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EDGE ITALIAN DOMINATION IN GRAPHS

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Abstract: An edge Italian dominating function (EIDF) of a graph $G = (V, E)$ is a function $f : E(G) \to \{0, 1, 2\}$ such that every edge e with $f(e) = 0$ is adjacent to some edge e' with $f(e') = 2$ or at least two edges e_1, e_2 with $f(e_1) = f(e_2) = 1$. The weight of an edge Italian dominating function is $\sum_{e \in E(G)} f(e)$. The edge Italian domination number of a graph G is defined as the minimum weight of an edge Italian dominating function of G and is denoted by $\gamma'_I(G)$. In this paper, we initiate a study on the edge Italian domination in graphs.

Keywords and Phrases: Roman Domination, Italian Domination, Edge Italian Domination, Edge Italian dominating function, Edge Italian Domination number.

2020 Mathematics Subject Classification: 05C70.

1. Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. A subset S of the vertex set V is called a dominating set of G if every vertex not in S is adjacent to some vertex in S. The domination number, $\gamma(G)$, of G is the minimum cardinality taken over all dominating sets of G.

Mitchell and Hedetniemi [7] introduced the concept of edge domination in graphs. A subset F of edges of a graph G is called an edge dominating set of G if every edge not in F is adjacent to some edge in F. The edge domination number of G, denoted by γ', is the minimum cardinality taken over all edge dominating sets of G.

Motivated by Stewart [10] on defending the Roman Empire, Cockayne et al. [3] introduced Roman Dominating Function. A function $f : V(G) \rightarrow \{0, 1, 2\}$ such that every vertex v with $f(v) = 0$ is adjacent to some vertex u with $f(u) = 2$ is called a Roman dominating function. The weight of a Roman dominating function is the value $\sum_{v\in V(G)} f(v)$. The Roman domination number of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of a Roman dominating function on G.

In order to reduce the cost of defending the Roman Empire, Henning and Hedetniemi [5] introduced the concept of weak Roman dominating function. Let $G = (V, E)$ be a graph. Define a function $f : V(G) \rightarrow \{0, 1, 2\}$. A vertex u with $f(u) = 0$ is said to be undefended with respect to f if it is not adjacent to a vertex with positive weight. The function f is called a weak Roman dominating function (WRDF) if each vertex u with $f(u) = 0$ is adjacent to a vertex v with $f(v) > 0$ such that the function $f' : V(G) \rightarrow \{0, 1, 2\}$, defined by $f'(u) = 1$, $f'(v) = f(v)-1$ and $f'(w) = f(w)$, if $w \in V - \{u, v\}$, has no undefended vertex. The weight of f is $\sum_{u\in V(G)} f(u)$. The minimum weight of a WRDF on G is called the weak Roman domination number and is denoted by $\gamma_r(G)$.

Roushini Leely Pushpam et al. [8] introduced edge version of Roman Domination. An edge Roman Dominating Function of a graph G is a function, $f : E(G) \rightarrow \{0, 1, 2\}$ such that every edge e with $f(e) = 0$ is adjacent to some edge e_1 with $f(e_1) = 2$. The edge Roman domination number of G, denoted by $\gamma'_R(G)$, is the minimum weight of an edge Roman dominating function of G.

Roushini Leely Pushpam and TNM Mai [9] introduced edge version of weak Roman domination. Let f be a function $f : E(G) \rightarrow \{0, 1, 2\}$. An edge x with $f(x) = 0$ is called undefended with respect to f if it is not incident to an edge with positive weight. f is called a weak edge Roman dominating function (WERDF) if each edge x with $f(x) = 0$ is incident to an edge y with $f(y) > 0$ such that the function, $f' : E(G) \rightarrow \{0, 1, 2\}$, defined by $f'(x) = 1$, by $f'(y) = f(y) - 1$ and $f'(z) = f(z)$, if $z \in E(G) - \{x, y\}$, has no undefended edge. The weight of f is $\sum_{x\in E(G)} f(x)$. The minimum weight of a WERDF on G is called the weak edge Roman domination number and is denoted by $\gamma'_{WR}(G)$.

An Italian dominating function of a graph G is a function $f : V(G) \rightarrow \{0, 1, 2\}$ such that every vertex v with $f(v) = 0$ is adjacent to some vertex u with $f(u) = 2$ or is adjacent to at least two vertices x, y with $f(x) = f(y) = 1$. The weight of an Italian dominating function is $\sum_{v\in V(G)} f(v)$. The minimum weight of such a function on G is called the Italian domination number of G and is denoted by $\gamma_I(G)$. Italian domination was first introduced as Roman $\{2\}$-domination by Chellali et al. [2]. It was further researched and renamed as Italian domination by Henning and Klostermeyer [6]. For the terms and definitions not explicitly defined here, refer Harary [4].

The following are some of the results connecting Domination, Roman Domination and its variations which will be used in the sequel.

Theorem 1.1. *For every graph* $G, \gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$ (Henning, Hedetniemi [5]).

Theorem 1.2. *For every graph* $G, \gamma(G) \leq \gamma_{\{R2\}}(G) \leq \gamma_R(G)$ *and* $\gamma_r(G) \leq \gamma_{\{R2\}}(G)$, *where* $\gamma_{\{R2\}}(G)$ *is* $\gamma_I(G)$ (Chellali M, Haynes T, Hedetniemi S T [2]).

Theorem 1.3. $\gamma'_R(P_n) = \lfloor \frac{2n}{3} \rfloor$ *and* $\gamma'_R(C_n) = \lceil \frac{2n}{3} \rceil$ (Roushini Leely Pushpam et al [8]).

Theorem 1.4. *For any connected graph* G *of even order* p, $\gamma'(K_p) = \frac{p}{2}$ *if and only if* G *is isomorphic to* K_p *or* $K_{p/2,p/2}$ (Arumugam, S., and S. Velammal [1]).

2. Edge Italian Dominating Function and Edge Italian Domination Number

In this paper, we introduce the edge variant of the Italian dominating function. An edge Italian dominating function (EIDF) of a graph $G = (V, E)$ is a function $f : E(G) \to \{0, 1, 2\}$ such that every edge e with $f(e) = 0$ is adjacent to some edge e' with $f(e') = 2$ or at least two edges e_1 and e_2 with $f(e_1) = f(e_2) = 1$. The weight of an edge Italian dominating function is $\sum_{e \in E(G)} f(e)$.

The edge Italian domination number of G, denoted by $\gamma'_I(G)$, is the minimum weight of all edge Italian dominating functions of G. Let E_0, E_1, E_2 be the partitions of the edge set E, such that $E_i = \{e \in E : f(e) = i\}$ for $i = 0, 1, 2$. We also denote the function $f : E(G) \to \{0, 1, 2\}$ by $f = (E_0, E_1, E_2)$.

We begin with an inequality connecting the edge domination number, the edge Italian domination number and the edge Roman domination number.

Theorem 2.1. *For any graph* G, $\gamma'(G) \leq \gamma'_I(G) \leq \gamma'_R(G)$.

Proof. Since every edge Roman dominating function is an edge Italian dominating function, it follows that $\gamma'_I(G) \leq \gamma'_R(G)$. To obtain the lower bound, consider the partitions E_0, E_1, E_2 of the edge set $E(G)$ in any edge Italian dominating function. Then $E_1 \cup E_2$ is a dominating set so that $\gamma'(G) \leq |E_1| + |E_2| \leq |E_1| + 2|E_2| = \gamma'_I(G)$. Hence, $\gamma'(G) \leq \gamma'_I(G) \leq \gamma'_R(G)$.

Chellali M, Haynes T, Hedetniemi S T [2] proved that every Italian dominating function is a weak Roman dominating function. We now present the edge version of it.

Theorem 2.2. *For every graph,* $G, \gamma'_{WR}(G) \leq \gamma'_I(G)$.

Proof. Let f be a an edge Italian dominating function with minimum weight

$\gamma'_I(G)$. Let $e \in E_0$ with $f(e) = 0$. Then either e is adjacent to e' with $f(e') = 2$ or e is adjacent to two edges x and y with $f(x) = 1$ and $f(y) = 1$. In the former case we can obtain a weak edge Roman dominating function g by reassigning the weights of e and e' such that $g(e) = 1$, $g(e') = 1$ and $g(e'') = f(e'')$, otherwise. In the latter case also, we can obtain a weak edge Roman dominating function g by reassigning the weights of e and x with $f(e) = 1$, $g(x) = 0$ and $g(y) = f(y) = 1$ and $g(z) = f(z)$, if $z \in E(G) - \{x, y\}$. Hence every edge Italian dominating function is a weak edge Roman dominating function and the result follows.

Theorem 2.3. *For any graph G, if $\gamma'_I(G) = 2$, then $diam(G) \leq 3$.*
Proof. Suppose, $\gamma'_I(G) = 2$. Then three cases to consider:
Case 1. If G has exactly two edges, then G is isomorphic to P_3 and $diam(G) = 2$.
Case 2. If G has exactly three edges, then G is isomorphic to P_4, $K_{1,3}$ or C_3. Then $diam(G) \leq 3$ for all these three graphs.
Case 3. If G has more than three edges, since $\gamma'_I(G) = 2$, either there exists an edge $e = uv$ with $f(e) = 2$ and all other edges have weight 0 and are adjacent to e or there are two edges e_1 and e_2 with $f(e_1) = 1$ and $f(e_2) = 1$ and all other edges have weight 0 and are adjacent to both e_1 and e_2. In any case, $diam(G) \leq 3$.

The converse of this theorem is not true. That is, all graphs with $diam(G) \leq 3$ need not have $\gamma'_I(G) = 2$. For example, let G be the graph obtained from the cycle, C_4 by adding a pendant edge to one of the vertices. Then $diam(G) = 3$ and $\gamma'_I(G) = 3$.

Theorem 2.4. *If G is a tree, then $\gamma'_I(G) = 2$ if and only if $2 \leq diam(G) \leq 3$.*
Proof. Let G be a tree with $\gamma'_I(G) = 2$. Then by theorem 2.3, $diam(G) \leq 3$. Since $\gamma'_I(G) = 2$, G has at least 2 edges. So, $diam(G) \geq 2$.

Conversely, let G be a tree with $2 \leq diam(G) \leq 3$. So, $\max\{d(u, v) : u, v \in V(G)\} \leq 3$. In fact, $diam(G) = 2$ or 3. If $diam(G) = 3$, then, the shortest path connecting u and v is of length 3. Let $ue_1w_1e_2w_2e_3v$ be the shortest path. Then no edge of G can be adjacent to u or v because in that case $diam(G)$ will be greater than 3. So all the edges of G must be adjacent to e_2. So, giving the weight 2 to the edge e_2 and the weight 0 to all other edges we get $\gamma'_I(G) = 2$.

If $diam(G) = 2$, since G is a tree, all the edges must be adjacent to each other. Giving the weight 2 to any one edge and weight 0 to all other edges or giving the weight 1 to any two edges and weight 0 to all other edges gives a minimum EIDF. Thus, $\gamma'_I(G) = 2$.

3. The Edge Italian Domination Number of some Special Types of Graphs

Theorem 3.1. *For the path graph P_n, $\gamma'_I(P_n) = \lceil \frac{n}{2} \rceil$, $n \geq 2$.*

Proof. Let $P_n = (v_1, e_1, v_2, e_2, ..., v_{n-1}, e_{n-1}, v_n)$; $e_i = \{v_i, v_{i+1}\}$ be a path graph. In any minimum EIDF, $f = (E_0, E_1, E_2)$ of P_n, between two edges in E_1 there can be at most one edge in E_0. In this case both the pendant edges must have the weight 1. Also, every edge in E_2 can be adjacent to at most two edges in E_0. In any case, $\sum f(e) \geq \lceil \frac{n}{2} \rceil$.

Now, define a function, $f : E(P_n) \to \{0, 1, 2\}$ as follows

Case 1. n is even

$$f(e_i) = \begin{cases} 1, & \text{if } i \text{ is odd} \\ 0, & \text{otherwise} \end{cases}$$

Since there are only $\frac{n}{2}$ such edges with weight, 1, we have $\sum f(e) = \frac{n}{2} \leq \lceil \frac{n}{2} \rceil$.

Case 2. n is odd

$$f(e_i) = \begin{cases} 1, & \text{if } i \text{ is odd or } i = n-1 \\ 0, & \text{otherwise} \end{cases}$$

Then, $\sum f(e) = \frac{(n-1)}{2} + 1 = \frac{n+1}{2} = \lceil \frac{n}{2} \rceil$. Thus, $\sum f(e) \leq \lceil \frac{n}{2} \rceil$ for all n. Therefore $\gamma'_I(P_n) = \lceil \frac{n}{2} \rceil$.

Theorem 3.2. *For the cycle graph C_n, $\gamma'_I(C_n) = \lceil \frac{n}{2} \rceil$.*

Proof. $C_n = (v_1, e_1, v_2, e_2, ..., v_n, e_n, v_1)$; $e_i = \{v_i, v_{i+1}\}$ be a cycle graph. Let $f = (E_0, E_1, E_2)$ be an EIDF on C_n. If $E_2 = \varnothing$, every edge in E_0 must be adjacent to two edges of E_1. If we assign the weights $1, 0, 1, 0, 1, 0...$ in order to the edges, at most two edges of E_1 can be adjacent to each other. Hence, $\sum f(e) = |E_1| \geq \lceil \frac{n}{2} \rceil$. If $E_2 \neq \varnothing$, since f is minimum every edge of E_2 can be adjacent to at most two edges of E_0. In this case, $\sum f(e) = |E_1| + 2|E_2| \geq \lceil \frac{n}{2} \rceil$.

Now, define $f : E(G) \to \{0, 1, 2\}$ by

$$f(e_i) = \begin{cases} 1, & \text{if } i \text{ is odd} \\ 0, & \text{otherwise} \end{cases}$$

Then, $\sum f(e) = \frac{n}{2} \leq \lceil \frac{n}{2} \rceil$, n is even and $\sum f(e) = \frac{(n+1)}{2} \leq \lceil \frac{n}{2} \rceil$, n is odd.
Thus $\sum f(e) \leq \lceil \frac{n}{2} \rceil$, for all n. Therefore $\gamma'_I(C_n) = \lceil \frac{n}{2} \rceil$.

Remark 3.3. *$\gamma'(C_5) = 2$, $\gamma'_I(C_5) = 3$, $\gamma'_R(C_5) = 4$. So, for $G = C_5$, $\gamma'(G) < \gamma'_I(G) < \gamma'_R(G)$.*

The Wheel graph W_n, $n \geq 3$, is the join of the graphs C_n and K_1 and hence is a graph with $n + 1$ vertices and $2n$ edges. It is formed by connecting a single vertex to all vertices of a cycle of length n. We call the vertices of C_n as rim vertices and the other single vertex as apex vertex.

Theorem 3.4. *For the Wheel graph W_n, $\gamma'_I(W_n) = \lceil \frac{n+2}{2} \rceil$, $n \geq 3$.*
Proof. Let $W_n = (V, E)$ be a Wheel graph. Let the edge set E be partitioned into two sets X, Y where, $X = \{x_1, x_2, ..., x_n\}$, the set of central edges joining the apex vertex to the rim vertices and $Y = \{y_1, y_2, ..., y_n\}$, the set of rim edges. Then the set Y form a cycle on n vertices and by theorem 3.2, $\gamma'_I(C_n) = \lceil \frac{n}{2} \rceil$. Also note that the minimum EIDF of C_n is an assignment of weights $(1, 0, 1, 0, ...)$ to y_i in order, so that $y_n = 0$ or 1 according as n is even or odd. If one of the central edges is assigned the weight 1 and all other edges the weight 0, we can get a minimum EIDF of W_n.
So, $\gamma'_I(W_n) = \lceil \frac{n}{2} \rceil + 1 = \lceil \frac{n+2}{2} \rceil$.

Theorem 3.5. *Let $G = K_{m,n}$ be a complete bipartite graph with $m \geq 2$, then for $m < n$, $\gamma'_I(K_{m,n}) = \begin{cases} n, & if \;\; n < 2m \\ 2m, & if \;\; n \geq 2m \end{cases}$ and $\gamma'_I(K_{n,n}) = n, \; \forall n$.*
Proof. Let $K_{m,n} = (V, E)$ be a complete bipartite graph. Let $X = \{x_1, x_2, ..., x_m\}$ and $Y = \{y_1, y_2, ..., y_n\}$ be a bipartition of the vertex set V.
Case 1. $m < n$ and $n < 2m$.
Define $f : E(K_{m,n}) \to \{0, 1, 2\}$ by $f(x_i y_i) = 1 \; \forall i = 1, 2, ..., m$ and $f(x_{m-k} y_{n-k}) = 1$, for $k = 0, 1, 2, ..., n - m - 1$ and $f(x_i y_j) = 0$, otherwise.
Then, $\sum f(e) \leq m + n - m = n$.

Consider the EIDF defined on E by assigning the weight 1 to at least one edge incident at each vertex, x_i, y_i and the weight 0 to all other edges. Then, there will be minimum n edges with weight 1.
So, $\sum f(e) \geq n$. Therefore, $\gamma'_I(K_{m,n}) = n$.
Case 2. $m < n$ and $n \geq 2m$.
Now define $f : E(K_{m,n}) \to \{0, 1, 2\}$ by $f(x_i y_i) = 1 \; \forall \; i = 1, 2, ..., m$ and $f(x_{m-k} y_{n-k})$ $= 1$, for $k = 0, 1, 2, ..., m - 1$ and $f(x_i y_j) = 0$, otherwise.
Then, $\sum f(e) \leq m + m = 2m$.

Next we define an EIDF on E in which one of the edges incident at each x_i is given the weight 2. Since all other edges of $K_{m,n}$ are incident at one of these x'_is they can be given the weight 0. Then f is minimum and hence $\sum f(e) \geq m + m = 2m$.
Case 3. m=n.
Define $f : E(K_{n,n}) \to \{0, 1, 2\}$ by $f(x_i y_i) = 1 \; \forall i = 1, 2, ..., n$ and $f(x_i y_j) = 0$, otherwise.
So, $\sum f(e) \leq n$.

Arumugam and Velammal [1] showed that $\gamma'(K_{n,n}) = n$. So there are n edges

in the minimum edge dominating set of $K_{n,n}$. Define an EIDF on E by assigning the weight 1 to each of these n edges and weight 0 to all other edges of $K_{n,n}$. Then, $\sum f(e) \geq n$.
Therefore, $\gamma_I'(K_{n,n}) = n$.

Proposition 3.6. *For the Star graph $K_{1,n}$, $\gamma_I'(K_{1,n}) = 2$, when $n \geq 2$.*
Proof. In the Star graph $K_{1,n}$, all the edges are adjacent each other. Assign the weight 1 to any two edges or weight 2 to one of the edges and weight 0 to all other edges. Then we get a minimum EIDF.
Hence, $\gamma_I'(K_{1,n}) = 2, \ \forall n \geq 2$.

Proposition 3.7. *For complete graphs K_n of even order, $\gamma_I'(K_n) = \frac{n}{2}$.*
Proof. Arumugam and Velammal [1] proved that $\gamma'(K_n) = \frac{n}{2}$, when n is even. So, the minimum edge dominating set of K_n has $\frac{n}{2}$ edges. Assign the weight 1 to each of these edges and weight 0 to all other edges. Then we get a minimum EIDF.
Hence, $\gamma_I'(K_n) = \frac{n}{2}$, when n is even.

References

[1] Arumugam, S., and Velammal S., Edge domination in graphs, Taiwanese journal of Mathematics, 2,2 (1998), 173-179.

[2] Chellali M, Haynes T. W., Hedetniemi S. T., and McRae A. A., Roman 2-domination, Discrete Appl. Math., 204 (2016), 22–28.

[3] Cockayne E. J., Dreyer Jr. P. M., Hedetniemi S. M., and Hedetniemi S. T., On Roman domination in graphs, Discrete Math., 278 (2004), 11–22.

[4] Harary Frank, Graph Theory, Addison Wesley, Reading Mass, (1969)

[5] Henning M. A., Hedetniemi S. T., Defending the Roman Empire-a new strategy, Discrete Math., 266 (2003), 239–251.

[6] Henning M. A. and Klostermeyer W. F., Italian domination in trees, Discrete Appl. Math, 217 (2017), 557–564.

[7] Mitchell S. and Hedetniemi S. T., Edge domination in trees, Congr. Numer., 19 (1977), 489–509.

[8] Roushini Leely Pushpam P., Malini Mai T. N. M., Edge Roman domination in graphs, J. Combin. Math. Combin. Comput, 69 (2009), 175-182.

[9] Roushini Leely Pushpam P., Malini Mai T. N. M., Weak edge Roman domination in graphs, Australas. J Comb., 51 (2011), 125-138.

[10] Stewart I., Defend the Roman Empire, Sci. Am., 281 (6) (1999), 136–139.