South East Asian J. of Mathematics and Mathematical Sciences
Vol. 17, No. 2 (2021), pp. 251-264

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

ANNIHILATOR 3-UNIFORM HYPERGRAPHS OF RIGHT TERNARY NEAR-RINGS

Teresa Arockiamary S, Meera C* and Santhi V**

Department of Mathematics,
Stella Maris College (Autonomous),
University of Madras, Chennai – 600086, Tamil Nadu, INDIA

E-mail : drtessys70@gmail.com

*Department of Mathematics,
Bharathi Women's College, Chennai - 600108, Tamil Nadu, INDIA

E-mail : eya278@gmail.com

**Department of Mathematics,
Presidency College, Chennai - 600005, Tamil Nadu, INDIA

E-mail : santhivaiyapuri2019@gmail.com

(**Received:** *Jan. 17, 2021* **Accepted:** *Jul. 15, 2021* **Published:** *Aug. 30, 2021)*

Abstract: The study of algebraic systems using graphs gives many interesting results. The ternary algebraic structures can be dealt with 3-uniform hypergraphs in which hyperedges are of size three. Right ternary near-ring, a generalization of near-ring in ternary context, was introduced by Daddi and Pawar in 2011. In this paper, annihilator 3-uniform hypergraph associated with the right ternary near-ring N denoted by $AH_3(N)$ is introduced. $AH_3(N)$ is seen to be empty when N is a constant RTNR and it is complete when N is a zero RTNR. If N is integral, then the nature of $AH_3(N)$ is studied. A necessary condition for $AH_3(N)$ to be complete is derived. Hypergraph invariants of $AH_3(\mathbb{Z}_n)$ are obtained. For certain RTNR, the existence of BIBD is verified.

Keywords and Phrases: 3-uniform hypergraph, Clique, Right ternary near-ring, Annihilator.

2020 Mathematics Subject Classification: 05C65, 20N10, 16Y30, 05E99.

1. Introduction

The properties of algebraic structures can be studied using tools of graph theory and is an interesting topic of research in recent years. The concept of zero-divisor graphs associated with zero-divisors of a commutative ring was initiated by Beck [2] in 1988. Badawi [1] introduced annihilator graph for a commutative ring. Tamizh chelvam [12] introduced and studied about three types of annihilating ideal graphs of near-rings. Zero-annihilator graph of a commutative ring was studied by Hojjat Mostafanasab [7].
In this paper, a right ternary near-ring N, introduced by Daddi and Pawar [6], is associated with a 3-uniform hypergraph denoted by $AH_3(N)$ using the concept of annihilator. A necessary condition for $AH_3(N)$ to be complete is proved and a criterion for $AH_3(N)$ to be nontrivial is derived. Hypergraph invariants of $AH_3(\mathbb{Z}_n)$ are obtained. It is shown that $AH_3(\mathbb{Z}_n)$ can be covered by cliques. Certain values of n are identified for which block designs exist in $AH_3(\mathbb{Z}_n)$.

2. Preliminaries

In this section, the basic definitions and results needed for the rest of the sections are given.

Definition 2.1. [3, 4, 5] *A hypergraph H is an ordered pair (V, E), where V is the set of vertices and E is a subset of the power set of V. H is called empty hypergraph if $V = \emptyset$ and $E = \emptyset$. H is said to be trivial if $V \neq \emptyset$ and $E = \emptyset$. A hypergraph H is called an r-uniform hypergraph if each hyperedge contains exactly r vertices. Also clique in H is a complete subhypergraph and the cardinality of largest maximal clique in H is called the clique number of H. The minimum and maximum degrees of hypergraph are denoted by δ and Δ respectively.*

Definition 2.2. [6, 9] *A right ternary near-ring(RTNR) is a nonempty set N with a binary operation $+$ and a ternary operation $[\ ]$ satisfying the conditions* :
(i) $(N, +)$ *is a group* (not necessarily abelian)
(ii) $(N, [\ ])$ *is a ternary semigroup* ($[[a\ b\ c]\ d\ e] = [a\ [b\ c\ d]\ e] = [a\ b\ [c\ d\ e]]$ for all $a, b, c, d, e \in N$)
(iii) *(Right distributive law)* $[(a + b)\ c\ d] = [a\ c\ d] + [b\ c\ d]$ *for all* $a, b, c, d \in N$.
Note that in an RTNR N, for every $x, y, z \in N$, (i) $[0\ x\ y] = 0$; (ii) $[-x\ y\ z] = -[x\ y\ z]$. The subsets $N_0 = \{t \in N \mid [t\ 0\ 0] = 0\}$ and $N_c = \{t \in N \mid [t\ 0\ 0] = t\}$ are called the zero-symmetric part and the constant part of N respectively. N is called a zero-symmetric RTNR if $N = N_0$ and it is called a constant RTNR if $N = N_c$. An RTNR N is called (i) an integral RTNR if N has no zero divisors. (ii) a zero

RTNR if $[N\ N\ N] = \{0\}$, where $[N\ N\ N] = \{[x\ y\ z] \mid x, y, z \in N\}$. For $x, y \in N$, the sets $[N\ x\ y] = \{[t\ x\ y] \mid t \in N\}$ and $[x\ N\ y], [x\ y\ N]$ etc. are defined in the same way. An element $e \in N$ is a right unital element if $[x\ e\ e] = x$, for every $x \in N$.

Definition 2.3. [9] *If N is an RTNR and $x,\ s \in N$, then* (i) *the annihilator of x with respect to s is $(0 : x)_s = \{t \in N \mid [t\ s\ x] = 0\}$ and* (ii) *the annihilator of x is $(0 : x) = \{t \in N \mid [t\ s\ x] = 0$ for all $s \in N\}$. It is to be noted that $(0 : x) = \cap_{s\in N}(0 : x)_s$ and x is said to have trivial annihilator if $(0 : x) = \{0\}$.*

Definition 2.4. [11, 9] *A design is a pair (X, A), where X is a set of points called elements and A is a collection of nonempty subsets of X called blocks. A 3-uniform hypergraph $H = (V, E)$ is said to have friendship property if for every three vertices $x,\ y,\ z \in V$, there exists a unique vertex w, called the universal friend, such that $xyw,\ xzw,\ yzw \in E$. For positive integers v, k and λ such that $v > k \geq 2$, a design (X, A) is called (v, k, λ) - balanced incomplete block design (abbreviated as (v, k, λ)-BIBD) if the following properties are satisfied* :
(i) $|X| = v$
(ii) *each block contains exactly k points*
(iii) *every pair of distinct points is contained in exactly λ blocks.*
The incidence matrix of (X, A), where $X = \{x_1, ..., x_v\}$ and $A = \{A_1, ..., A_b\}$, is the $v \times b$, 0 - 1 matrix $M = (m_{i,j})$ defined by the rule $m_{i,j} = \begin{cases} 1 & if\ \ x_i \in A_j \\ 0 & if\ \ x_i \notin A_j \end{cases}$.

3. Main Results: Annihilator 3-uniform hypergraph of RTNR

In this section, annihilator 3-uniform hypergraph of RTNR is defined and some of the properties are illustrated with examples.

Definition 3.1. *An annihilator 3-uniform hypergraph associated with an RTNR N denoted by $AH_3(N)$ is defined as a 3-uniform hypergraph whose vertex set is the set of all elements of N having nontrivial annihilators and three distinct vertices x, y and z are adjacent whenever the intersection of their annihilators is not $\{0\}$. In other words, $AH_3(N) = (V, E)$, where $V = N\backslash T$, $T = \{x \in N \mid (0 : x) = \{0\}\}$ and $E = \{xyz \mid (0 : x) \cap (0 : y) \cap (0 : z) \neq \{0\}, x \neq y \neq z\}$.*

Example 3.2. Consider $N = D_8 = \{0, a, 2a, 3a, b, a + b, 2a + b, 3a + b\}$, which forms a near-ring under the addition $(+)$ and the multiplication $(\cdot)$ corresponding to Scheme 134 : $(0, 1, 14, 5, 15, 21, 17, 23), p : 418$, Pilz [10]. Let the ternary product $[\ ]$ be defined by $[x\ y\ z] = (x \cdot y) \cdot z$ for all $x,\ y,\ z \in N$. Then $(N, +, [\ ])$ is an RTNR and $AH_3(N)$ is a complete hypergraph on $V = \{0, 2a, b, a+b, 2a+b, 3a+b\}$, since $(0 : 0) = N$; $(0 : a) = (0 : 3a) = \{0\}$; $(0 : 2a) = (0 : b) = (0 : 2a + b) =$

$\{0, 2a, a+b, 3a+b\}$; $(0 : a+b) = (0 : 3a+b) = \{0, 2a, b, 2a+b\}$.

Lemma 3.3. *Let N be an RTNR. Then $AH_3(N)$ is*
(i) *an empty hypergraph if N is a constant RTNR.*
(ii) *a complete hypergraph if N is a zero RTNR.*
Proof. Let N be an RTNR. Then
(i) If N is a constant RTNR, then for any $x, s \in N$, $(0 : x)_s = \{t \in N \mid [t\ s\ x] = 0\}$ $= \{t \in N \mid [[t\ 0\ 0]\ s\ x] = 0\} = \{t \in N \mid [t\ 0\ [0\ s\ x]] = 0\} = \{0\}$ so that $(0 : x) = \cap_{s \in N}(0 : x)_s = \{0\}$. Thus $V = \emptyset$ and $E = \emptyset$ in $AH_3(N)$, proving (i).
(ii) If N is a zero RTNR, then $[x\ y\ z] = 0$ for every $x, y, z \in N$. Therefore for any $x, s \in N$, $(0 : x)_s = \{t \in N \mid [t\ s\ x] = 0\} = N$ so that $(0 : x) = N$. Hence $V = N$ and $(0 : x) \cap (0 : y) \cap (0 : z) \neq \{0\}$, for every $x, y, z \in V$. Thus $AH_3(N)$ is complete.

Lemma 3.4. *Let N be an integral RTNR. Then $AH_3(N)$ is trivial if N is zero-symmetric.*
Proof. Suppose N is zero-symmetric. Then $(0 : 0) = \{t \mid [t\ s\ 0] = 0\} = N$ and so $0 \in V$. If N is integral, then for $x (\neq 0) \in N, (0 : x) = \{t \in N \mid [t\ s\ x] = 0$ for every $s \in N\} = \{0\}$. Hence $AH_3(N)$ is trivial.

Lemma 3.5. *Let N be an RTNR with $n(n \geq 3)$ elements. Then $|V| \leq n - m$ if N has m right unital elements.*
Proof. Let N be an RTNR with $n(n \geq 3)$ elements and let $e \in N$ be a right unital element. Then $(0 : e)_e = \{x \in N \mid [x\ e\ e] = 0\} = \{0\}$ so that $(0 : e) = \{0\}$. Therefore $e \notin V$. Hence if there are m right unital elements, then there can be at the most $n - m$ vertices.

Lemma 3.6. *Let N be a commutative RTNR. Then the following assertions hold:*
(i) *$AH_3(N)$ is trivial if every nonzero element in N has trivial annihilator.*
(ii) *$AH_3(N)$ is nontrivial if there exists $x (\neq 0) \in N$, which does not have additive self-inverse and $(0 : x) \neq \{0\}$.*
Proof. Let N be a commutative RTNR. Then for every $x \in N$, $[x\ 0\ 0] = [0\ 0\ x] = 0$. Therefore N is zero-symmetric and so $0 \in V$. $\rightarrow$ (1)
(i) If $(0 : x) = \{0\}$ for every $x \neq 0 \in N$, then $AH_3(N)$ is trivial by (1).
(ii) Let $x (\neq 0) \in N$ be such that $-x \neq x$ and $(0 : x) \neq \{0\}$.
It is now claimed that $(0 : x) = (0 : (-x))$.
For, if $s \in N$ is given, then $(0 : (-x))_s = \{t \in N \mid [t\ s\ (-x)] = 0\} = (0 : x)_s$, proving the claim.
Hence $(0 : 0) \cap (0 : x) \cap (0 : (-x)) \neq \{0\}$. Thus $0x(-x)$ is a hyperedge in $AH_3(N)$, proving (ii).

The following theorem gives a necessary condition for $AH_3(N)$ to be complete.

Theorem 3.7. *Let N be an RTNR with $n(n \geq 3)$ elements whose annihilators are N. Then $AH_3(N)$ is complete.*

Proof. Let N be an RTNR with $n(n \geq 3)$ elements and for every $x \in N$, $(0 : x) = N$. Then it is obvious that $V = N$ and for any x, y, $z \in N$, $(0 : x) \cap (0 : y) \cap (0 : z) = N \neq \{0\}$ and therefore $xyz \in E$. Hence $AH_3(N)$ is complete.

A necessary and sufficient condition for $AH_3(N)$ to be nontrivial is derived in the following theorem.

Theorem 3.8. *Let N be a commutative RTNR. Then $AH_3(N)$ is nontrivial if and only if $[N\ x\ z] = [N\ y\ z] = \{0\}$ for some $x, y, z \in N$.*

Proof. Let N be a commutative RTNR. Then $(0 : 0) = N$ and so $0 \in V$. Now, suppose that $AH_3(N)$ is nontrivial. Then there exists at least one hyperedge $0xy$, where x and y are nonzero elements such that $(0 : x) \cap (0 : y) \neq \{0\}$. If there exists $z(\neq 0) \in (0 : x) \cap (0 : y)$, then $[z\ s\ x] = [z\ s\ y] = 0$ for all $s \in N$, which implies $[Nxz] = [Nyz] = \{0\}$.

Conversely, suppose that $[N\ x\ z] = [N\ y\ z] = \{0\}$ for some nonzero x, y, $z \in N$. Then $[s\ x\ z] = [s\ y\ z] = 0$ for all $s \in N$, which implies $z \in (0 : x) \cap (0 : y)$, as N is commutative. Hence 0, x, $y \in V$ are distinct vertices and they satisfy $(0 : 0) \cap (0 : x) \cap (0 : y) \neq \{0\}$ so that $0xy$ is a hyperedge in $AH_3(N)$, proving the theorem.

4. Special Cases

Some of the properties of annihilator 3-uniform hypergraph of $\mathbb{Z}_n$ are established in this section.

4.1. Annihilator 3-uniform hypergraph of $\mathbb{Z}_n$

Consider $AH_3(\mathbb{Z}_n)$, where $n \geq 3$ and $\mathbb{Z}_n$ is the RTNR with the usual addition modulo n and ternary multiplication induced by multiplication modulo n. Throughout this section, $AH_3(\mathbb{Z}_n)$ is denoted by (V, E) and the cardinality of V and E by $|V|$ and $|E|$ respectively.

Lemma 4.1.1. *The following assertions hold in $\mathbb{Z}_n$:*

(i) $(0 : 1) = \{0\}$ (ii) $(0 : 0) = \mathbb{Z}_n$ (iii) *For any $x \in \mathbb{Z}_n$, $(0 : x) = (0 : x)_1$.*

Proof. (i) $(0 : 1)_1 = \{x \in \mathbb{Z}_n \mid [x\ 1\ 1] = 0\} = \{0\}$ so that $(0 : 1) = \{0\}$.

(ii) $[t\ s\ 0] = [0\ s\ t] = 0$ for every t, $s \in \mathbb{Z}_n$. Therefore $(0 : 0) = \mathbb{Z}_n$.

(iii) It is obvious that $(0 : x) \subseteq (0 : x)_1$, for every $x \in \mathbb{Z}_n$. Now if $t \in (0 : x)_1$, then $[t\ 1\ x] = 0$ and so for every $s \in \mathbb{Z}_n$, $[t\ s\ x] = [t\ [s\ 1\ 1]\ x] = [[t\ 1\ x]\ s\ 1] = 0$, which shows $t \in (0 : x)$. Therefore $(0 : x)_1 \subseteq (0 : x)$, proving (iii).

In what follows some of the properties of annihilators in $\mathbb{Z}_n$ are proved which are useful in the sequel of this section.

Lemma 4.1.2. *Let $x \in \mathbb{Z}_n^\star$. Then $(0:x) = (0:c)$, where $c = (x,n)$, the g.c.d of x and n.*
Proof. Let $x \in \mathbb{Z}_n^\star$ and $(x,n) = c$. Then there exist integers l and m such that $lx + mn = c$. Now $t \in (0:x) \Rightarrow t \in (0:x)_1 \Rightarrow [t\ 1\ x] = 0 \Rightarrow t \cdot x = 0$ (where $\cdot$ denotes the multiplication modulo n) $\Rightarrow [t\ l\ x] = 0 \Rightarrow [t\ 1\ c] = 0 \Rightarrow t \in (0:c)$. Thus $(0:x) \subseteq (0:c)$.
Also $t \in (0:c) \Rightarrow t \in (0:c)_1 \Rightarrow [t\ 1\ c] = 0 \Rightarrow t \cdot c = 0 \Rightarrow tkc = 0$ (for an integer k such that $x = kc$) $\Rightarrow t \cdot x = 0 \Rightarrow [t\ 1\ x] = 0 \Rightarrow t \in (0:x)$. Thus $(0:c) \subseteq (0:x)$, proving the result.

In the following lemma it is proved that the annihilator of a divisor $d(\neq 1)$ of n consists of all multiples of $\frac{n}{d}$.

Lemma 4.1.3. *If $d \mid n$ and $d \neq 1$, then $(0:d) \neq \{0\}$.*
Moreover, $(0:d) = \{kl | k \in \{1, 2, \cdots, d\}\} = \langle l \rangle$ (say), where $l = \frac{n}{d}$.
Proof. Let $d \mid n$ and $d \neq 1$. Then $ld = n$ for some $l \in \mathbb{Z}_n^\star$, which implies $[l\ 1\ d] = 0$ $\Rightarrow l \in (0:d)_1$=$(0:d) \Rightarrow (0:d)$ is nontrivial. Also $t \in (0:d) \Rightarrow t \in (0:d)_1 \Rightarrow$ $[t\ 1\ d] = 0 \Rightarrow t \cdot d = 0 \Rightarrow td = kn$, $k \in \{1, \cdots, d\} \Rightarrow t \in \langle l \rangle$, proving the result.
The following lemma establishes some of the relations between annihilators of two different divisors of n.

Lemma 4.1.4. *Let d_1 and d_2 be two divisors of n. Then the following assertions hold:*
(i) *If $d_1 \neq d_2$, then $(0:d_1) \neq (0:d_2)$.*
(ii) *If $d_1 \mid d_2$, then $(0:d_1) \subset (0:d_2)$.*
(iii) *If $(d_1, d_2) = 1$, then $(0:d_1) \cap (0:d_2) = \{0\}$.*
(iv) *If $(d_1, d_2) = r$, then $(0:r) \subset (0:d_1) \cap (0:d_2)$.*
Proof. Let d_1 and d_2 be two divisors of n.
(i) If $d_1 \neq d_2$, then by Lemma 4.1.3, $(0:d_1) = \langle l_1 \rangle$ and $(0:d_2) = \langle l_2 \rangle$, where $l_1 d_1 = n$, $l_2 d_2 = n$ and $l_1 \neq l_2$. Hence $(0:d_1) \neq (0:d_2)$.
(ii) If $d_1 | d_2$, then $d_2 = kd_1$, $k \neq 1$. Hence $t \in (0:d_1) = (0:d_1)_1$, which implies $[t\ 1\ d_1] = 0 \Rightarrow t \cdot d_1 = 0 \Rightarrow (t \cdot k) \cdot d_1 = 0 \Rightarrow t \cdot d_2 = 0 \Rightarrow [t\ 1\ d_2] = 0$ $\Rightarrow t \in (0:d_2)_1 = (0:d_2)$. Also $|(0:d_1)| = d_1 < d_2 = |(0:d_2)|$. Thus $(0:d_1) \subset (0:d_2)$.
(iii) If $(d_1, d_2) = 1$, then there exist integers r and s such that $rd_1 + sd_2 = 1$.
Suppose $t \in (0:d_1) \cap (0:d_2)$. Then $[t\ 1\ d_1] = 0$ and $[t\ 1\ d_2] = 0$. Now $trd_1 + tsd_2 = t$ and so $t = 0$, proving (iii).

(iv) If $(d_1, d_2) = r \neq 1$, then $r \mid d_1$ and $r \mid d_2$. Hence $(0 : r) \subset (0 : d_1) \cap (0 : d_2)$ by (ii).

Definition 4.1.5. *On* $\mathbb{Z}_n^\star = \{1, 2, \cdots, n\text{–}1\}$, *define a relation* $\sim$ *by* $x \sim y$ *if and only if* $(x, n) = (y, n)$. *Obviously,* $\sim$ *is an equivalence relation on* $\mathbb{Z}_n^\star$ *and the equivalence class of* $x \in \mathbb{Z}_n^\star$ *under* $\sim$ *is given by* $[x]_\sim = \{y \in \mathbb{Z}_n^\star | (x, n) = (y, n)\}$.

Remark 4.1.6. *The equivalence relation* $\sim$ *provides a partition of* $\mathbb{Z}_n^\star$.

Lemma 4.1.7. *For any* n, $\mathbb{Z}_n^\star = \cup_{d|n}[d]_\sim$, *where* $[d]_\sim = \{x \in \mathbb{Z}_n^\star | (x, n) = d\}$.
Proof. If $x \in \mathbb{Z}_n^\star$, then by the above remark, $\mathbb{Z}_n^\star = \cup_{x \in \mathbb{Z}_n^\star}[x]_\sim$. If $(x, n) = 1$, then $x \in [1]_\sim$. If $(x, n) = d$, then $x \in [d]_\sim$. Thus $\mathbb{Z}_n^\star = \cup_{x \in \mathbb{Z}_n^\star}[x]_\sim = [1]_\sim \cup (\cup_{(x,n)=d}[d]_\sim) = \cup_{d|n}[d]_\sim$.

Lemma 4.1.8. $\mathbb{Z}_n^\star = [1]_\sim \cup (\cup_{d|n}[d]_\sim)$, *where* $[d]_\sim = \{x \in \mathbb{Z}_n^\star | (0 : x) = (0 : d)\}$.
In particular, $\mathbb{Z}_n^\star = [1]_\sim = \{x \in \mathbb{Z}_n^\star | (0 : x) = \{0\}\}$, *if* n *is prime.*
Proof. From Lemma 4.1.7, $\mathbb{Z}_n^\star = \cup_{d|n}[d]_\sim$, where $[d]_\sim = \{x \in \mathbb{Z}_n^\star | (x, n) = d\} = \{x \in \mathbb{Z}_n^\star | (0 : x) = (0 : d)\}$, using Lemma 4.1.2.
If n is prime, then $\mathbb{Z}_n^\star = [1]_\sim = \{x \in \mathbb{Z}_n^\star | (0 : x) = (0 : 1)\} = \{x \in \mathbb{Z}_n^\star | (0 : x) = \{0\}\}$.

The following lemma is proved with the help of the notions given above.

Lemma 4.1.9. *In* $AH_3(N)$, $|V| = \begin{cases} n - \phi(n) & \text{if } n \text{ is not prime} \\ 1 & \text{if } n \text{ is prime} \end{cases}$.

Proof. If n is not prime, then $V = \{0\} \cup \left(\cup_{d|n}[d]_\sim, d \neq 1\right) = \mathbb{Z}_n \backslash [1]_\sim$. Hence $|V| = n - \phi(n)$. If n is prime, then $|V| = 1$ as $\phi(n) = n - 1$.

Note 4.1.10. For a composite number n, if F denotes the set of all proper divisors of n, then obviously, $d \in F$ implies $d \notin [1]_\sim$. Hence $V = \{0\} \cup \left(\cup_{d|n}[d]_\sim, d \neq 1\right) = \{0\} \cup (\cup_{d \in F}[d]_\sim)$.

Lemma 4.1.11. $AH_3(\mathbb{Z}_n)$ *is* (i) *trivial if* n *is prime* (ii) *nontrivial if* $n (n \geq 6)$ *is not prime.*
Proof. It can be seen from Note 4.1.10 that $V = \{0\} \cup (\cup_{d \in F}[d]_\sim)$. Now,
(i) If n is prime, then $F = \emptyset$. Therefore $V = \{0\}$ and so $AH_3(\mathbb{Z}_n)$ is trivial.
(ii) If n is not prime, then $F \neq \emptyset$. If $d_1 \in F$, then $d_2 = \frac{n}{d_2} \in F$.
Let $d_1 < d_2$. Then $d_1 + d_1 \in \mathbb{Z}_n$ and $(0 : d_1) \subset= (0 : (d_1 + d_1)) \subset (0 : 0)$.
Therefore $0 d_1 (d_1 + d_1)$ is a hyperedge in $AH_3(\mathbb{Z}_n)$, showing that it is nontrivial.

Notation 4.1.12. Given $n \geq 4$, (i) let $F = \{d \mid d \mid n, d \neq 1, d \neq n\}$;
$P = \{p \in F \mid p \text{ is prime}\}$; $D = \{d \in F \mid d \text{ is composite}\}$;
$D_p = \{d \in D \mid p \mid d\}$, for $p \in P$. Then $F = P \cup D$, where $D = \cup_{p \in P} D_p$.
(ii) for $p \in P$, let $M_p = \left\{p, 2p, \cdots, \left(\frac{n}{p} - 1\right) p\right\}$.

Remark 4.1.13. *If $d \in D_p$, $p \in P$, then $(0:d) \supset (0:p)$.*

The following lemma shows that V can be described in terms of M_p, $p \in P$.

Lemma 4.1.14. *In $AH_3(\mathbb{Z}_n)$, $V = \{0\} \cup (\cup_{p\in P} M_p)$, where $M_p = [p]_\sim \cup \big(\cup_{d\in D_p}[d]_\sim\big)$.*
Proof. Let $p \in P$. Then it is observed that $M_p = [p]_\sim \cup \big(\cup_{d\in D_p}[d]_\sim\big)$.
Also, $[p]_\sim = \{x \in \mathbb{Z}_n^\star \mid (x,n) = p\} = \{kp \mid (k,n) = 1\} \subseteq M_p$; Also if $d \in D_p$, then $d = lp$, $l \neq 1$ and $[d]_\sim = \{x \in \mathbb{Z}_n^\star \mid (x,n) = d\} = \{kp \mid (k,n) = l\} \subseteq M_p$.
Therefore $[p]_\sim \cup \big(\cup_{d\in D_p}[d]_\sim\big) \subseteq M_p$.
Now, let $x \in M_p$. Then $x = kp$, where either $(k,n) = 1$ or $(k,n) \neq 1$.
If $(k,n) = 1$, then $x \in [p]_\sim$ since $(x,n) = (kp,n) = p$.
If $(k,n) = l \neq 1$, then $x \in [lp]_\sim$ since $(x,n) = (kp,n) = lp$, where $lp \in D_p$.
Therefore $M_p \subseteq [p]_\sim \cup \big(\cup_{d\in D_p}[d]_\sim\big)$. Hence $M_p = [p]_\sim \cup \big(\cup_{d\in D_p}[d]_\sim\big)$.
Now, using Notation 4.1.12, $F = P \cup \{d \in D_p \mid p \in P\}$. Thus by Note 4.1.10 and the above observation $V = \{0\} \cup (\cup_{d\in F}[d]_\sim) = \{0\} \cup (\cup_{p\in P} M_p)$.

Illustration 4.1.15. Note that in $AH_3(\mathbb{Z}_{12})$, $F = P \cup D$, where $P = \{2,3\}$, $D = \{4,6\}$ and $D_2 = \{4,6\}; D_3 = \{6\}$. It can be seen that $V = \{0\} \cup (M_2 \cup M_3)$, where $M_2 = [2]_\sim \cup ([4]_\sim \cup [6]_\sim)$ and $M_3 = [3]_\sim \cup [6]_\sim$.

Lemma 4.1.16. *Let $p \in P$. Then $\{0\} \cup M_p$ forms a complete subhypergraph in $AH_3(\mathbb{Z}_n)$ with $\frac{n}{p}$ vertices.*
Proof. Let x, y, $z \in \{0\} \cup M_p$, where $M_p = [p]_\sim \cup \big(\cup_{d\in D_p}[d]_\sim\big)$. Then the following cases arise:

(i) $x = 0$; y, $z \in [p]_\sim$ (ii) $x = 0$; y, $z \in [d]_\sim$
(iii) $x = 0$; $y \in [p]_\sim$; $z \in [d]_\sim$ (iv) $x \in [p]_\sim$; y, $z \in [d]_\sim$
(v) x, $y \in [p]_\sim$; $z \in [d] \sim$ (vi) x, y, $z \in [p]_\sim$ (vii) x, y, $z \in [d]_\sim$

By Remark 4.1.13, $\{0\} \neq (0:p) \subset (0:d)$ for every $d \in D_p$. Therefore in case (i) and case (vi), $(0:x) \cap (0:y) \cap (0:z) = (0:p) \neq \{0\}$. Hence $xyz \in E$.
In case (ii) - (v) and case (vii), $(0:x) \cap (0:y) \cap (0:z) \supset (0:p) \neq \{0\}$. Therefore $xyz \in E$. Thus $\{0\} \cup M_p$ forms a complete subhypergraph.
Also, it is obvious that $|\{0\} \cup M_p| = \frac{n}{p}$ from Notation 4.1.12(ii). Hence the proof.

Lemma 4.1.17. *In $AH_3(\mathbb{Z}_n)$, $\{0\} \cup M_p$ forms a maximal clique for every $p \in P$.*
Proof. Let $p \in P$. Then from the previous lemma it is observed that $\{0\} \cup M_p$ forms a clique. If x, $y \notin \{0\} \cup M_p$, then obviously $(x,p) = (y,p) = 1$. Therefore $(0:x) \cap (0:y) \cap (0:p) = \{0\}$ and hence $xyp \notin E$. Thus $\{0\} \cup M_p$ forms a maximal clique.

Remark 4.1.18. (i) *Let $|P| = k$. Then there are k maximal cliques formed by $\{0\} \cup M_p$, where $p \in P$, each of which has $\frac{n}{p}$ vertices and they cover $AH_3(\mathbb{Z}_n)$.*

(ii) *The clique number of* $AH_3(\mathbb{Z}_n)$ *is* $\frac{n}{p}$, *where* p *is the smallest prime factor of* n.

Illustration 4.1.19. The annihilator 3-uniform hypergraph of $\mathbb{Z}_{12}$ is shown in Figure. 1, in which each triangle represents a hyperedge and there are two maximal cliques, namely, the subhypergraphs on $\{0\} \cup M_3$(dotted lines) and $\{0\} \cup M_2$.

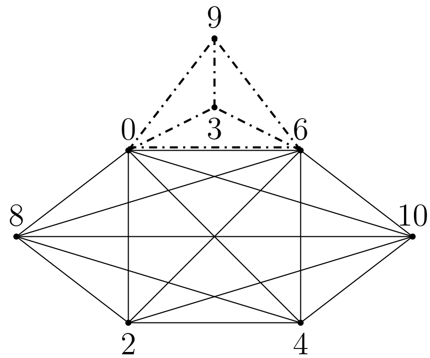

Figure 1: $AH_3(\mathbb{Z}_{12})$

Lemma 4.1.20. $AH_3(\mathbb{Z}_n)$ *has an isolated vertex if* $n = 2q$, q *is prime and* $n \geq 6$.
Proof. Let $n(n \geq 6)$ be such that $n = 2q$, q is prime. Then $V = \{0\} \cup M_2 \cup M_q$, where $M_2 = \{2, 4, \cdots, 2(q-1)\}$ and $M_q = \{q\}$. Notice that for any $x(\neq 0) \in V$, $(0 : x) \cap (0 : q) = \{0\}$ since $(2, q) = 1$. Thus q is an isolated vertex.

Illustration 4.1.21. In $AH_3(\mathbb{Z}_{14})$, $V = \{0\} \cup M_2 \cup M_7$, where $M_2 = \{2, 4, 6, 8, 10, 12\}$, $M_7 = \{7\}$ and there is no hyperedge containing 7.

Lemma 4.1.22. *Let* $n(n \geq 6)$ *be a composite number. Then* $AH_3(\mathbb{Z}_n)$ *is connected except when* $n = 2q$, q *is prime.*
Proof. Let $x, y \in V$. Then the proof is given by considering the number of prime factors of n.
case (i) If n has only one prime factor, then $n = p^\alpha$, $\alpha \geq 2$. It is noted that $V = \{0\} \cup M_p$, which forms a complete hypergraph by Lemma 4.1.17. Therefore $AH_3(\mathbb{Z}_n)$ is complete and hence is connected.
case (ii) If n has only two prime factors, then $n = p^\alpha q^\beta$, $\alpha \geq 1$, $\beta \geq 1$. Now, $V = \{0\} \cup M_p \cup M_q$.
If $x, y \in M_p$ or $x, y \in M_q$, then by Lemma 4.1.17, there is a hyperedge $0xy$. Hence $AH_3(\mathbb{Z}_n)$ is connected.
Suppose $x \in M_p$ and $y \in M_q$. Consider the following subcases.
(a) $n = p^\alpha q^\beta$, $p = 2$, $\alpha = 1$, $\beta = 1$.

That is, $n = 2q$ and $V = \{0\} \cup M_2 \cup M_q$. Then as in Lemma 4.1.20, q is isolated. Therefore $AH_3(\mathbb{Z}_n)$ is not connected.
(b) $n = p^\alpha q^\beta$, $p = 2$, $\alpha = 1$, $\beta \geq 2$.
That is, $n = 2q^\beta$, $\beta \geq 2$ and $V = \{0\} \cup M_2 \cup M_q$, where $M_2 = \{2, 4, \cdots, 2(q^\beta - 1)\}$ and $M_q = \{q, 2q, \cdots, (2q^{\beta-1} - 1)q\}$. Hence by Lemma 4.1.17, for every $u = 2k \in M_2$ and $v = lq \in M_q$, there exist hyperedges $h_1 = 0xu$ and $h_2 = 0yv$, showing that $AH_3(\mathbb{Z}_n)$ is connected.
(c) $n = p^\alpha q^\beta$, $p = 2$, $\alpha \geq 2$, $\beta \geq 1$.
Now, $V = \{0\} \cup M_2 \cup M_q$ and $M_2 = \{2, 4, \cdots, 2(2^{\alpha-1}q^\beta - 1)\}$;
$M_q = \{q, 2q, \cdots, (p^\alpha q^{\beta-1} - 1)q\}$. Therefore by Lemma 4.1.17, for every $u = 2k$ and $v = lq$, there exist hyperedges $h_1 = 0xu$ and $h_2 = 0yv$, showing that $AH_3(\mathbb{Z}_n)$ is connected.
(d) $n = p^\alpha q^\beta$, $p \neq 2$, $\alpha \geq 1$, $\beta \geq 1$.
Now, $V = \{0\} \cup M_p \cup M_q$ and $M_p = \{p, 2p, \cdots, (p^{\alpha-1}q^\beta - 1)p\}$;
$M_q = \{q, 2q, \cdots, (p^\alpha q^{\beta-1} - 1)q\}$. Therefore as in (c), for every $u = kp \in M_p$ and $v = lq \in M_q$, there exist hyperedges $h_1 = 0xu$ and $h_2 = 0yv$. Hence $AH_3(\mathbb{Z}_n)$ is connected.
case (iii) If n has three or more prime factors, then a similar argument is carried out to prove that $AH_3(\mathbb{Z}_n)$ is connected. Thus, $AH_3(\mathbb{Z}_n)$ is connected except when $n = 2q$, q is prime.

Lemma 4.1.23. *$AH_3(\mathbb{Z}_n)$ is complete if and only if n has only one prime factor.*
Proof. Let n have only one prime factor. Then $n = p^\alpha$, $\alpha \geq 2$. Then $V = \{0\} \cup M_p$ forms a complete hypergraph by Lemma 4.1.17. Therefore $AH_3(\mathbb{Z}_n)$ is complete.
Conversely, assume that $AH_3(\mathbb{Z}_n)$ is complete. Let if possible p and q be prime factors of n. Then $(0 : p) \cap (0 : q) = \{0\}$ and therefore there is no hyperedge in $AH_3(\mathbb{Z}_n)$ containing p and q, a contradiction to the assumption. Thus there can be only one prime factor for n. Hence the proof.

Lemma 4.1.24. *$AH_3(\mathbb{Z}_n)$ is connected and the diameter is 2.*
Proof. Let x, $y \in V$, where $V=\{0\} \cup (\cup_{p \in P} M_p)$. Then
Case (i) if $x, y \in \{0\} \cup M_p$, for $p \in P$, then by Lemma 4.1.17, there is a hyperedge $0xy$. Therefore the distance between x and y is 1 in this case.
Case (ii) if $x \in M_p$ and $y \in M_q$ for p, $q (p \neq q) \in P$, then by Lemma 4.1.17, for every $u = kp$ and $v = lq$, there are hyperedges $h_1 = 0xu$, $h_2 = 0yv \in E$. Therefore the distance between x and y is 2 in this case. Hence the proof.

The remaining part of this section provides the enumeration of hyperedges in $AH_3(\mathbb{Z}_n)$, for certain values of n, using cliques.

Lemma 4.1.25. *If $AH_3(\mathbb{Z}_n)$, $n = p^{\alpha}$, then $|E| = p^{\alpha-1}C_3$.*
Proof. Let $n = p^{\alpha}$. Then by Lemma 4.1.23, $AH_3(\mathbb{Z}_{p^{\alpha}})$ is complete. Therefore $|E| = p^{\alpha-1}C_3$ since $V = \{0\} \cup M_p$, where $M_p = \{p, 2p, \ldots, (p^{\alpha-1} - 1)p\}$.

Lemma 4.1.26. *In $AH_3(\mathbb{Z}_n)$, if $n = 2q$, then $|E| = qC_3$.*
Proof. Let $n = 2q$. Then $V = \{0\} \cup M_2 \cup M_q$ and as seen in Lemma 4.1.20, $\{0\} \cup M_2$ has q vertices and q is isolated. Therefore the number of possible hyperedges in $AH_3(\mathbb{Z}_n)$ is qC_3.

Lemma 4.1.27. *In $AH_3(\mathbb{Z}_n)$, if $n = pq (2 \neq p < q)$, then $|E| = pC_3 + qC_3$.*
Proof. Let $n = pq(2 \neq p < q)$. Then $V = \{0\} \cup M_p \cup M_q$, where
$M_p = \{p, 2p, \cdots, (q-1)p\}$; $M_q = \{q, 2q, \cdots, (p-1)q\}$; $M_p \cap M_q = \emptyset$. Obviously if $x \in M_p$ and $y \in M_q$, then $(0 : x) \cap (0 : y) = \{0\}$. Hence the possible number of hyperedges in $AH_3(\mathbb{Z}_n)$ is $|E| = pC_3 + qC_3$.

Lemma 4.1.28. *In $AH_3(\mathbb{Z}_{2^2q})$, $(q \geq 3)$, $|E| = 2qC_3 + 4C_3$.*
Proof. Let $n = 2^2q(q \geq 3)$. Then $V = \{0\} \cup M_2 \cup M_q$, where
$M_2 = \{2, 4, \cdots, (q-1)2, 2q, 2(q+1), \cdots, 2(2q-1)\}$; $M_q = \{q, 2q, 3q\}$. Note that $M_2 \cap M_q = \{2q\}$. Hence the total number of hyperedges in $AH_3(\mathbb{Z}_n)$ is $|E| = 2qC_3 + 4C_3$.

Lemma 4.1.29. *In $AH_3(\mathbb{Z}_{p^2q})$, $(q \geq 3)$, $|E| = pqC_3 + p^2C_3 - pC_3$.*
Proof. Let $n = p^2q(q \geq 3)$. Then $V = \{0\} \cup M_p \cup M_q$, where
$M_p = \{p, 2p, \cdots, (q-1)p, qp, (q+2)p, \cdots, (pq-1)p\}$;
$M_q = \{q, 2q, \cdots, (p-1)q, pq, (p+1)q, \cdots, (p^2-1)q\}$. Hence the subhypergraphs induced by $\{0\} \cup M_p$ and $\{0\} \cup M_q$ have pqC_3 and p^2C_3 hyperedges respectively. Now, $d \in M_p \cap M_q \Rightarrow pq|d \Rightarrow d \in \{pq, 2pq, \cdots, (p-1)pq\}$, since $p^2q = n$ and so $|M_p \cap M_q| = p - 1$. Hence pC_3 hyperedges are counted twice in the above enumeration process. Thus by eliminating repeated hyperedges, $|E| = pqC_3 + p^2C_3 - pC_3$.

Lemma 4.1.30. *In $AH_3(\mathbb{Z}_n)$, if $n = pqr$, then*
$|E| = pqC_3 + prC_3 + qrC_3 - pC_3 - qC_3 - rC_3$.
Proof. Let $n = pqr$. Then $V = \{0\} \cup M_p \cup M_q \cup M_r$. Notice that $|M_p \cap M_q| = r-1$; $|M_q \cap M_r| = p - 1$; $|M_p \cap M_r| = q - 1$; $M_p \cap M_q \cap M_r = \emptyset$. Therefore by a similar process of computation as in previous lemma, after eliminating repeated hyperedges, $|E| = pqC_3 + prC_3 + qrC_3 - pC_3 - qC_3 - rC_3$.

Illustration 4.1.31. The above process of enumeration is illustrated for $n = 30$.
For $n = 30$, $V = \{0\} \cup M_2 \cup M_3 \cup M_5$, where
$M_2 = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28\}$;
$M_3 = \{3, 6, 9, 12, 15, 18, 21, 24, 27\}$; $M_5 = \{5, 10, 15, 20, 25\}$.
Therefore $|E| = 15C_3 + 10C_3 + 6C_3 - 3C_3 - 5C_3$.

4.2. Existence of BIBDs in $AH_3(N)$, $N = \mathbb{Z}_n$, for certain values of n

In this section, the special RTNR $N = (\mathbb{Z}_n, +_n, [\ ])$, where $[\ ]$ is defined as $[x\ y\ z] = \begin{cases} x & \text{if } y = z = n-1 \\ 0 & \text{otherwise} \end{cases}$, for $x,\ y,\ z \in N$, is considered and certain values of n are identified for which block designs exist in $AH_3(N)$ and the properties of BIBD are verified. It is observed that block designs exist in $AH_3(\mathbb{Z}_n)$, for $n = 5, 9, 11$.

Example 4.2.1. In $(V, E) = AH_3(N)$, where $N = \mathbb{Z}_5$, $V = \mathbb{Z}_5 \setminus \{4\} = \{0, 1, 2, 3\}$ and there is only one quad given by **123**(012, 013, 023). Also $AH_3(N)$ is a 3-uniform friendship hypergraph with universal friend 0. It is observed that all the 4 vertices occur in $r = 3$ hyperedges and any two distinct vertices occur in $\lambda = 2$ hyperedges. The incidence matrix M satisfies $MM^t = (r–\lambda)I + \lambda J$, where I is the unit matrix of order $|V| \times |V|$ and J is a $|V| \times |V|$ matrix with entries 1. Moreover $|V|r = 3|E|$ and $\lambda(|V| - 1) = 2r$. Thus (V, E) is a (4, 3, 2)–BIBD.

Example 4.2.2. In $(V, E) = AH_3(N)$, where $N = \mathbb{Z}_9$, $V = \mathbb{Z}_9 \setminus \{8\}$ and there are 14 quads which are given by

137(017, 013, 037), **124**(014, 012, 024), **235**(025, 023, 035),
346(036, 034, 046), **457**(047, 045, 057), **156**(016, 015, 056),
267(027,, 026, 067), **123**(621, 631, 623), **257**(125, 127, 157),
467(146, 147, 167), **347**(234, 237, 247), **456**(245, 246, 256),
567(356, 357, 367).

Thus, the annihilator 3-uniform hypergraph is a friendship 3-uniform hypergraph. It is easy to verify the properties of BIBD as in previous case and $AH_3(N)$ is seen to be a (8, 3, 6)-BIBD.

Example 4.2.3. In $(V, E) = AH_3(N)$, where $N = \mathbb{Z}_{11}$, $V = \mathbb{Z}_{11} \setminus \{10\}$ and there are 120 hyperedges and 30 quads. The annihilator 3-uniform hypergraph is a friendship 3-uniform hypergraph and $AH_3(N)$ is a (10, 3, 8)-BIBD.

5. Conclusion

In this paper, it is proved that $AH_3(N)$ is empty if N is constant RTNR and it is complete if N is a zero RTNR. $AH_3(\mathbb{Z}_n)$ is seen to be nontrivial only when n is composite. A necessary and sufficient condition for $AH_3(\mathbb{Z}_n)$ to be complete is found as $n = p^k$ whereas it is connected except for $n = 2q$, q is prime. The clique number for $AH_3(\mathbb{Z}_n)$ is found. Enumeration of hyperedges in $AH_3(\mathbb{Z}_n)$ is done for certain values of n by using cliques. It is observed that $AH_3(\mathbb{Z}_n)$, where $\mathbb{Z}_n$ is special RTNR, exhibits $(n-1, 3, n-3)$–BIBD for some values of n.

References

[1] Badawi, A., The annihilator ideal graph of a commutative ring, Comm. Algebra, 42 (2014), 108-121.

[2] Beck, I., Coloring of commutative rings, J. Algebra, 116 (1988), 208-226.

[3] Berge, C., Hypergraphs, North-Holland, Amsterdam, (1989).

[4] Bretto, A., Hypergraph Theory Introduction, Springer, (2013).

[5] Buss, E., Han, Hiep, Schacht and Mathis, Minimum vertex degree conditions for loose Hamilton cycles in 3-uniform hypergraphs, J. Combin. Theory Ser. B, 103(6) (2013), 658-678.

[6] Daddi, V. R. and Pawar Y. S., Right Ternary Near-rings, Bull. Calcutta Math. Soc., 103(1) (2011), 21-30.

[7] Hojjat Mostafanasab, Zero-Annihilator Graphs of Commutative Rings, Kragujevac Journal of Mathematics, (2017).

[8] Li, P. C., Van Rees, G. H. J., Seo, S. H., Singhi, N. M., Friendship 3-hypergraphs, Discrete Math., 312 (2012), 1892-1899.

[9] Meera, C., An Algebraic Study on Right Ternary Near-rings and N-groups and fuzzy soft Right Ternary Near-rings and N-subgroups with some Applications (Ph.D, thesis), (2016).

[10] Pilz, G., Near-rings, North-Holland, Amsterdam, (1983).

[11] Stinson, D. R., Combinatorial Designs: Constructions and Analysis, Springer, (2004).

[12] Tamizh Chelvam, T. and Rammurthy, S., On annihilator graphs of near-rings, Palestine Journal of Mathematics, 5(Special Issue: 1) (2016), 100-107.