South East Asian J. of Mathematics and Mathematical Sciences
Vol. 17, No. 2 (2021), pp. 107-122

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

FIXED POINTS IN MENGER SPACE FOR FAINTLY COMPATIBLE, RECIPROCAL CONTINUOUS AND COMPATIBILITY OF TYPE (K) MAPPINGS

Rajesh Shrivastava, Arihant Jain* and Archana Yadav

Department of Mathematics,
Govt. Shyama Prasad Mukherji Science and Commerce College,
Bhopal, Madhya Pradesh - 462008, INDIA

E-mail : rajeshraju0101@rediffmail.com

*School of Studies in Mathematics,
Vikram University, Ujjain, Madhya Pradesh - 456010, INDIA

E-mail : arihant2412@gmail.com

(**Received:** *Sep. 01, 2020* **Accepted:** *May 15, 2021* **Published:** *Aug. 30, 2021)*

Abstract: In this paper, the concept of compatibility of type (K) and faintly compatibility in Menger space has been applied to prove a common fixed point theorem for six self-maps which generalizes the result of Jain et al. [3]. We also give examples in support of our result.

Keywords and Phrases: Menger space, Common fixed points, Reciprocal continuous maps, Compatible maps of type (K) and Faintly compatible.

2020 Mathematics Subject Classification: Primary 47H10, Secondary 54H25.

1. Introduction

The notion of probabilistic metric space (briefly, PM-space) had been coined by Menger [7] in 1942, as a generalization of metric space. Such a probabilistic generalization of metric spaces appears to be well adapted for the investigation of physical quantities and physiological thresholds. It is also of fundamental importance in probabilistic functional analysis. A common fixed point theorem is a statement containing a set of conditions sufficient to ensure the existence of a common fixed point of a number of self maps in a metric space. Most of these theorems mainly contain a commutativity condition, a condition on the ranges of the maps, some continuity conditions and a contractive or possibly a Lipschitz type or a Banach type or a Boyd and Wong type condition. Most of the authors used one or more of these conditions or their various forms and obtained some remarkable success thereby enriching the literature on fixed point theory.

The fixed point theory progress in PM-spaces was due to Schweizer and Sklar [12], Sehgal and Bharucha-Reid [13], Mishra [8], Sharma et al. [15] and Jain et al. [2] in the fields of compatible maps, semi-compatible maps, weak compatible maps and occasionally weak compatible maps. Similarly, progress of the fixed point theory in metric space was due to Jungck and Rhoades [6], Sessa [9] and Jungck [5]. Bisht and Shahzad [1] introduced the notion of faintly compatible maps and also established some interesting common fixed point theorems for non-commuting maps under both contractive and non-contractive conditions.

In the present paper, a fixed point theorem for six self-maps has been proved using the concept of faintly compatibility and compatibility of type (K) which generalizes the result of Jain et al. [3].

2. Preliminaries

Definition 2.1. [8] *A mapping $\mathcal{F} : R \to R^+$ is called a distribution if it is non-decreasing left continuous with* $\inf\{\mathcal{F}(t)|t \in R\} = 0$ *and* $\sup\{\mathcal{F}(t)|t \in R\} = 1$.

We shall denote by L the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & t \le 0 \\ 1, & t > 0. \end{cases}$$

Definition 2.2. [8] *A mapping $t : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if it satisfies the following conditions:*
(t-1) $t(a,1) = a, t(0,0) = 0$;
(t-2) $t(a,b) = t(b,a)$;
(t-3) $t(c,d) \ge t(a,b)$; *for* $c \ge a, d \ge b$,
(t-4) $t\big(t(a,b),c\big) = t\big(a,t(b,c)\big)$ *for all* $a,b,c,d \in [0,1]$.

Definition 2.3. [8] *A probabilistic metric space (PM-space) is an ordered pair $(X,\mathcal{F})$ consisting of a non-empty set X and a function $\mathcal{F} : X \times X \to L$, where L is the collection of all distribution functions and the value of $\mathcal{F}$ at $(u,v) \in X \times X$ is represented by $F_{u,v}$. The function $F_{u,v}$ assumed to satisfy the following conditions:*
(PM-1) $F_{u,v}(x) = 1$, *for all* $x > 0$, *if and only if* $u = v$;
(PM-2) $F_{u,v}(0) = 0$;

(PM-3) $F_{u,v} = F_{v,u}$;
(PM-4) *If* $F_{u,v}(x) = 1$ *and* $F_{v,w}(y) = 1$ *then* $F_{u,w}(x+y) = 1$,
for all $u, v, w \in X$ *and* $x, y > 0$.

Definition 2.4. [8] *A Menger space is a triplet* $(X, \mathcal{F}, t)$ *where* $(X, \mathcal{F})$ *is a PM-space and* t *is a* t*-norm such that the inequality*
(PM-5) $F_{u,w}(x+y) \geq t\big\{F_{u,v}(x), F_{v,w}(y)\big\}$, *for all* $u, v, w \in X, x, y \geq 0$.

Definition 2.5. [12] *A sequence* $\{x_n\}$ *in a Menger space* $(X, \mathcal{F}, t)$ *is said to be convergent and converges to a point* x *in* X *if and only if for each* $\varepsilon > 0$ *and* $\lambda > 0$, *there is an integer* $M(\varepsilon, \lambda)$ *such that*

$$F_{x_n,x}(\varepsilon) > 1 - \lambda \ \text{for all} \ n \geq M(\varepsilon, \lambda).$$

Further, the sequence $\{x_n\}$ *is said to be Cauchy sequence if for* $\varepsilon > 0$ *and* $\lambda > 0$, *there is an integer* $M(\varepsilon, \lambda)$ *such that*

$$F_{x_n,x_m}(\varepsilon) > 1 - \lambda \ \text{for all} \ m, n \geq M(\varepsilon, \lambda).$$

A Menger PM-space $(X, \mathcal{F}, t)$ *is said to be complete if every Cauchy sequence in* X *converges to a point in* X.

A complete metric space can be treated as a complete Menger space in the following way:

Proposition 2.1. [8] *If* (X, d) *is a metric space then the metric* d *induces mappings* $\mathcal{F} : X \times X \to L$, *defined by* $F_{p,q}(x) = H\big(x - d(p,q)\big), p, q \in X$, *where*

$$H(k) = 0, \ \text{for} \ k \leq 0 \ \text{and} \ H(k) = 1, \ \text{for} \ k > 0.$$

Further if, $t : [0,1] \times [0,1] \to [0,1]$ *is defined by* $t(a,b) = \min\{a,b\}$. *Then* $(X, \mathcal{F}, t)$ *is a Menger space. It is complete if* (X, d) *is complete.*
The space $(X, \mathcal{F}, t)$ so obtained is called the induced Menger space.

Definition 2.6. [2] *Self mappings* A *and* S *of a Menger space* $(X, \mathcal{F}, t)$ *are said to be weak compatible if they commute at their coincidence points, i.e.* $Ax = Sx$ *for* $x \in X$ *implies* $ASx = SAx$.

Definition 2.7. [8] *Self mappings* A *and* S *of a Menger space* $(X, \mathcal{F}, t)$ *are said to be compatible if* $F_{ASx_n,SAx_n}(x) \to 1$ *for all* $x > 0$, *whenever* $\{x_n\}$ *is a sequence in* X *such that* $Ax_n, Sx_n \to u$ *for some* u *in* X *as* $n \to \infty$.
Keeping in view of [4], we define the compatible mappings of type (K) in Menger space as follows :

Definition 2.8. *Self mappings* S *and* T *of a Menger space* $(X, \mathcal{F}, t)$ *are said to* *be compatible mappings of type (K) if* $F_{SSx_n,Tt}(x) \to 1$ *and* $F_{TTx_n,St}(x) \to 1$ *for all* $x > 0$*, whenever* $\{x_n\}$ *is a sequence in* X *such that* $Sx_n, Tx_n \to t$ *for some* t *in* X *as* $n \to \infty$*.*

Keeping in view of [1], we define the conditional compatible mappings in Menger space as follows :

Definition 2.9. *Two self-maps* A *and* S *of a Menger space* $(X, \mathcal{F}, t)$ *are said to be conditionally compatible if whenever the set of sequences* $\{x_n\}$ *satisfying* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n$ *is non-empty, there exists a sequence* $\{z_n\}$ *in* X *such that* $\lim_{n\to\infty} Az_n = \lim_{n\to\infty} Sz_n = t$*, for some* $t \in X$ *and* $F_{ASz_n,SAz_n}(x) \to 1$ *as* $n \to \infty$*.*

Keeping in view of [11], we define the reciprocal continuous mappings in Menger space as follows :

Definition 2.10. *Two self maps* A *and* S *of a Menger space* $(X, \mathcal{F}, t)$ *are said to be reciprocally continuous if* $F_{ASx_n,Ax(x)} \to 1$ *and* $F_{SAx_n,Sx}(x) \to 1$ *as* $n \to \infty$*, whenever* $\{x_n\}$ *is a sequence in* X *such that* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$*, for some* $x \in X$*.*

Keeping in view of [1], we define the faintly compatible mappings in Menger space as follows :

Definition 2.11. *Two self maps* A *and* S *of a Menger space* $(X, \mathcal{F}, t)$ *are said to be faintly compatible if* (A, S) *is conditionally compatible and* A *and* S *commute on a non-empty subset of the set of coincidence points, whenever the set of coincidence points is non-empty.*

Example 2.1. Let (X, d) be a usual metric space where $X = [0, 4]$ and $(X, \mathcal{F}, t)$ be the induced Menger space with $F_{x,y} = \frac{t}{t+d(x,y)}$ for all $t > 0$.

Let A, S be the self maps of X given by $Ax = 1$, if $x \leq 1, Ax = 2$, if $x > 1$. $Sx = 2 - x$, if $x \leq 1, Sx = 4$, if $x > 1$.

Let $x_n = 1$. Now $Ax_n \to 1, Sx_n \to 1$ and $ASx_n \to 1, SAx_n \to 1$ and so $F_{ASx_n,SAx_n}(x) \to 1$. Therefore, (A, S) is conditionally compatible. Also $A1 = S1$ and $AS1 = SA1$. Hence, (A, S) is faintly compatible.

Example 2.2. Let (X, d) be a metric space where $X = [0, 1]$ and $(X, \mathcal{F}, t)$ be the induced Menger space with $F_{x,y}(t) = \frac{t}{t+d(x,y)}$ for all $t > 0$.

Define self maps I and L as follows:

$$I(x) = x \text{ for all } x \in X \text{ and } L(x) = \left\{ \begin{array}{ll} x, & \text{if } 0 \leq x < \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} \leq x \leq 1 \end{array} \right\}.$$

Taking $x_n = \frac{1}{2} - \frac{1}{n}$, we get $Ix_n = x_n = \frac{1}{2} - \frac{1}{n}$ and $Lx_n = \frac{1}{2} - \frac{1}{n}$,

$$\lim_{n\to\infty} Lx_n = \frac{1}{2}$$
$$\lim_{n\to\infty} Ix_n = \frac{1}{2}.$$

Also
$$\lim_{n\to\infty} LLx_n = \lim_{n\to\infty} L\left(\frac{1}{2} - \frac{1}{n}\right) = \frac{1}{2},$$
$$\lim_{n\to\infty} IIx_n = \lim_{n\to\infty} I\left(\frac{1}{2} - \frac{1}{n}\right) = \frac{1}{2}.$$

So
$$\lim_{n\to\infty} IIx_n = Lx,$$
$$\lim_{n\to\infty} LLx_n = Ix.$$

Therefore, by definition, (I, L) is compatible mapping of type (K).
Now, we give an example of maps which are faintly compatible but not occasionally weakly compatible.

Example 2.3. Let (X, d) be a metric space where $X = R$ and $(X, \mathcal{F}, t)$ be the induced Menger space with $F_{x,y}(t) = \frac{t}{t+d(x,y)}$ for all $t > 0$.
Define $A, S : R \to R$ by $Ax = x^2$ and $Sx = 2x - x^2$ for all $x \in R$.
Let $x_n = 1$. Now as $n \to \infty, Ax_n \to 1, Sx_n \to 1$ and $ASx_n \to 1, SAx_n \to 1$ and so $F_{ASx_n,SAx_n}(x) \to 1$.
Therefore, $[A, S]$ is conditionally compatible.
Also $A1 = S1$ and $AS1 = SA1$. Hence $[A, S]$ is faintly compatible.
Now, for $x = 0, Ax = Sx$.
For $x = 2, Ax \neq Sx$
For $x = 0, 2, AS0 = SA0$ and $AS2 \neq SA2$.
Hence, $[A, S]$ is faintly compatible but not occasionally weakly compatible.
Now, we give an example of maps which are compatible maps of type (K) but not compatible maps of type (P).

Example 2.4. Let (X, d) be a metric space where $X \in R^+$ and $(X, \mathcal{F}, t)$ be the induced Menger space with $F_{x,y}(t) = \frac{2t^2}{t+|x,y|}$ for all $t > 0,\ x, y \in X$.
Define self maps S and T as follows:

$$S(x) = \left\{ \begin{array}{ll} \frac{1}{3}, & 0 \leq x < \frac{1}{2} \\ \frac{1}{2}, & \frac{1}{2} \leq x \leq 1 \end{array} \right\}.$$

And $Tx = 1 - x$
Take $x_n = \frac{1}{2} + \frac{1}{n}, n \geq 3$.

$$\lim_{n\to\infty} Sx_n = \frac{1}{2}$$
$$\lim_{n\to\infty} Tx_n = \frac{1}{2}$$

$$\lim_{n\to\infty} SSx_n = \lim_{n\to\infty} S\left(\frac{1}{2}\right) = \frac{1}{2}$$

$$\lim_{n\to\infty} TTx_n = \lim_{n\to\infty} T\left(\frac{1}{2}\right) = \frac{1}{2}.$$

So,

$$\lim_{n\to\infty} SSx_n = Tx$$
$$\lim_{n\to\infty} TTx_n = Sx.$$

Hence (S, T) is compatible of type (K).
Now, $\lim_{n\to\infty} F_{SSx_n,TTx_n}(t) = F_{1/2,1/2}(t) = \frac{2t^2}{t+|x+y|} \neq 1$ for $t > 0$.
Hence the pair (S, T) is not compatible of type (P).

Lemma 2.1. [16] *Let $\{x_n\}$ be a sequence in a Menger space $(X, \mathcal{F}, t)$ with continuous t-norm t and $t(a, a) \geq a$. If there exists a constant $k \in (0, 1)$ such that $F_{x_n,x_{n-1}}(kt) \geq F_{x_{n-1},x_n}(t)$ for all $t \geq 0$ and $n = 1, 2, 3, ...$ then $\{x_n\}$ is a Cauchy sequence in X.*

Lemma 2.2. [16] *Let $(X, \mathcal{F}, t)$ be a Menger space. If there exists a constant $k \in (0, 1)$ such that $F_{x,y}(kt) \geq F_{x,y}(t)$ for all $x, y \in X$ and $t > 0$, then $x = y$.*

A class of implicit relation. Let Φ be the set of all real continuous functions $\phi : (R^+)^4 \to R$, non-decreasing in the first argument with the property:
a. For $u, v \geq 0$, $\phi(u, v, v, u) \geq 0$ or $\phi(u, v, u, v) \geq 0$ implies that $u \geq v$.
b. $\phi(u, u, 1, 1) \geq 0$ implies that $u \geq 1$.

Example 2.5. Define $\phi(t_1, t_2, t_3, t_4) = 18t_1 - 16t_2 + 8t_3 - 10t_4$. Then $\phi \in \Phi$.

3. Main Result

Theorem 3.1. *Let A, B, L, M, S and T be self-mappings on a complete Menger space $(X, \mathcal{F}, t)$ with $t(a, a) \geq a$, for some $a \in [0, 1]$, satisfying:*
(3.1.1) *$L(X) \subseteq ST(X), M(X) \subseteq AB(X)$;*
(3.1.2) *$ST(X)$ and $AB(X)$ are complete subspaces of X;*

(3.1.3) *either AB or L is continuous;*
(3.1.4) *(L, AB) is compatible maps of type (K);*
(3.1.5) *(M, ST) is reciprocal continuous and faintly compatible;*
(3.1.6) *for some $\phi \in \Phi$, there exists $k \in (0,1)$ such that for all $x, y \in X$ and $t > 0$,*

$$\phi\Big(F_{Lx,My}(kt), F_{ABx,STy}(t), F_{Lx,ABx}(t), F_{My,STy}(kt)\Big) \geq 0.$$

Then A, B, L, M, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$. From condition (3.1.1) there exist $x_1, x_2 \in X$ such that $Lx_0 = STx_1 = y_0$ and $Mx_1 = ABx_2 = y_1$ Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in Xsuch that

$$Lx_{2n} = STx_{2n+1} = y_{2n} \text{ and } Mx_{2n+1} = ABx_{2n+2} = y_{2n+1} \text{ for } n = 0, 1, 2....$$

Step 1. Putting $x = x_{2n}$ and $y = x_{2n+1}$ in (3.1.6), we get

$$\phi\Big(F_{Lx_{2n},Mx_{2n+1}}(kt), F_{ABx_{2n},STx_{2n+1}}(t), F_{Lx_{2n},ABx_{2n}}(t), F_{Mx_{2n+1},STx_{2n+1}}(kt)\Big) \geq 0.$$

Letting $n \to \infty$, we get

$$\phi\Big(F_{y_{2n},y_{2n+1}}(kt), F_{y_{2n-1},y_{2n}}(t), F_{y_{2n},y_{2n-1}}(t), F_{y_{2n+1},y_{2n}}(kt)) \geq 0.$$

Using (a), we get

$$F_{y_{2n},y_{2n+1}}(kt) \geq F_{y_{2n-1},y_{2n}}(t).$$

Therefore, for all n even or odd, we have

$$F_{y_n,y_{n+1}}(kt) \geq F_{y_{n-1},y_n}(t).$$

Therefore, by Lemma 2.1, $\{y_n\}$ is a Cauchy sequence in X, which is complete. Hence, $\{y_n\} \to z \in X$. Also its subsequence converges as follows:

$$\{L_{x_{2n}}\} \to z, \ \{AB_{x_{2n}}\} \to z, \ \{M_{x_{2n+1}}\} \to z \text{ and } \{ST_{x_{2n+1}}\} \to z.$$

Case 1. When AB is continuous.

Let $\{x_n\}$ be a sequence of X such that

$$\lim_{n\to\infty} ABx_{2n} = \lim_{n\to\infty} Lx_{2n} = z \text{ for some } z \in Z.$$

Then by definition of compatible maps of type (K), we have

$$\lim_{n\to\infty} ABABx_{2n} = Lz.$$

If AB is continuous, $\lim_{n\to\infty} ABABx_{2n} = AB\Big(\lim_{n\to\infty} ABx_{2n}\Big) = ABz$. This implies that $ABz = Lz$.

Step 2. Putting $x = ABx_{2n}$ and $y = x_{2n+1}$ in (3.1.6), we get

$$\phi\Big(F_{LABx_{2n},Mx_{2n+1}}(kt), F_{ABABx_{2n},STx_{2n+1}}(t), F_{LABx_{2n},ABABx_{2n}}(t), F_{Mx_{2n+1},ST_{x_{2n+1}}}(kt)\Big) \geq 0.$$

Letting $n \to \infty$, we get

$$\phi\Big(F_{ABz,z}(kt), F_{ABz,z}(t), F_{ABz,ABz}(t), F_{z,z}(kt)\Big) \geq 0$$
$$\phi\Big(F_{ABz,z}(kt), F_{ABz,z}(t), 1, 1\Big) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi\Big(F_{ABz,z}(t), F_{ABz,z}(t), 1, 1\Big) \geq 0.$$

Using (b), we get

$$F_{ABz,z}(t) \geq 1, \text{ for all } t > 0$$

which gives $ABz = z$.

Step 3. Putting $x = z$ and $y = x_{2n+1}$ in (3.1.6), we get

$$\phi\Big(F_{Lz,Mx_{2n+1}}(kt), F_{ABz,STx_{2n+1}}(t), F_{Lz,ABz}(t), F_{Mx_{2n+1},STx_{2n+1}}(kt)\Big) \geq 0.$$

Letting $n \to \infty$, we get

$$\phi\Big(F_{Lz,z}(kt), F_{ABz,z}(t), F_{Lz,ABz}(t), F_{z,z}(kt)\Big) \geq 0$$
$$\phi\Big(F_{Lz,z}(kt), 1, F_{Lz,z}(t), 1\Big) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi\Big(F_{Lz,z}(t), 1, F_{Lz,ABz}(t), 1\Big) \geq 0.$$

Using (a), we get

$$F_{Lz,z}(t) \geq 1, \text{ for all } t > 0,$$

which gives $z = Lz$. Thus, we have $z = Lz = ABz$.

Step 4. Putting $x = Bz$ and $y = x_{2n+1}$ in (3.1.6), we get

$$\phi\Big(F_{LBz,Mx_{2n+1}}(kt), F_{ABBz,STx_{2n+1}}(t), F_{LBz,ABBz}(t), F_{Mx_{2n+1},STx_{2n+1}}(kt)\Big) \geq 0.$$

Letting $n \to \infty$, we get

$$\phi\Big(F_{Bz,z}(kt), F_{Bz,z}(t), F_{Bz,Bz}(t), F_{z,z}(kt)\Big) \geq 0$$
$$\phi\Big(F_{Bz,z}(kt), F_{Bz,z}(t), 1, 1\Big) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi\Big(F_{Bz,z}(t), F_{Bz,z}(t), 1, 1\Big) \geq 0.$$

Using (b), we have

$$F_{Bz,z}(t) \geq 1, \text{ for all } t > 0,$$

which gives $z = Bz$. Since $z = ABz$, we also have $z = Az$. Therefore, $z = Az = Bz = Lz$.

Step 5. As $L(X) \subseteq ST(X)$, there exists $v \in X$ such that $z = Lz = STv$. Putting $x = x_{2n}$ and $y = v$ in (3.1.6), we get

$$\phi\Big(F_{Lx_{2n},Mv}(kt), F_{ABx_{2n},STv}(t), F_{Lx_{2n},ABx_{2n}}(t), F_{Mv,STv}(kt)\Big) \geq 0.$$

Letting $n \to \infty$, we get

$$\phi\Big(F_{Lz,Mv}(kt), F_{z,STv}(t), F_{z,z}(t), F_{Mv,z}(kt)\Big) \geq 0$$
$$\phi\Big(F_{z,Mv}(kt), 1, 1, F_{Mv,z}(kt)\Big) \geq 0.$$

Using (a), we have

$$F_{z,Mv}(t) \geq 1, \text{ for all } t > 0$$

which gives $z = Mv$. Therefore, $z = Mv = STv$. As (M, ST) is faintly compatible, there is a sequence $\{z_n\}$ in X satisfying $\lim_{n\to\infty} Mz_n = \lim_{n\to\infty} STz_n = v$ for some $v \in X$ such that

$$\lim_{n\to\infty} F_{MSTz_n,STMz_n}(t) = 1.$$

As (M, ST) is reciprocal continuous, we get

$$\lim_{n\to\infty} MSTz_n = Mv, \ \lim_{n\to\infty} STMz_n = STv$$

and so $Mv = STv$. Again as (M, ST) is faintly compatible, we get

$$MSTv = STMv.$$

Thus, $STz = Mz$.

Step 6. Putting $x = x_{2n}$ and $y = z$ in (3.1.6), we get

$$\phi\Big(F_{Lx_{2n},Mz}(kt), F_{ABx_{2n},STz}(t), F_{Lx_{2n},ABx_{2n}}(t), F_{Mz,STz}(kt)) \geq 0.$$

Letting $n \to \infty$, we get

$$\phi(F_{z,Mz}(kt), F_{z,Mz}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi\Big(F_{z,Mz}(t), F_{z,Mz}(t), 1, 1\Big) \geq 0.$$

Using (b), we have

$$F_{z,Mz}(t) \geq 1, \text{ for all } t > 0$$

which gives $z = Mz = STz$.

Step 7. Putting $x = x_{2n}$ and $y = Tz$ in (3.1.6) and using Step 5, we get

$$\phi\Big(F_{Lx_{2n},MTz}(kt), F_{ABx_{2n},STTz}(t), F_{Lx_{2n},ABx_{2n}}(t), F_{MTz,STTz}(kt)\Big) \geq 0.$$

Letting $n \to \infty$, we get

$$\phi\Big(F_{z,T_z}(kt), F_{z,Tz}(t), F_{z,z}(t), F_{Tz,Tz}(kt)\Big) \geq 0$$
$$\phi\Big(F_{z,Tz}(kt), F_{z,Tz}(t), 1, 1\Big) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi\Big(F_{z,Tz}(t), F_{z,Tz}(t), 1, 1\Big) \geq 0.$$

Using (b), we have

$$F_{z,Tz}(t) \geq 1, \text{ for all } t > 0$$

which gives $z = Tz$. Since $Tz = STz$, we also have $z = Sz$.
Hence,

$$Az = Bz = Lz = Mz = Tz = Sz = z.$$

Hence, the six self maps have a common fixed point in this case.

Case 2. When L is continuous.

Let $\{x_n\}$ be a sequence of X such that

$$\lim_{n\to\infty} ABx_{2n} = \lim_{n\to\infty} Lx_{2n} = z \text{ for some } z \in Z.$$

Then by definition of compatible maps of type (K), we have

$$\lim_{n\to\infty} LLx_{2n} = ABz.$$

If L is continuous, $\lim_{n\to\infty} LLx_{2n} = L\Big(\lim_{n\to\infty} Lx_{2n}\Big) = Lz$.
This implies that $ABz = Lz$.

Step 8. Putting $x = z$ and $y = x_{2n+1}$ in (3.1.6), we get

$$\phi\Big(F_{Lz,Mx_{2n+1}}(kt), F_{ABz,STx_{2n+1}}(t), F_{Lz,ABz}(t), F_{Mx_{2n+1},STx_{2n+1}}(kt)\Big) \geq 0.$$

Letting $n \to \infty$, we get

$$\phi\Big(F_{Lz,z}(kt), F_{Lz,z}(t), F_{Lz,Lz}(t), F_{z,z}(kt)\Big) \geq 0$$
$$\phi\Big(F_{Lz,z}(kt), F_{Lz,z}(t), 1, 1\Big) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi\Big(F_{Lz,z}(t), F_{Lz,z}(t), 1, 1\Big) \geq 0.$$

Using (b), we have

$$F_{Lz,z}(t) \geq 1, \text{ for all} t > 0$$

which gives $z = Lz$.
Therefore, $z = Lz = ABz$.

Step 9. Putting $x = Bz$ and $y = x_{2n+1}$ in (3.1.6), we get

$$\phi\Big(F_{LBz,Mx_{2n+1}}(kt), F_{ABBz,STx_{2n+1}}(t), F_{LBz,ABBz}(t), F_{Mx_{2n+1},STx_{2n+1}}(kt)\Big) \geq 0.$$

Letting $n \to \infty$, we get

$$\phi\Big(F_{Bz,z}(kt), F_{Bz,z}(t), F_{Bz,Bz}(t), F_{z,z}(kt)\Big) \geq 0$$
$$\phi\Big(F_{Bz,z}(kt), F_{Bz,z}(t), 1, 1\Big) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi\Big(F_{Bz,z}(t), F_{Bz,z}(t), 1, 1\Big) \geq 0.$$

Using (b), we have

$$F_{Bz,z}(t) \geq 1, \text{ for all} t > 0$$

which gives $z = Bz$.
Since $z = ABz$, we also have

$$z = Az.$$

Therefore, $z = Az = Bz = Lz$.

Step 10. As $L(X) \subseteq ST(X)$, there exists $v \in X$ such that $z = Lz = STv$.
Putting $x = x_{2n}$ and $y = v$ in (3.1.6), we get

$$\phi\Big(F_{Lx_{2n},Mv}(kt), F_{ABx_{2n},STv}(t), F_{Lx_{2n},ABx_{2n}}(t), F_{Mv,STv}(kt)\Big) \geq 0.$$

Letting $n \to \infty$, we get

$$\phi\Big(F_{z,Mv}(kt), F_{z,STv}(t), F_{z,z}(t), F_{Mv,z}(kt)\Big) \geq 0$$
$$\phi\Big(F_{z,Mv}(kt), 1, 1, F_{Mv,z}(kt)\Big) \geq 0.$$

Using (a), we have

$$F_{z,Mv}(kt) \geq 1, \text{ for all } t > 0$$

which gives $z = Mv$.
Therefore, $z = Mv = STv$
As (M, ST) is faintly compatible, there is a sequence $\{z_n\}$ in X satisfying

$$\lim_{n\to\infty} Mz_n = \lim_{n\to\infty} STz_n = v \text{ for some } v \in X \text{ such that}$$
$$\lim_{n\to\infty} F_{MSTz_n,STMz_n}(t) = 1.$$

As (M, ST) is reciprocal continuous, we get

$$\lim_{n\to\infty} MSTz_n = Mv, \lim_{n\to\infty} STMz_n = STv$$

and so $Mv = STv$.
Again as (M, ST) is faintly compatible, we get

$$MSTv = STMv.$$

Thus $STz = Mz$.

Step 11. Putting $x = x_{2n}$ and $y = z$ in (3.1.6), we get

$$\phi\Big(F_{Lx_{2n},Mz}(kt), F_{ABx_{2n},STz}(t), F_{Lx_{2n},ABx_{2n}}(t), F_{Mz,STz}(kt)\Big) \geq 0.$$

Letting $n \to \infty$, we get

$$\phi\Big(F_{z,Mz}(kt), F_{z,Mz}(t), 1, 1\Big) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi\Big(F_{z,Mz}(t), F_{z,Mz}(t), 1, 1\Big) \geq 0.$$

Using (b), we have

$$F_{z,Mz}(t) \geq 1, \text{ for all } t > 0$$

which gives $z = Mz = STz$.

Step 12. Putting $x = x_{2n}$ and $y = Tz$ in (3.1.6) and using Step 5, we get

$$\phi\Big(F_{Lx_{2n},MTz}(kt), F_{ABx_{2n},STTz}(t), F_{Lx_{2n},ABx_{2n}}(t), F_{MTz,STTz}(kt)\Big) \geq 0.$$

Letting $n \to \infty$, we get

$$\phi\Big(F_{z,Tz}(kt), F_{z,Tz}(t), F_{z,z}(t), F_{Tz,Tz}(kt)\Big) \geq 0$$
$$\phi\Big(F_{z,Tz}(kt), F_{z,Tz}(t), 1, 1\Big) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi\Big(F_{z,Tz}(t), F_{z,Tz}(t), 1, 1\Big) \geq 0.$$

Using (b), we have

$$F_{z,Tz}(t) \geq 1, \quad \text{for all } \ t > 0.$$

Thus, $F_{z,Tz}(t) = 1$, we have $z = Tz$.
Since $Tz = STz$, we also have $z = Sz$.
Hence, $Az = Bz = Lz = Mz = Tz = Sz = z$.
Hence, the six self maps have a common fixed point in this case also.

Uniqueness. Let w be another common fixed point of A, B, L, M, S and T, then

$$w = Aw = Bw = Lw = Mw = Sw = Tw.$$

Putting $x = z$ and $y = w$ in (3.1.6), we get

$$\phi\Big(F_{Lz,Mw}(kt), F_{ABz,STw}(t), F_{Lz,ABz}(t), F_{Mw,STw}(kt)\Big) \geq 0.$$

Letting $n \to \infty$, we get

$$\phi\Big(F_{z,w}(kt), F_{z,w}(t), F_{z,z}(t), F_{w,w}(kt)\Big) \geq 0$$
$$\phi\Big(F_{z,w}(kt), F_{z,w}(t), 1, 1\Big) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi\Big(F_{z,w}(t), F_{z,w}(t), 1, 1\Big) \geq 0.$$

Using (b), we have

$$F_{z,w}(t) \geq 1, \text{ for all } t > 0.$$

Thus, $F_{z,w}(t) = 1$,
i.e., $z = w$.
Therefore, z is a unique common fixed point of A, B, L, M, S and T.
This completes the proof.
On taking $B = T = I$ (the identity map) on X in Theorem 3.1, we get the following corollary.
Corollary 3.1. *Let A, L, M and S be self-mappings on a complete Menger space $(X, \mathcal{F}, t)$ with $t(a, a) \geq a$, for some $a \in [0, 1]$, satisfying :*
(3.1.7) *$L(X) \subseteq S(X)$, $M(X) \subseteq A(X)$;*
(3.1.8) *$S(X)$ and $A(X)$ are complete subspaces of X;*
(3.1.9) *either A or L is continuous;*
(3.1.10) *(L, A) is compatible maps of type (K);*
(3.1.11) *(M, S) is reciprocal continuous and faintly compatible;*
(3.1.12) *for some $\phi \in \Phi$, there exists $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$,*

$$\phi\Big(F_{Lx,My}(kt), F_{Ax,Sy}(t), F_{Lx,Ax}(t), F_{My,Sy}(kt)\Big) \geq 0.$$

Then A, L, M and S have a unique common fixed point in X.

Now, we give an example of Corollary 3.1.

Example 3.1. Let (X, d) be a metric space where $X = [2, 20]$ and $(X, \mathcal{F}, t)$ be the induced Menger space with $F_{x,y}(t) = \frac{t}{t+d(x,y)}$ for all $t > 0$.
Define A, L, M and $S : X \to X$ as follows:

$$A(x) = \begin{cases} 2, & x = 2 \\ 6, & x > 2 \end{cases}, \qquad L(x) = \begin{cases} 2, & x = 2 \\ 3, & x > 2 \end{cases},$$

$$M(x) = \begin{cases} 2, & x = 2 \text{ or } x > 5 \\ 6, & 2 < x \leq 5 \end{cases} \quad \text{and} \quad S(x) = \begin{cases} 2, & x = 2 \\ 12, & 2 < x \leq 20 \\ x - 5, & x > 5 \end{cases}.$$

Then, for the constant sequence $\{x_n\} = 2$, the pair (M, S) is reciprocally continuous and faintly compatible mappings and (L, A) is compatible mapping of type (K). Also, these mappings satisfy all the conditions of the above corollary and have a unique common fixed point $x = 2$.

4. Conclusion

Theorem 3.1 is a generalization of the result of Jain et al. [3] in the sense that the conditions of compatible maps of type (P) and occasionally weakly compatible have been replaced by compatible maps of type (K) and faintly compatible mappings.

Acknowledgement

Authors are thankful to the anonymous referees for their valuable comments for improving the presentation of the paper.

References

[1] Bisht, R. K. and Shahzad, N., Faintly compatible mappings and common fixed points, Fixed point theory and applications, 156 (2013), 1-9.

[2] Jain, A. and Chaudhary, B., On common fixed point theorems for semi - compatible and occasionally weakly compatible mappings in Menger space, Int. J. Res. Rev. Applied Sciences, 14 (3), (2013), 662-670.

[3] Jain, A., Gupta, V. K., Badshah, V. H. and Sharma, A. K., Fixed points in Menger space for occasionally weakly compatible and compatibility of type (P), Int. J. Adv. Tech. Engg. Science, 7 (2014), 102-111.

[4] Jha, K., Popa, V. and Manandhar, K. B., A common fixed point theorem for compatible mapping of type (K) in metric space, Int. J. Math. Sci. Eng. Appl., 8(1) (2014), 383-391.

[5] Jungck, G., Compatible mappings and common fixed points, Int. J. Math. Math. Sci., 9(4) (1986), 771-779.

[6] Jungck, G. and Rhoades, B. E., Fixed points for set valued functions without continuity, Indian J. Pure Appl. Math., 29 (1998), 227-238.

[7] Menger, K., Statistical metrics, Proc. Nat. Acad. Sci. USA, 28 (1942), 535-537.

[8] Mishra, S. N., Common fixed points of compatible mappings in PM-spaces, Math. Japon., 36(2) (1991), 283-289.

[9] Pant, B. D. and Chauhan, S., Common fixed point theorems for semi - compatible mappings using implicit relation, Int. J. Math. Anal., 3(28) (2009), 1389-1398.

[10] Pant, R. P., A common fixed point theorem under a new condition, Indian J. Pure Appl. Math. 30(2) (1999), 147-152.

[11] Pant, R. P. and Bisht, R. K., Occasionally weakly compatible mappings and fixed points, Bull. Belg. Math. Soc. Simon Stevin, 19 (2012), 655-661.

[12] Schweizer, B. and Sklar, A., Statistical metric spaces, Pacific J. Math., 10 (1960), 313-334.

[13] Sehgal, V. M. and Bharucha-Reid, A. T., Fixed points of contraction maps on probabilistic metric spaces, Math. System Theory, 6 (1972), 97- 102.

[14] Sessa, S., On a weak commutativity condition of mappings in fixed point consideration, Publ. Inst. Math. Beograd, 32(46) (1982), 146-153.

[15] Sharma, A., Jain, A. and Choudhari, S., Weak compatibility in Menger spaces, Int. J. Algebra, 5(18) (2011), 855-865.

[16] Singh, S. L. and Pant, B. D., Common fixed point theorems in probabilistic metric spaces and extension to uniform spaces, Honam. Math. J., 6 (1984), 1-12.