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THE COMPLETE PRODUCT OF TWO FUZZY GRAPHS AND ITS RELATIONSHIP WITH FUZZY GRAPH ISOMORPHISM

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Abstract: Fuzzy graph was introduced by Kaufmann [7] in 1973. In this paper, we introduced the concept of the complete product of two fuzzy graphs with an Illustrative example. We proved the result that If $G : (\sigma, \mu) = (U, E_U)$, $H : (\tau, \vartheta) = (V, E_V)$, $G' : (\sigma', \mu') = (U', E_{U'})$ and $H' : (\tau', \vartheta') = (V', E_{V'})$ are any four fuzzy graphs such that $G : (\sigma, \mu) \cong G' : (\sigma', \mu')$ and $H : (\tau, \vartheta) \cong H' : (\tau', \vartheta')$ under the fuzzy graph isomorphisms f and h respectively, then $G \times_P H \cong G' \times_P H'$. As the proof is too long, we have demonstrated the result in two by parting into two hypotheses.

Keywords and Phrases: Fuzzy relation, Fuzzy graph, Uniform vertex fuzzy graph, Fuzzy graph isomorphism, The complete product of two fuzzy graphs.

2020 Mathematics Subject Classification: 05C70, 05C72.

1. Introduction

The concept of a Graph is introduced for the first time by Leonhard Euler [4] in the year 1736. It is quite well known that graphs are simply models of relations. A graph is a convenient way of representing the information involving relationship between objects. The objects are represented by vertices and relations by edges. When there is vagueness in the description of the objects or in its relationships or in both, it is natural that we need to design a 'Fuzzy Graph Model'. Application of fuzzy relations are widespread and important in the field of clustering analysis, neural networks, computer networks, pattern recognition, decision making and expert systems. In each of these, the basic mathematical structure is that of a fuzzy graph. We know that a graph is a symmetric binary relation on a nonempty set V. Similarly, a fuzzy graph is a symmetric binary fuzzy relation on a fuzzy subset. The first definition of a fuzzy graph was by Kaufmann [7] in 1973, based on Zadeh's fuzzy relations [13]. But it was Azriel Rosenfeld [9] who considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs in 1975. Later on, so many Mathematicians produced various results on Fuzzy Graphs in various aspects.

In this paper, we introduced the concept of the complete product of two fuzzy graphs with an Illustrative example. We proved the result that If $G : (\sigma, \mu) = (U, E_U)$, $H : (\tau, \vartheta) = (V, E_V)$, $G' : (\sigma', \mu') = (U', E_{U'})$ and $H' : (\tau', \vartheta') = (V', E_{V'})$ are any four fuzzy graphs such that $G : (\sigma, \mu) \cong G' : (\sigma', \mu')$ and $H : (\tau, \vartheta) \cong H' : (\tau', \vartheta')$ under the fuzzy graph isomorphisms f and h respectively, then $G \times_P H \cong G' \times_P H'$. As the proof is too long, we have demonstrated the result in two by parting into two hypotheses.

2. Preliminaries

In this section we have given the definitions and examples related to the concepts which are discussed in the introduction.

Definition 2.1. [5] *A graph G is a triplet consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates two vertices with each edge. The two vertices called its endpoints (not necessarily distinct). Graphically, we represent a graph by drawing a point for each vertex and representing each edge by a curve joining its endpoints.*

Definition 2.2. [9, 13] *Let S be a non-empty set. Let $\mu : S \times S \to [0, 1]$ and $\sigma : S \to [0, 1]$ be any two fuzzy sets. Then μ is said to be a fuzzy relation on σ if $\mu(x, y) \leq \sigma(x) \cap \sigma(y) \forall x, y \in S$.*
Note: $\sigma(x) \cap \sigma(y) = \min\{\sigma(x), \sigma(y)\}$.

Definition 2.3. [9, 13] *(i) A fuzzy relation μ on a fuzzy set σ is called symmetric if $\mu(x, y) = \mu(y, x), \forall x, y \in S$. (ii) A fuzzy relation μ on a fuzzy set σ is called reflexive if $\mu(x, x) = \sigma(x), \forall x \in S$.*

Definition 2.4. [9, 13] *A fuzzy graph G is a pair of functions $G : (\sigma, \mu)$ where σ*

is a fuzzy subset on a non-empty set V and σ is a symmetric fuzzy relation on σ. The underlying crisp graph of $G : (\sigma, \mu)$ is denoted by $G*(V, E)$ where $E \subset V \times V$.

Example 2.5. Here we have given an example for fuzzy graph.

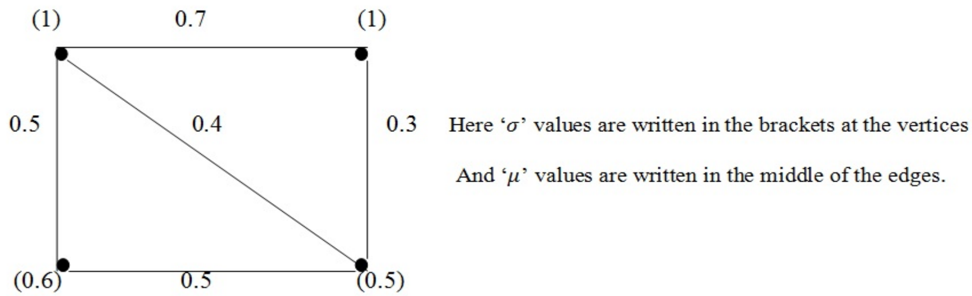

Figure 1: Fuzzy graph

Note 2.7: In any fuzzy graph $G : (\sigma, \mu)$ on $G*(V, E)$, we assume that V is finite and μ is reflexive on σ. Also, we ignore the loops and parallel edges.

Definition 2.8. [3] *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G*(V, E)$. Let $\sigma* = \{x \in V/\sigma(x) > 0\}$. Then G is said to be a Uniform vertex fuzzy graph if $\sigma(x) = k \forall x \in \sigma*$ where k is some positive real such that $0 \leq k \leq 1$.*

Example 2.9. Here we have given an example for Uniform vertex fuzzy graph.

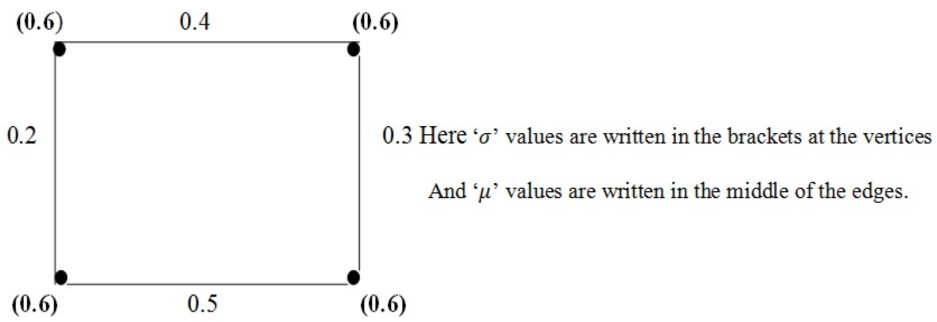

Figure 2: Uniform vertex fuzzy graph

3. The Complete Product of Two Fuzzy Graphs

Definition 3.1. *Let $G : (\sigma, \mu)$ and $H : (\tau, \vartheta)$ be any two fuzzy graphs with $G* : (U, E_U)$ and $H* : (V : E_V)$. Then the Complete product of G and H is defined as a fuzzy graph*

$G \times_P H : (\sigma \times_P \tau, \mu \times_P \vartheta) = (U \times V, E)$

where $E = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7 \cup E_8$.
Such that
$E_1 = \{((u_1, v_1), (u_2, v_2)) : u_1 = u_2 \text{ and } (v_1, v_2) \in E_V\}$ *and*
$E_2 = \{((u_1, v_1), (u_2, v_2)) : u_1 = u_2 \text{ and } (v_1, v_2) \notin E_V\}$ *and*
$E_3 = \{((u_1, v_1), (u_2, v_2)) : v_1 = v_2 \text{ and } (u_1, u_2) \in E_U\}$ *and*
$E_4 = \{((u_1, v_1), (u_2, v_2)) : v_1 = v_2 \text{ and } (u_1, u_2) \notin E_U\}$ *and*
$E_5 = \{((u_1, v_1), (u_2, v_2)) : (u_1, u_2) \in E_U \text{ and } (v_1, v_2) \notin E_V\}$ *and*
$E_6 = \{((u_1, v_1), (u_2, v_2)) : (u_1, u_2) \notin E_U \text{ and } (v_1, v_2) \in E_V\}$ *and*
$E_7 = \{((u_1, v_1), (u_2, v_2)) : (u_1, u_2) \in E_U \text{ and } (v_1, v_2) \in E_V\}$
$E_8 = \{((u_1, v_1), (u_2, v_2)) : (u_1, u_2) \notin E_U \text{ and } (v_1, v_2) \notin E_V\}$
With $\sigma \times_P \tau(u, v) = \sigma(u) \wedge \tau(v)\ \forall (u, v) \in U \times V$ *and*

$$(\mu \times_P \vartheta)(w) = \begin{cases} \sigma(u_1) \wedge \vartheta(v_1, v_2), & \text{if } w \in E_1 \\ \sigma(u_1) \wedge \tau(v_1) \wedge \tau(v_2), & \text{if } w \in E_2 \\ \mu(u_1, u_2) \wedge \tau(v_1), & \text{if } w \in E_3 \\ \sigma(u_1) \wedge \sigma(u_2) \wedge \tau(v_1), & \text{if } w \in E_4 \\ \mu(u_1, u_2) \wedge \tau(v_1) \wedge \tau(v_2), & \text{if } w \in E_5 \\ \sigma(u_1) \wedge \sigma(u_2) \wedge \vartheta(v_1, v_2), & \text{if } w \in E_6 \\ \mu(u_1, u_2) \wedge \vartheta(v_1, v_2), & \text{if } w \in E_7 \\ \sigma(u_1) \wedge \sigma(u_2) \wedge \tau(v_1) \wedge \tau(v_2), & \text{if } w \in E_8 \end{cases}$$

where $w = ((u_1, v_1), (u_2, v_2))$.

Example 3.2. Consider the following two fuzzy graphs

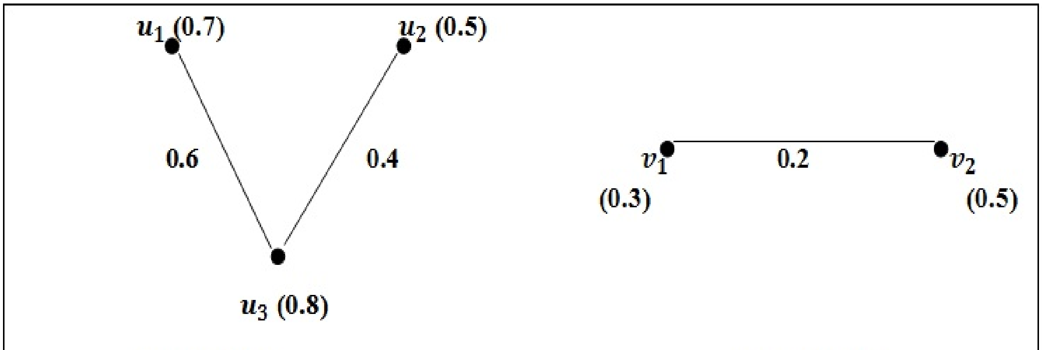

Figure 3: G Figure 4: H

where $w_1 = (u_1, v_1) = 0.5$, $w_2 = (u_1, v_2) = 0.3$, $w_3 = (u_2, v_1) = 0.5$, $w_4 = (u_2, v_2) = 0.3$, $w_5 = (u_3, v_1) = 0.5$, $w_6 = (u_3, v_2) = 0.3$.

Note. The crisp graph $(G \times_P H))^*$ of a complete product of two fuzzy graphs G and H is always a complete graph as it contains an edge between every pair of vertices.

Now we are going to prove some important properties of $G \times_P H$.

The following diagram represents the complete product of the above two fuzzy graphs G and H.

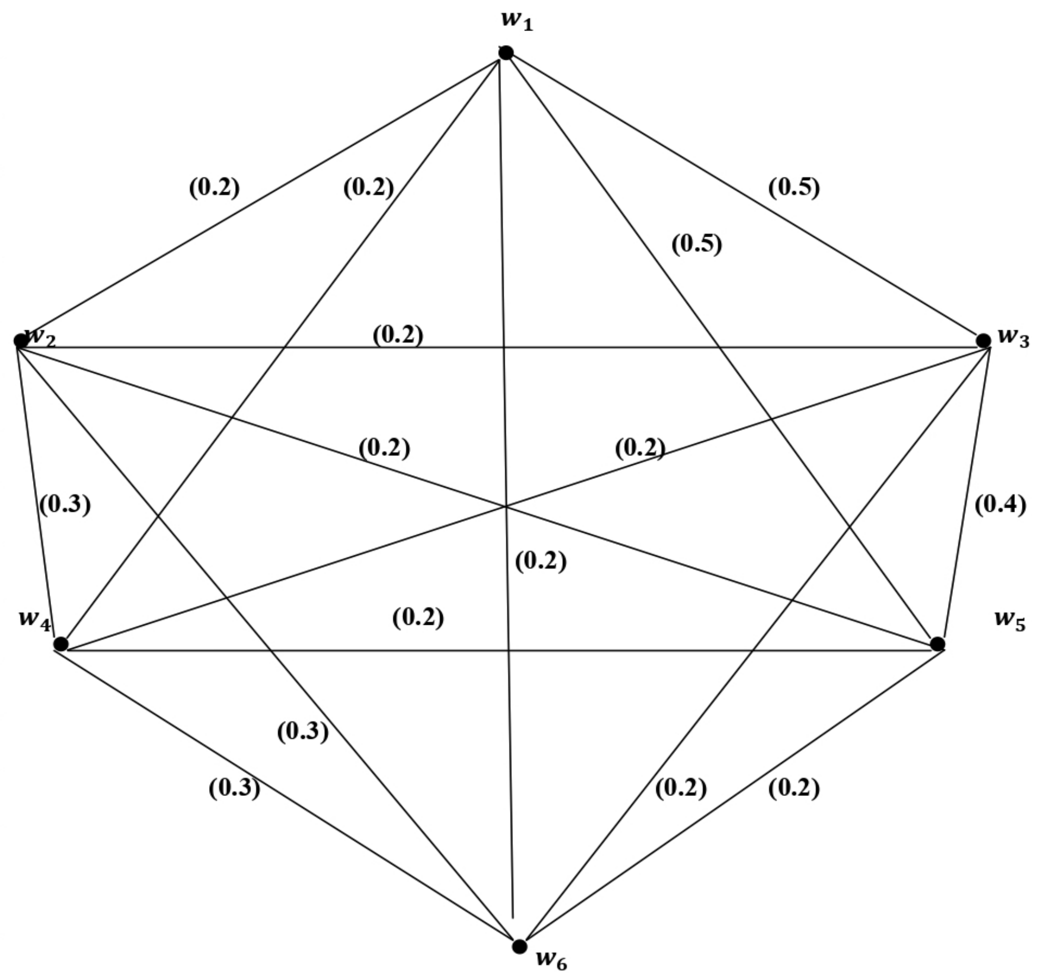

Figure 5: $\mathbf{G \times_P H}$

Theorem 3.3. *Let $G : (\sigma, \mu) = (U, E_U)$, $H : (\tau, \vartheta) = (V, E_V)$, $G' : (\sigma'\mu') = (U', E_{U'})$ and $H' : (\tau'\vartheta') = (V', E_{V'})$ are any four fuzzy graphs such that $G : (\sigma, \mu) \cong G' : (\sigma', \mu')$ and $H : (\tau, \vartheta) \cong H' : (\tau', \vartheta')$ under the fuzzy graph isomorphisms f and h respectively.*
Let $E = E_1 \cup\ E_2 \cup\ E_3 \cup\ E_4 \cup\ E_5 \cup\ E_6 \cup E_7 \cup E_8$ and $E' = E_1' \cup\ E_2' \cup\ E_3' \cup\ E_4' \cup$

$E'_5 \cup$ E'_6 $\cup E'_7$ $\cup E'_8$ be the edge sets of $G \times_P H$ and $G' \times_P H'$ respectively. Then $w = ((u_1, v_1), (u_2, v_2)) \in E_i$ implies $w' = ((f(u_1), h(v_1)), (f(u_2), h(v_2))) \in E'_i$ $\forall$ $i = 1, 2, ..., 8$.

Proof.

Case I. Let $w = ((u_1, v_1), (u_2, v_2)) \in E_1$. Then $u_1 = u_2$ and $(v_1, v_2) \in E_V$. $\Rightarrow u_1 = u_2$ and $\vartheta(v_1, v_2) \neq 0$. $\Rightarrow f(u_1) = f(u_2)$ and $\vartheta'(h(v_1), h(v_2)) \neq 0$. (since $\vartheta'(h(v_1), h(v_2)) = \vartheta(v_1, v_2)) \Rightarrow f(u_1) = f(u_2)$ and $\vartheta'(h(v_1), h(v_2)) \in E_{V'} \Rightarrow w' = ((f(u_1), h(v_1)), (f(u_2), h(v_2))) \in E'_1$.

Case II. Let $w = ((u_1, v_1), (u_2, v_2)) \in E_2$. Then $u_1 = u_2$ and $(v_1, v_2) \notin E_V$. $\Rightarrow u_1 = u_2$ and $(v_1, v_2) = 0$. $\Rightarrow f(u_1) = f(u_2)$ and $\vartheta'(h(v_1), h(v_2)) = 0$. (since $\vartheta'(h(v_1), h(v_2)) = \vartheta(v_1, v_2)) \Rightarrow f(u_1) = f(u_2)$ and $\vartheta'(h(v_1), h(v_2)) \notin E_{V'}$. $\Rightarrow w' = ((f(u_1), h(v_1)), (f(u_2), h(v_2))) \in E'_2$.

Case III. Let $w = ((u_1, v_1), (u_2, v_2)) \in E_3$. Then $v_1 = v_2$ and $(u_1, u_2) \in E_U$. $\Rightarrow w' = ((f(u_1), h(v_1)), (f(u_2), h(v_2))) \in E'_3$ (The proof is similar to that of case I).

Case IV. Let $w = ((u_1, v_1), (u_2, v_2)) \in E_4$. Then $v_1 = v_2$ and $(u_1, u_2) \notin E_U$. $\Rightarrow w' = ((f(u_1), h(v_1)), (f(u_2), h(v_2))) \in E'_4$ (The proof is similar to that of case II).

Case V. Let $w = ((u_1, v_1), (u_2, v_2)) \in E_5$. Then $(u_1, u_2) \in E_U$ and $(v_1, v_2) \notin E_V$. $\Rightarrow \mu(u_1, u_2) \neq 0$ and $\vartheta(v_1, v_2) = 0$. $\Rightarrow \mu(u_1, u_2) = \mu'(f(u_1), f(u_2))$ and $\vartheta'(h(v_1), h(v_2)) = \vartheta(v_1, v_2) = 0$. $\Rightarrow (f(u_1), f(u_2)) \in E_{U'}$ and $(h(v_1), h(v_2)) \notin E_{V'}$. $\Rightarrow w' = ((f(u_1), h(v_1)), (f(u_2), h(v_2))) \in E'_5$.

Case VI. Let $w = ((u_1, v_1), (u_2, v_2)) \in E_6$. Then $(u_1, u_2) \notin E_U$ and $(v_1, v_2) \notin E_V$. So $w' = ((f(u_1), h(v_1)), (f(u_2), h(v_2))) \in E'_6$ (The proof is similar to that of case V).

Case VII. Let $w = ((u_1, v_1), (u_2, v_2)) \in E_7$. Then $(u_1, u_2) \in E_U$ and $(v_1, v_2) \in E_V$. $\Rightarrow \mu(u_1, u_2) \neq 0$ and $\vartheta(v_1, v_2) \neq 0$. $\Rightarrow \mu'(f(u_1), f(u_2)) \neq 0$ and $\vartheta'(h(v_1), h(v_2)) \neq 0$. $\Rightarrow (f(u_1), f(u_2)) \in E_{U'}$ and $(h(v_1), h(v_2)) \in E_{V'}$. $\Rightarrow w' = ((f(u_1), h(v_1)), (f(u_2), h(v_2))) \in E'_7$.

Case VIII. Let $w = ((u_1, v_1), (u_2, v_2)) \in E_8$. Then $(u_1, u_2) \notin E_U$ and $(v_1, v_2) \notin E_V$. $\Rightarrow \mu(u_1, u_2) = 0$ and $\vartheta(v_1, v_2) = 0$. $\Rightarrow \mu'(f(u_1), f(u_2)) = 0$ and $\vartheta'(h(v_1), h(v_2)) = 0$. $\Rightarrow (f(u_1), f(u_2)) \notin E_{U'}$ and $(h(v_1), h(v_2)) \notin E_{V'}$. $\Rightarrow w' = ((f(u_1), h(v_1)), (f(u_2), h(v_2))) \in E'_8$. Hence the proof.

Theorem 3.4. *Let $G : (\sigma, \mu) = (U, E_U)$, $H : (\tau, \vartheta) = (V, E_V)$, $G' : (\sigma'\mu') = (U', E_{U'})$ and $H' : (\tau'\vartheta') = (V', E_{V'})$ are any four fuzzy graphs such that $G : (\sigma, \mu) \cong G' : (\sigma', \mu')$ and $H : (\tau, \vartheta) \cong H' : (\tau', \vartheta')$ under the fuzzy graph isomorphisms f and h respectively.*

Let $E = E_1 \cup$ $E_2 \cup$ $E_3 \cup$ $E_4 \cup$ $E_5 \cup$ $E_6 \cup E_7$ $\cup E_8$ and $E' = E'_1 \cup$ $E'_2 \cup$ $E'_3 \cup$ $E'_4 \cup$ $E'_5 \cup$ E'_6 $\cup E'_7$ $\cup E'_8$ be the edge sets of $G \times_P H$ and $G' \times_P H'$ respectively. Then

$G \times_P H \cong G' \times_P H'$.

Proof. Define $f \times_P h : U \times V \to U' \times V'$ such that $f \times_P h(u, v) = (f(u), h(v))$. Now $f \times_P h(u_1, v_1) = f \times_P h(u_2, v_2) \Leftrightarrow (f(u_1), h(v_1)) = (f(u_2), h(v_2)) \Leftrightarrow f(u_1) = f(u_2)$ and $h(v_1) = h(v_2) \Leftrightarrow u_1 = u_2$ and $v_1 = v_2$ (since f and h are one to one mappings). By the above observation it is clear that $f \times_P h$ is a well-defined and one to one mapping. Now let $(u', v') \in U' \times V'$. then as f and h are surjective mappings $\exists u \in U$ and $v \in V$ such that $f(u) = u'$ and $h(v) = v'$. Now $f \times_P h(u, v) = (f(u), h(v)) = (u', v')$. Therefore $f \times_P h$ is surjective.

To show that $(\sigma \times_P \tau)(u, v) = (\sigma' \times_P \tau')(f(u), h(v))$

$$(\sigma' \times_P \tau')(f(u), h(v)) = \sigma'(f(u)) \wedge \tau'(h(v)) = \sigma(u) \wedge \tau(v) = (\sigma \times_P \tau)(u, v)$$

(since f, h are isomorphisms)

To show that $(\mu \times_P \vartheta)((u_1, v_1), (u_2, v_2)) = (\mu' \times_P \vartheta')((f(u_1), h(v_1)), (f(u_2), h(v_2)))$

Case I. Let $w = ((u_1, v_1), (u_2, v_2)) \in E_1$.

Then

$$(\mu \times_P \vartheta)(w) = \sigma(u_1) \wedge \vartheta(v_1, v_2) \tag{3.1}$$

since $w \in E_1$, by Lemma-3, we have

$$\begin{aligned} w' &= ((f(u_1), h(v_1)), (f(u_2), h(v_2))) \in E_1' \\ \Rightarrow (\mu' \times_P \vartheta')(w') &= (\mu' \times_P \vartheta')((f(u_1), h(v_1)), (f(u_2), h(v_2))) \\ &= \sigma'(f(u_1)) \wedge \vartheta'(h(v_1), h(v_2)) \\ &= \sigma(u_1) \wedge \vartheta(v_1, v_2) \\ &= (\mu \times_P \vartheta)(w). \quad \text{(by using (3.1))} \end{aligned}$$

Therefore $(\mu \times_P \vartheta)((u_1, v_1), (u_2, v_2)) = (\mu' \times_P \vartheta')((f(u_1), h(v_1)), (f(u_2), h(v_2)))$

Case II. Let $w = ((u_1, v_1), (u_2, v_2)) \in E_2$.

Then

$$(\mu \times_P \vartheta)(w) = \sigma(u_1) \wedge \tau(v_1)\tau(v_2) \tag{3.2}$$

since $w \in E_2$, by Lemma-3, we have

$$\begin{aligned} w' &= ((f(u_1), h(v_1)), (f(u_2), h(v_2))) \in E_2' \\ \Rightarrow (\mu' \times_P \vartheta')(w') &= (\mu' \times_P \vartheta')((f(u_1), h(v_1)), (f(u_2), h(v_2))) \\ &= \sigma'(f(u_1)) \wedge \tau'(h(v_1))\tau'(h(v_2)) \\ &= \sigma(u_1) \wedge \tau(v_1) \wedge \tau(v_2) \\ &= (\mu \times_P \vartheta)(w). \quad \text{(by using (3.2))} \end{aligned}$$

Therefore $(\mu \times_P \vartheta)((u_1, v_1), (u_2, v_2)) = (\mu' \times_P \vartheta')((f(u_1), h(v_1)), (f(u_2), h(v_2)))$

Case III. Let $w = ((u_1, v_1), (u_2, v_2)) \in E_3$.
Then

$$(\mu \times_P \vartheta)(w) = \mu(u_1, u_2) \wedge \tau(v_1) \tag{3.3}$$

since $w \in E_3$, by Lemma-3, we have

$$\begin{aligned} w' &= ((f(u_1), h(v_1)), (f(u_2), h(v_2))) \in E'_3 \\ \Rightarrow (\mu' \times_P \vartheta')(w') &= (\mu' \times_P \vartheta')((f(u_1), h(v_1)), (f(u_2), h(v_2))) \\ &= \mu'(f(u_1), f(u_2)) \wedge \tau'(h(v_1)) \\ &= \mu(u_1, u_2) \wedge \tau(v_1) \\ &= (\mu \times_P \vartheta)(w). \quad \text{(by using (3.3))} \end{aligned}$$

Therefore $(\mu \times_P \vartheta)((u_1, v_1), (u_2, v_2)) = (\mu' \times_P \vartheta')((f(u_1), h(v_1)), (f(u_2), h(v_2)))$

Case IV. Let $w = ((u_1, v_1), (u_2, v_2)) \in E_4$.
Then

$$(\mu \times_P \vartheta)(w) = \sigma(u_1) \wedge \sigma(u_2) \wedge \tau(v_1) \tag{3.4}$$

since $w \in E_4$, by Lemma-3, we have

$$\begin{aligned} w' &= ((f(u_1), h(v_1)), (f(u_2), h(v_2))) \in E'_4 \\ \Rightarrow (\mu' \times_P \vartheta')(w') &= (\mu' \times_P \vartheta')((f(u_1), h(v_1)), (f(u_2), h(v_2))) \\ &= \sigma'(f(u_1)) \wedge \sigma'(f(u_2)) \wedge \tau'(h(v_1)) \\ &= \sigma(u_1) \wedge \sigma(u_2) \wedge \tau(v_1) \quad \text{(since } f \text{ and } h \text{ are isomorphisms)} \\ &= (\mu \times_P \vartheta)(w). \quad \text{(by using (3.4))} \end{aligned}$$

Therefore $(\mu \times_P \vartheta)((u_1, v_1), (u_2, v_2)) = (\mu' \times_P \vartheta')((f(u_1), h(v_1)), (f(u_2), h(v_2)))$

Case V. Let $w = ((u_1, v_1), (u_2, v_2)) \in E_5$.
Then

$$(\mu \times_P \vartheta)(w) = \mu(u_1, u_2) \wedge \tau(v_1) \wedge \tau(v_2) \tag{3.5}$$

since $w \in E_5$, by Lemma-3, we have

$$\begin{aligned} w' &= ((f(u_1), h(v_1)), (f(u_2), h(v_2))) \in E'_5 \\ \Rightarrow (\mu' \times_P \vartheta')(w') &= (\mu' \times_P \vartheta')((f(u_1), h(v_1)), (f(u_2), h(v_2))) \\ &= \mu'(f(u_1), f(u_2)) \wedge \tau'(h(v_1)) \wedge \tau'(h(v_2)) \\ &= \mu(u_1, u_2) \wedge \tau(v_1) \wedge \tau(v_2) \quad \text{(since } f \text{ and } h \text{ are isomorphisms)} \\ &= (\mu \times_P \vartheta)(w). \quad \text{(by using (3.5))} \end{aligned}$$

Therefore $(\mu \times_P \vartheta)((u_1, v_1), (u_2, v_2)) = (\mu' \times_P \vartheta')((f(u_1), h(v_1)), (f(u_2), h(v_2)))$

Case VI. Let $w = ((u_1, v_1), (u_2, v_2)) \in E_6$.
Then

$$(\mu \times_P \vartheta)(w) = \sigma(u_1) \wedge \sigma(u_2) \wedge \vartheta(v_1, v_2) \tag{3.6}$$

since $w \in E_6$, by Lemma-3, we have

$$\begin{aligned}
w' &= ((f(u_1), h(v_1)), (f(u_2), h(v_2))) \in E'_6 \\
\Rightarrow (\mu' \times_P \vartheta')(w') &= (\mu' \times_P \vartheta')((f(u_1), h(v_1)), (f(u_2), h(v_2))) \\
&= \sigma'(f(u_1)) \wedge \sigma'(f(u_2)) \wedge \vartheta'(h(v_1))(h(v_2)) \\
&= \sigma(u_1) \wedge \sigma(u_2) \wedge \vartheta(v_1, v_2) \quad \text{(since } f \text{ and } h \text{ are isomorphisms)} \\
&= (\mu \times_P \vartheta)(w). \quad \text{(by using (3.6))}
\end{aligned}$$

Therefore $(\mu \times_P \vartheta)((u_1, v_1), (u_2, v_2)) = (\mu' \times_P \vartheta')((f(u_1), h(v_1)), (f(u_2), h(v_2)))$

Case VII. Let $w = ((u_1, v_1), (u_2, v_2)) \in E_7$.
Then

$$(\mu \times_P \vartheta)(w) = \mu(u_1, u_2) \wedge \vartheta(v_1, v_2) \tag{3.7}$$

since $w \in E_7$, by Lemma-3, we have

$$\begin{aligned}
w' &= ((f(u_1), h(v_1)), (f(u_2), h(v_2))) \in E'_7 \\
\Rightarrow (\mu' \times_P \vartheta')(w') &= (\mu' \times_P \vartheta')((f(u_1), h(v_1)), (f(u_2), h(v_2))) \\
&= \mu'(f(u_1), f(u_2)) \wedge \vartheta'(h(v_1))(h(v_2)) \\
&= \sigma(u_1, u_2) \wedge \vartheta(v_1, v_2) \quad \text{(since } f \text{ and } h \text{ are isomorphisms)} \\
&= (\mu \times_P \vartheta)(w). \quad \text{(by using (3.7))}
\end{aligned}$$

Therefore $(\mu \times_P \vartheta)((u_1, v_1), (u_2, v_2)) = (\mu' \times_P \vartheta')((f(u_1), h(v_1)), (f(u_2), h(v_2)))$

Case VIII. Let $w = ((u_1, v_1), (u_2, v_2)) \in E_8$.
Then

$$(\mu \times_P \vartheta)(w) = \sigma(u_1)\sigma(u_2) \wedge \tau(v_1) \wedge \tau(v_2) \tag{3.8}$$

since $w \in E_8$, by Lemma-3, we have

$$\begin{aligned}
w' &= ((f(u_1), h(v_1)), (f(u_2), h(v_2))) \in E'_8 \\
\Rightarrow (\mu' \times_P \vartheta')(w') &= (\mu' \times_P \vartheta')((f(u_1), h(v_1)), (f(u_2), h(v_2))) \\
&= \mu'(f(u_1)) \wedge \sigma'(f(u_2)) \wedge \tau'(h(v_1)) \wedge \tau'(h(v_2)) \\
&= \sigma(u_1)\sigma(u_2) \wedge \tau(v_1) \wedge \tau(v_2) \quad \text{(since } f \text{ and } h \text{ are isomorphisms)} \\
&= (\mu \times_P \vartheta)(w). \quad \text{(by using (3.8))}
\end{aligned}$$

Therefore $(\mu \times_P \vartheta)((u_1, v_1), (u_2, v_2)) = (\mu' \times_P \vartheta')((f(u_1), h(v_1)), (f(u_2), h(v_2)))$
Therefore, in all the above cases we have shown that

$$(\mu \times_P \vartheta)((u_1, v_1), (u_2, v_2)) = (\mu' \times_P \vartheta')((f(u_1), h(v_1)), (f(u_2), h(v_2)))$$

Hence $G \times_P H \cong G' \times_P H'$ under the fuzzy graph isomorphism $f \times_P h$.

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