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DOMINATION POLYNOMIALS OF THE JEWEL GRAPH AND ITS COMPLEMENT

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Abstract: Let $G = (V(G), E(G))$ be a simple graph. The Jewel graph J_n is a graph with vertex set $V(J_n) = \{u, v, x, y, u_i : 1 \leq i \leq n\}$ and edge set $E(J_n) = \{ux, uy, xy, xv, yv, uu_i, vu_i : 1 \leq i \leq n\}$. The domination polynomial of a graph G of order n is the polynomial $D(G, x) = \sum_{i=\gamma(G)}^{n} d(G, i)x^i$, where $d(G, i)$ is the number of dominating sets of G of cardinality i. In this paper, we present various domination polynomials of the Jewel graph J_n. Also we determine the same results for the complement of the Jewel graph.

Keywords and Phrases: Domination polynomial, Jewel graph..

2020 Mathematics Subject Classification: 05C31, 05C69.

1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. A set $D \subseteq V$ is a dominating set if every vertex in $V - D$ is adjacent to a vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. A dominating set with cardinality $\gamma(G)$ is called a γ-set. For a detailed treatment of this parameter the reader is referred to [8].

A dominating set $D_i \subseteq V(G)$ is an independent dominating set [5] if the induced subgraph $\langle D_i \rangle$ has no edges. Independent domination number is the minimum size of an independent dominating set of G and is denoted by $i(G)$. A dominating set $D_t \subseteq V(G)$ is a total dominating set [5] if the induced subgraph $\langle D_t \rangle$ has no isolated vertices. The total domination number $\gamma_t(G)$ of a graph G is the minimum cardinality of a total dominating set. A dominating set $D_c \subseteq V(G)$ is a connected dominating set [10] if the induced subgraph $\langle D_c \rangle$ is connected. The connected domination number $\gamma_c(G)$ is the minimum cardinality of a connected dominating set. A dominating set $D_{cl} \subseteq V(G)$ is a dominating clique [6] if the induced subgraph$\langle D_{cl} \rangle$ is a complete graph. The clique domination number $\gamma_{cl}(G)$ of G is the minimum cardinality of a dominating clique. For a detailed treatment of the above parameters the reader is referred to [9].

The complement $\bar{G}$ of a graph G also has $V(G)$ as its vertex set, but two vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in G.

2. Preliminaries

Domination Polynomial in graphs was first introduced by Arocha and Llano [3] in 2000. Later in 2014, Alikhani and Peng made a slight modification of that definition and investigated its properties in [2]. The definition that we follow in this paper is that of Alikhani and Peng. Alikhani and others have contributed a lot to this concept by publishing many research articles and made this concept to reach greater heights in the field of graph theory.

Definition 2.1. [2] *The domination polynomial of a graph G of order n is the polynomial $D(G,x) = \sum_{i=\gamma(G)}^{n} d(G,i)x^i$, where $d(G,i)$ is the number of dominating sets of G of cardinality i.*

We need the following theorems to prove our main results. They have been proved by S. Alikhani and Y. H. Peng [2].

Theorem 2.1. [2] *If a graph G consists of m components $G_1, \ldots, G_m$ then $D(G,x) = D(G_1,x) \ldots D(G_m,x)$.*

Theorem 2.2. [2] *Let G be a graph with $|V(G)| = n$. If G is connected, then $d(G,n) = 1$ and $d(G,n-1) = n$.*

Lemma 2.1. [2] *For every $n \in N$, $D(K_n,x) = (1+x)^n - 1$.*

Theorem 2.3. [2] *Let G_1 and G_2 be graphs of order n_1 and n_2 respectively. Then $D(G_1+G_2,x) = ((1+x)^{n_1} - 1)((1+x)^{n_2} - 1) + D(G_1,x) + D(G_2,x)$.*

Theorem 2.4. [11] Let G be a complete graph K_n of n vertices. Then $D_i(K_n,x) = nx$.

Theorem 2.5. [11] *For any two graphs G_1 and G_2, $D_i(G_1+G_2,x) = D_i(G_1) + D_i(G_2)$.*

Theorem 2.6. [11] *Let $G = G_1 \cup G_2$. Then $D_i(G,x) = D_i(G_1,x)D_i(G_2,x)$.*

Theorem 2.7. [4] *For a complete graph K_n with $n \geq 2$ vertices,* $D_t(K_n, x) = \sum_{i=2}^{n} \binom{n}{i} x^i$.

Theorem 2.8. [4] *Let G_1 and G_2 be two connected graphs without isolated vertices. Then $D_t(G_1 + G_2, x) = D_t(K_{|V_1|,|V_2|}, x) + D_t(G_1, x) + D_t(G_2, x)$, where $V(G_1) = V_1$ and $V(G_2) = V_2$.*

Theorem 2.9. [4] *For a complete bipartite graph $G \cong K_{r,s}$ with $r, s \geq 2$ vertices,* $D_t(G, x) = D_t(K_{r+s}, x) - D_t(K_r, x) - D_t(K_s, x)$.

Theorem 2.10. [4] *Let $G_1, G_2, \ldots, G_k$ be nontrivial connected graphs. Then* $D_t(G_1 \cup G_2 \cup \ldots \cup G_k, x) = \prod_{i=1}^{k} D_t(G_i, x)$.

3. Main Results

Definition 3.1. *The Jewel graph J_n is a graph with vertex set $V(J_n) = \{u, v, x, y, u_i : 1 \leq i \leq n\}$ and edge set $E(J_n) = \{ux, uy, xy, xv, yv, uu_i, vu_i : 1 \leq i \leq n\}$.*

Example 1. The Jewel graph J_2 is shown in Figure 1.

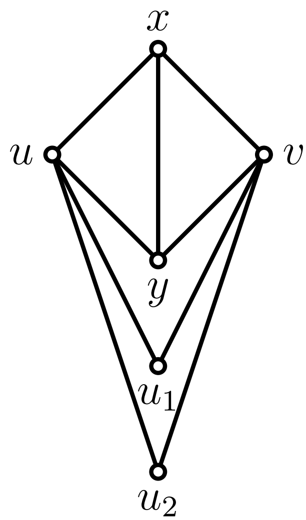

Figure 1: J_2

Observation 3.1. *The complement $\overline{J_n}$ of the Jewel graph J_n has two components. Let the vertex set be $V(\overline{J_n}) = V(J_n)$ and edge set be $E(\overline{J_n}) = \{uv, xu_i, yu_i : 1 \leq i \leq n\}$. It can be written as $\overline{J_n} = (K_n + 2K_1) \cup P_2$ or $\overline{J_n} = (K_{n-1} + P_3) \cup P_2$.*

Example 2. The complement of the Jewel graph J_2, $\overline{J_2}$ is shown in Figure 2.

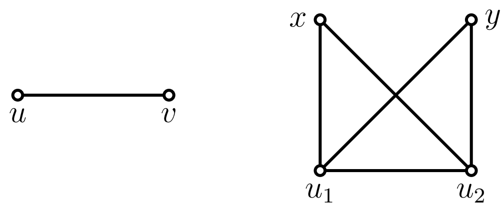

Figure 2: $\overline{J_2}$

Theorem 3.1. *Let J_n be the Jewel graph and $\overline{J_n}$ be its Complement. Then*
(i) $D(J_n, x) = (1+x)^{n+4} - (1+x)^{n+2} + x^{n+2} + 2x^{n+1} - 2x$.

(ii) $D(\overline{J_n}, x) = ((1+x)^{n+2} - 1 - 2x)(x^2 + 2x)$.

Proof. (i) The Jewel graph has $n+4$ vertices. $S = \{u, v\}$ is a minimum dominating set and so $\gamma(J_n) = 2$. Let i be a natural number, $2 \le i \le n+4$. Any subset $D \subseteq V(J_n)$ with $|D| \ge 2$ containing at least one of u, v is a dominating set of J_n. Among the subsets of $V(J_n) - \{u, v\}$, there are only three sets which are dominating namely, $\{u_1, u_2, \ldots, u_n, x, y\}$, $\{u_1, u_2, \ldots, u_n, x\}$ and $\{u_1, u_2, \ldots, u_n, y\}$. The first is of cardinality $n+2$ and the other two are of cardinality $n+1$.
Hence for $2 \le i \le n$, $d(J_n, i) = \binom{n+4}{i} - \binom{n+2}{i}$
$d(J_n, n+1) = \binom{n+4}{n+1} - \left(\binom{n+2}{n+1} - 2\right)$
$d(J_n, n+2) = \binom{n+4}{n+2}$
By Theorem 2.2, for $i = n+3$ and $n+4$, $d(J_n, i) = \binom{n+4}{i}$ so,

$$\begin{aligned}
D(J_n, x) &= \sum_{i=2}^{n} \left(\binom{n+4}{i} - \binom{n+2}{i}\right) x^i + \left(\binom{n+4}{n+1} - \left(\binom{n+2}{n+1} - 2\right)\right) \\
&\quad x^{n+1} + \sum_{i=n+2}^{n+4} \binom{n+4}{i} x^i \\
&= (1+x)^{n+4} - 1 - (n+4)x - (\binom{n+2}{2} x^2 + \binom{n+2}{3} x^3 + \ldots \\
&\quad + \binom{n+2}{n} x^n) - n x^{n+1} \\
&= (1+x)^{n+4} - 1 - (n+4)x - (1+x)^{n+2} + 1 + (n+2)x - \\
&\quad n x^{n+1} + \binom{n+2}{n+1} x^{n+1} + \binom{n+2}{n+2} x^{n+2} \\
D(J_n, x) &= (1+x)^{n+4} - (1+x)^{n+2} + x^{n+2} + 2x^{n+1} - 2x.
\end{aligned}$$

(ii) The complement of the Jewel graph J_n is $\overline{J_n} = (K_n + 2K_1) \cup P_2$

$D(K_n, x) = (1 + x)^n - 1$
$D(P_2, x) = x^2 + 2x$
By applying Theorem 2.3, with $G_1 = K_n$ and $G_2 = 2K_1$, we get
$D(K_n + 2K_1) = ((1 + x)^{n+2} - 1 - 2x)$
Now, by Theorem 2.1, we have the result.

Theorem 3.2. *Let J_n be the Jewel graph and $\overline{J_n}$ be its Complement. Then*
(i) $D_i(J_n, x) = 2x^{n+1} + x^2$.
(ii) $D_i(\overline{J_n}, x) = (x^2 + (n + 2)x)(2x) = 2x^3 + (n + 2)2x^2$.
Proof. (i) Note that $S = \{u, v\}$ is the only minimum independent dominating set and so $i(J_n) = 2$. Both u and v are adjacent with all the other vertices. So, the subsets $D \subseteq V(J_n)$ containing at least one of u, v is not a independent dominating set. Among the subsets of $V(J_n) - \{u, v\}$ there are only two sets which are independent dominating namely, $\{u_1, u_2, \ldots, u_n, x\}$ and $\{u_1, u_2, \ldots, u_n, y\}$ with cardinality $n + 1$. Hence $D_i(J_n, x) = 2x^{n+1} + x^2$.

(ii) By applying Theorem 2.5, with $G_1 = K_n$ and $G_2 = 2K_1$ we get
$D_i(K_n + 2K_1) = nx + x^2 + 2x$ and by Theorem 2.6, we have the result.

Theorem 3.3. *Let J_n be the Jewel graph and $\overline{J_n}$ be its Complement. Then*
(i) $D_t(J_n, x) = (1 + x)^{n+4} - (1 + x)^{n+2} - x^2 - 2x$.
(ii) $D_t(\overline{J_n}, x) = \left(\sum_{i=2}^{n+2} \binom{n+2}{i} x^i - x^2\right) x^2$

Proof. (i) Any subset $D \subseteq V(J_n)$ with $|D| = 2$ containing one of u, v is a minimum total dominating set and so $\gamma_t(J_n) = 2$. Both u and v are adjacent with all the other vertices but these two are not adjacent. So $\{u, v\}$ is not a total dominating set and any subset $D \subseteq V(J_n)$ with $|D| > 2$ containing both u and v is a total dominating set of J_n. Among the subsets of $V(J_n) - \{u, v\}$, there is no total dominating set. Hence for $i = 2$, $d(J_n, i) = \binom{n+4}{2} - \binom{n+2}{2} - 1$,
And for $3 \leq i \leq n + 2, d(J_n, i) = \binom{n+4}{i} - \binom{n+2}{i}$ Clearly for $i = n + 3$ and $n + 4$, $d(J_n, i) = \binom{n+4}{i}$ So,

$$\begin{aligned}
D_t(J_n, x) &= \left(\binom{n+4}{2} - \binom{n+2}{2} - 1\right) x^2 + \sum_{i=3}^{n+2} \left(\binom{n+4}{i} - \binom{n+2}{i}\right) x^i \\
&\quad + \sum_{i=n+3}^{n+4} \binom{n+4}{i} \\
&= (1 + x)^{n+4} - (n + 4)x - 1 - (1 + x)^{n+2} + (n + 2)x - x^2 \\
D_t(J_n, x) &= (1 + x)^{n+4} - (1 + x)^{n+2} - x^2 - 2x
\end{aligned}$$

(ii) The complement of the Jewel graph J_n is $\overline{J_n} = (K_{n-1} + P_3) \cup P_2$.
$D_t(P_3, x) = x^3 + 2x^2$. By Theorem 2.7, $D_t(K_{n-1}, x) = \sum_{i=2}^{n} \binom{n-1}{i} x^i$. By Theorem 2.8, $D_t(K_{n-1} + P_3, x) = D_t(K_{n-1,3}, x) + D_t(K_{n-1}, x) + D_t(P_3, x)$ Now by Theorem 2.9, we have $D_t(K_{n-1} + P_3, x) = D_t(K_{n-1+3}, x) - D_t(K_{n-1}, x) - D_t(K_3, x) + D_t(K_{n-1}, x) + D_t(P_3, x) = D_t(K_{n+2}, x) - D_t(K_3, x) + D_t(P_3, x) = \sum_{i=2}^{n+2} \binom{n+2}{i} x^i - x^2$.
Now by Theorem 2.10, we have the result.

Theorem 3.4. *If J_n is the Jewel graph, then Connected domination polynomial of Jewel graph J_n is*

$D_c(J_n, x) = (1 + x)^{n+4} - (1 + x)^{n+2} - x^2 - 2x.$
Proof. The proof is similar.

Theorem 3.5. *If J_n is the Jewel graph, then Clique domination polynomial of Jewel graph J_n is*

$D_{cl}(J_n, x) = 2x^3 + (2n + 4)x^2.$
Proof. Let $S = \{u, y\}$ is a minimum clique dominating set and so $\gamma_{cl}(J_n) = 2$. For $i = 2$, any subset $D \subseteq V(J_n)$ with $|D| = 2$ containing one of u, v is a clique dominating set. So, $d(J_n, 2) = \binom{n+4}{2} - \binom{n+2}{2} - 1$. For $i = 3$, there are only two sets which are clique dominating namely, $\{u, x, y\}$ and $\{v, x, y\}$. So, $d(J_n, 3) = 2$. Trivially there are no clique dominating sets for $i \geq 4$. Thus $D_{cl}(J_n, x) = 2x^3 + (2n + 4)x^2$.

Remark 3.1. *Since the complement of the Jewel graph is not connected, connected domination polynomial and clique domination polynomial of this graph does not exist.*

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References

[1] Alikhani S., On the domination polynomial of some graph operations, ISRN Combin. (2013), Article ID 146595.

[2] Alikhani S. and Peng Y. H., Introduction to domination polynomial of a graph, Ars Combinatoria, 114 (2014), 257-266.

[3] Arocha J. L. and Llano B., Mean value for the matching and dominating polynomial, Discuss. Math. Graph theory, 20(1) (2000), 57-70.

[4] Chaluvaraju B. and Chaitra V., Total domination polynomial of a graph, Journal of Informatics and Mathematical Sciences, Vol. 6 (2014), 87-92.

[5] Cockayne E. J. and Hedetniemi S. T., Towards a theory of domination in graphs, Networks, 7 (1977), 247-261.

[6] Cozzens M. B. and Kelleher l., Dominating Cliques in graphs, Discrete Mathematics, 86 (1990), 101-116.

[7] Harary F., Graph Theory, Addison-Wesley, Reading-Mass, (1969).

[8] Haynes T. W., Hedetniemi S. T., Slater P. J., Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1998.

[9] Kulli V. R., Theory of domination in graphs, Vishwa international publications, (2010)

[10] Sampathkumar E. and Walikar H. B., The connected domination number of a graph, Jour. Math. Phy. Sci., Vol. 13, No. 6, (1979).

[11] Shivaswamy P. M., Soner N. D. and Alwardi Anwar, Independent Dominating Polynomial in Graphs, IJSIMR Vol. 2, (2014), 757-763.