS(K_{1,n})$ is an integer cordial graph.

Example 3.3. Integer cordial labeling of $DS(K_{1,5})$ is shown in **Figure 3**.

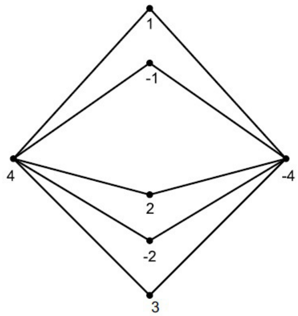

Figure 3

Theorem 3.4. *$DS(B_{n,n})$ is an integer cordial graph.*

Proof. Consider $B_{n,n}$ with $V(B_{n,n}) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$, where u_i, v_i are pendant vertices. Here $V(B_{n,n}) = V_1 \cup V_2$, where $V_1 = \{u_i, v_i : 1 \leq i \leq n\}$ and $V_2 = \{u, v\}$. Now in order to obtain $DS(B_{n,n})$ from G, we add w_1, w_2 corresponding to V_1 and V_2. Then $|V(DS(B_{n,n}))| = 2n+4$ and $|E(DS(B_{n,n}))| = \{uv, uw_2, vw_2\} \cup \{uv_i, vv_i, w_1u_i, w_1v_i : 1 \leq i \leq n\}$. So, $|E(DS(B_{n,n}))| = 4n + 3$.
We define $g : V \rightarrow [-\lfloor \frac{p}{2} \rfloor, \ldots, \lfloor \frac{p}{2} \rfloor]$ as follows:
$g(u) = 2$.
$g(v) = -2$.
$g(w_1) = 1$.
$g(w_2) = -1$.
$g(u_i) = (i + 2)$, $1 \leq i \leq n$.
$g(v_i) = -(i + 2)$, $1 \leq i \leq n$.
Therefore, $|e_g(0) - e_g(1)| \leq 1$.
Hence $DS(B_{n,n})$ is an integer cordial graph.

Example 3.4. Integer cordial labeling of $DS(B_{5,5})$ is shown in **Figure 4**.

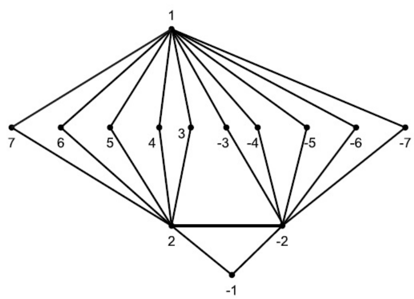

Figure 4

Theorem 3.5. *$S'(B_{n,n})$ is an integer cordial graph.*
Proof. Consider $B_{n,n}$ with vertex set $\{u, v, u_i, v_i, 1 \leq i \leq n\}$, where u_i, v_i are pendant vertices. In order to obtain $S'(B_{n,n})$, add u', v', u'_i, v'_i vertices corresponding to u, v, u_i, v_i, where $1 \leq i \leq n$. If $G^* = S'(B_{n,n})$ then $|V(G^*)| = 4n + 4$ and $|E(G^*)| = 6n + 3$.
We define $g : V \to [-\lfloor \frac{p}{2} \rfloor, \ldots, \lfloor \frac{p}{2} \rfloor]$ as follows:
$g(u) = 1$,
$g(u_i) = (n + i + 2)$, $1 \leq i \leq n$.
$g(u') = 2$,
$g(u'_i) = i + 2$, $1 \leq i \leq n$.
$g(v) = -1$, $g(v_i) = -(n + i + 2)$, $1 \leq i \leq n$.
$g(v') = -2$, $g(v'_i) = -(i + 2)$, $1 \leq i \leq n$.
Therefore $|e_g(0) - e_g(1)| \leq 1$.
Hence, $S'(B_{n,n})$ is an integer cordial.

Example 3.5. Integer cordial labeling of $S'(B_{6,6})$ is shown in **Figure 5**.

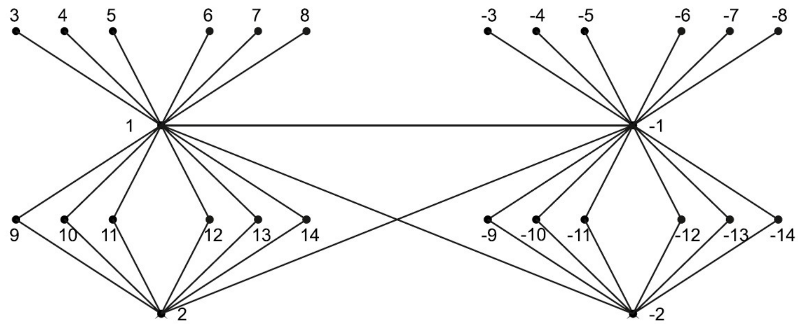

Figure 5

Theorem 3.6. *$S'(K_{1,n})$ is an integer cordial graph.*
Proof. Let $v_1, v_2, v_3, \ldots, v_n$ be the pendant vertices and v be the apex vertex of $K_{1,n}$ and $u, u_1, u_2, u_3, \ldots, u_n$ are added vertices corresponding to $v, v_1, v_2, v_3, \ldots, v_n$ to obtain $S'(K_{1,n})$. Let G^* be the graph $S'(K_{1,n})$ then $|V(G^*)| = 2n + 2$ and $|E(G^*)| = 3n$.
We define $g : V \to [-\lfloor \frac{p}{2} \rfloor, \ldots, \lfloor \frac{p}{2} \rfloor]$ as follows:
Case 1: when n is even
$g(v) = 1$,

$$g(v_i) = \begin{cases} -(\frac{i+2}{2}) & \text{i is even; } 1 \leq i \leq n \\ (\frac{i+3}{2}) & \text{i is odd; } 1 \leq i \leq n. \end{cases}$$

$g(u) = -1,$

$$g(u_i) = \begin{cases} -(\frac{n+i+2}{2}) & \text{i is even; } 1 \leq i \leq n \\ (\frac{n+i+3}{2}) & \text{i is odd; } 1 \leq i \leq n. \end{cases}$$

Case 2: when n is odd

$g(v) = 1,$

$$g(v_i) = \begin{cases} -(\frac{i+2}{2}) & \text{i is even; } 1 \leq i \leq n \\ (\frac{i+3}{2}) & \text{i is odd; } 1 \leq i \leq n. \end{cases}$$

$g(u) = -1,$

$$g(u_i) = \begin{cases} (\frac{n+i+3}{2}) & \text{i is even; } 1 \leq i \leq n \\ -(\frac{n+i+2}{2}) & \text{i is odd; } 1 \leq i \leq n. \end{cases}$$

Thus, in all cases we have $|e_g(0) - e_g(1)| \leq 1$.
Hence $S'(K_{1,n})$ is an integer cordial graph.

Example 3.6. Integer cordial labeling of $S'(K_{1,7})$ is shown in **Figure 6**.

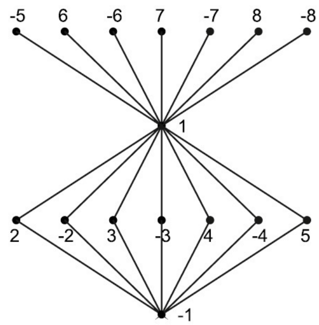

Figure 6

Theorem 3.7. $B^2_{4,4}$ *is an integer cordial graph.*
Proof. Consider $B_{n,n}$ with vertex set $\{u, v, u_i, v_i/1 \leq i \leq n\}$ where u_i, v_i are pendant vertices. Let G^* be the graph $B^2_{n,n}$ then $|V(G^*)| = 2n + 2$ and $|E(G^*)| = 4n + 1$.
We define $g : V \to [-\lfloor \frac{p}{2} \rfloor, \ldots, \lfloor \frac{p}{2} \rfloor]$ as follows:
$g(u) = 1,$
$g(u_i) = i + 1,\ 1 \leq i \leq n.$
$g(v) = -1,\ g(v_i) = -(i + 1),\ 1 \leq i \leq n.$
Therefore $|e_g(0) - e_g(1)| \leq 1$.
Hence, $B^2_{n,n}$ is an integer cordial graph.

Example 3.7. Integer cordial labeling of $B^2_{4,4}$ is shown in **Figure 7**.

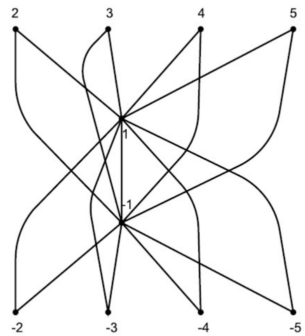

Figure 7

4. Conclusion

It is very interesting to investigate graph or graph families which admit integer cordial labeling. Here, We have investigated seven new graphs related to star and bistar graphs which admit integer cordial labeling. It will add new horizon to the research work in the area tethering two branches - labeling of graphs and number theory.

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