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FIXED POINT OF KANNAN RESULTS IN FUZZY METRIC SPACES

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Abstract: In this paper we prove some fixed point theorems of R. Kannan [Bull. Calcutta Math. Soc., 60(1968), 71-76] in fuzzy metric spaces in the sense of I. Kramosil and J. Michalek, [Kybernetica, 11(1975), 336-344]. The mappings are used as defined by V. Gregori and A. Sapena [Fuzzy Sets and Systems, 125(2002), 245-252].

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1. Introduction

The notion of fuzzy set was introduced by Zadeh [15]. Subsequently many authors used the fuzzy concept in various fields and in this fuzzy Mathematics, fuzzy metric space plays a vital role. The notion of fuzzy metric spaces has been introduced in different ways by many authors (refer [3], [10], [16]). George and Veeramani [2] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [10] and also defined the Hausdorff topology of fuzzy metric space and consequently they showed every metric induces a fuzzy metric. Grabiec [5] proved the contraction principle in the setting of fuzzy metric space introduced by Kramosil and Michalek [10]. Gregori and Sapena [7] obtained fixed point results for fuzzy metric spaces in the sense of George and Veeramani [2] and also for Kramosill and Michalek's fuzzy metric space which is complete in Grabiec's sense. Imdad and Ali [8] introduced the notion R-weakly commuting of type (P) and generalized some fixed point results in fuzzy metric space. In this paper we prove some fixed point theorems of Kannan ([9, Theorems 3 and 4]) on fuzzy metric spaces in the sense of Kramosil and Michalek [10] and the mappings are defined by Gregori and Sapena [7].

Definition 1.1. [11] *A binary operation* $* : [0,1] \times [0,1] \longrightarrow [0,1]$ *is a continuous* t*-norm if* $([0,1], *)$ *is a topological monoid with unit 1 such that* $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$ *and* $a, b, c, d \in [0,1]$.

Definition 1.2. [10] *The 3-tuple* $(X, M, *)$ *is said to be a fuzzy metric space if* X *is an arbitrary set,* $*$ *is a continuous* t*-norm and* M *is a fuzzy set on* $X^2 \times [0, \infty)$ *satisfying the following conditions:*

(i) $M(x, y, 0) = 0,$

(ii) $M(x, y, t) = 1$ *for all* $t > 0$ *iff* $x = y,$

(iii) $M(x, y, t) = M(y, x, t),$

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s),$

(v) $M(x, y, .) : [0, \infty) \to [0, 1]$ *is left continuous*

(vi) $\lim_{t\to\infty} M(x, y, t) = 1$

Definition 1.3. [5] *A sequence* $\{x_n\}$ *in a fuzzy metric space* $(X, M, *)$ *is called Cauchy sequence (G- Cauchy sequence) if* $lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ *for all* $t > 0$ *and* $p > 0$.

Definition 1.4. [5] *A sequence* $\{x_n\}$ *in a fuzzy metric space* $(X, M, *)$ *is said to converge to* x *if* $lim_{n\to\infty} M(x_n, x, t) = 1$ *for all* $t > 0$.

Definition 1.5. [5] *A fuzzy metric space* $(X, M, *)$ *is called complete if for every Cauchy sequence in* X *converges in* X.

Definition 1.6. [5] *A pair of self mappings (S, T) of a fuzzy metric space* $(X, M, *)$ *is called R-weakly commuting of type (P) if there exists* $R > 0$ *such that* $M(SSx, TTx, t) \geq M(Sx, Tx, t/R)$ *for all* $x \in X$ *and* $t > 0$.

Definition 1.7. [7] *Let* $(X, M, *)$ *be a fuzzy metric space. A mapping* $, T : X \to X$ *is called fuzzy contractive mapping if there exists* $k \in (0, 1)$ *such that*
$\frac{1}{M(Tx,Ty,t)} - 1 \leq k(\frac{1}{M(x,y,t)} - 1)$ *for all* $x, y \in X,$ *and* $t > 0$.

Theorem 1.8. [7] *Let $(X, M, *)$ be a G-complete fuzzy metric space and let $T : X \to X$ be a fuzzy contractive mapping. Then T has a unique fixed point.*

In this paper, $(X, M, *)$ will be fuzzy metric space in Kramosil and Michalek's sense and the results are proved in Grabiec's sense.

2. Main Results

The following theorem shows that existence of unique common fixed point for pair of contractive type mappings.

Theorem 2.1. *Let $(X, M, *)$ be a complete fuzzy metric space with $t * t \geq t$ and let T and S be any two self mappings on X such that*

(i) $\frac{1}{M(Tx,Sy,t)} - 1 \leq k(\frac{1}{M(x,y,t)} - 1)$ *for all $x, y \in X$, $x \neq y$ and $0 < k < 1$.*

(ii) there exists α with $0 < \alpha < 1$ such that $\frac{1}{M(Sx,Sy,t)} - 1 \leq \alpha(\frac{1}{M(x,y,t)} - 1)$ *for all $x, y \in X$.*

Then T and S have a unique common fixed point in X.

Proof. Construct a sequence $\{x_n\}$ in X such that $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$.
Our claim is that T and S have a unique common fixed point in X.
Suppose that T and S have distinct fixed points x_1 and x_2 in X. That is, $x_1 = Tx_1$ and $x_2 = Sx_2$. Then

$$\frac{1}{M(x_1, x_2, t)} - 1 = \frac{1}{M(Tx_1, Sx_2, t)} - 1 \leq k(\frac{1}{M(x_1, x_2, t)} - 1)$$

This is a contradiction to the fact that $M(x, y, .)$ is non decreasing.
Since S is a contraction by Theorem 1.8, S has a unique fixed point y in X.
Therefore if T has a fixed point, then it should coincide with that of S.
let us assume that $x_{2n} \neq x_{2n+1}$. For if $x_{2n} = x_{2n+1}$, then T has a fixed point. Let x_{2n} and x_{2n+1}be any two members of construction of sequence $\{x_n\}$ in X. Then it follows

$$\begin{aligned} \frac{1}{M(x_{2n+1}, x_{2n+2}, t)} - 1 &= \frac{1}{M(Tx_{2n}, Sx_{2n+1}, t)} - 1 \\ &\leq k(\frac{1}{M(x_{2n}, x_{2n+1}, t)} - 1) \leq \ldots \\ &\leq k^{2n}(\frac{1}{M(x_0, x_1, t)} - 1) \to 0 \quad \text{as} \quad n \to \infty \end{aligned}$$

Hence $\lim_n M(x_n, x_{n+1}, t) = 1$ for all $t > 0$
Then for a fixed positive integer p we have

$$M(x_n, x_{n+p}, t) \geq M(x_n, x_{n+1}, \frac{t}{p}) * \cdots * M(x_{n+p-1}, x_{n+p}, \frac{t}{p}) \to 1 * \cdots * 1 = 1$$

Therefore $\{x_n\}$ is a G-Cauchy sequence. Since X is complete, $\{x_n\}$ converges to $x \in X$.
Our claim is that T has a fixed point x in X. Now,

$$\frac{1}{M(x_{2n+2}, Tx, t)} - 1 = \frac{1}{M(Sx_{2n+1}, Tx, t)} - 1 \leq k(\frac{1}{M(x_{2n+1}, x, t)} - 1)$$

Letting $n \to \infty$ on both sides implies
$M(x, Tx, t) \geq 1$. Hence $Tx = x$
Therefore T has a fixed point x in X.
Thus S and T have a unique common fixed point in X.

Remark 2.2. *Theorem 2.1 is a corollary of Theorem 1.8 if we take $T = S$.*

The above theorem obtains only sufficient conditions for existence of a common fixed point for T and S, but the conditions are not necessary as shown in the following example.

Example 2.3. Let $X = [0, 1]$ with $d(x, y) = |x - y|$ for all $x, y \in X$.
Define $T, S : X \to X$ by $T(x) = 1 - x, S(x) = \frac{3}{4} - \frac{x}{2}$ for all $x \in X$.
Let $M(x.y, t) = \frac{t}{t+d(x,y)}$ for all $x \in X$ and $t > 0$. Here $T(\frac{1}{2}) = S(\frac{1}{2}) = \frac{1}{2}$ is the only fixed point of both T and S in X. Further
$\frac{1}{M(S(x),S(y),t)} - 1 = \alpha(\frac{1}{M(x,y,t)} - 1)$ is true for $\alpha = 1/2$. Now
For $k = 1/4, x = 2/5$ and $y = 1/3$, $\frac{1}{M(T(x),S(y),t)} - 1 = k(\frac{1}{M(x,y,t)} - 1)$
Hence the conditions of the above theorem are not necessary.

The following theorem provides sufficient condition for existence of fixed point in a non complete fuzzy metric space.

Theorem 2.4. *Let $(X, M, *)$ be a metric space and let T be a self mapping on X such that*

(i) $\frac{1}{M(Tx,Ty,t)} - 1 \leq k(\frac{1}{M(x,y,t)} - 1)$ *for all $x, y \in X$.*

(ii) T is continuous at x in X.

(iii) there exists a point x_0 in X such that the sequence of iterates $\{T^n x_0\}$ has a subsequence $\{T^{n_i} x_0\}$ converging to a point x in X.

Then x is a unique fixed point of T in X.

Proof. Let $x_0 \in X$ be as in (iii)
Then for the sequence $\{x_n\}$ in X such that $x_n = T^n x_0$ for all $n \in N$.
Since $\{x_{n_i}\} \to x$ and T is continuous at x, $Tx_{n_i} \to Tx$ as $i \to \infty$. Now

$$\frac{1}{M(Tx_{n_i}, Tx, t)} - 1 \leq k(\frac{1}{M(x_{n_i}, x, t)} - 1) \to 0 \quad \text{as} \quad i \to \infty.$$

Hence $\lim_n M(Tx_{n_i}, Tx, t) = 1$ for each $t > 0$.
Therefore $\lim_n x_{n_{i+1}} = T(x)$.
Thus $T(x) = x$.
For uniqueness, let y be another fixed point of T in X. Then for $t > 0$ we have

$$\begin{aligned}
\frac{1}{M(x,y,t)} - 1 &= \frac{1}{M(Tx,Ty,t)} - 1 \leq k(\frac{1}{M(x,y,t)} - 1) \\
&= k(\frac{1}{M(Tx,Ty,t)} - 1) \\
&\leq k^2(\frac{1}{M(x,y,t)} - 1) \leq \ldots \\
&\leq k^n(\frac{1}{M(x,y,t)} - 1) \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Hence x is a unique fixed point of T in X.

The next theorem ascertains sufficient condition for existence of fixed point in everywhere dense subset of a fuzzy metric space.

Theorem 2.5. *Let $(X, M, *)$ be a fuzzy metric space. If T is continuous self mappings on X such that*

(i) $\frac{1}{M(Tx,Ty,t)} - 1 \leq k(\frac{1}{M(x,y,t)} - 1)$ *for all $x, y \in E$, $k \in (0,1)$, where E denotes an every where dense subset of X.*

(ii) there exists a point $x_0 \in X$ such that the sequence of iterates $T^n(x_0)$ has a subsequence $\{T^{n_i}x_0\}$ converging to a point $x \in X$.

(iii) If $\{x_n\}$ and $\{y_n\}$ are any two sequences in E such that $x_n \to x$ and $y_n \to y$ in E as $n \to \infty$, then $M(x_n, y_n, t) \to M(x, y, t)$ as $n \to \infty$ for all $t > 0$. Then T has a unique fixed point in X.

Proof. The proof is true for all $x, y \in E$ by Theorem 2.4. So it is enough to prove this result is true for all $x, y \in X - E$.

Let $x, y \in X - E$. Then there exist sequences $\{x_n\}$ and $\{y_n\}$ in E such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Since T is continuous, $T(x_n) \to T(x)$ and $T(y_n) \to T(y)$ as $n \to \infty$.

$$\begin{aligned} \frac{1}{M(Tx, Ty, t)} - 1 &= \lim_{n\to\infty} \frac{1}{M(Tx_n, Ty_n, t)} - 1 \\ &\leq k \lim_{n\to\infty} (\frac{1}{M(x_n, y_n, t)} - 1) \\ &= k(\frac{1}{M(x, y, t)} - 1) \end{aligned}$$

This holds for any points $x, y \in X - E$.
There fore all conditions of Theorem 2.4 are satisfied. Hence the proof follows from Theorem 2.4.

The following theorem ascertains existence of fixed point by using R-weakly commuting of type(P) in non complete fuzzy metric space.

Theorem 2.6. *Let $(X, M, *)$ be a fuzzy metric space and K a nonempty subset of X. If S, T are R- weakly commuting self mapping of type(P) of satisfying the condition $\frac{1}{M(Sx,Sy,t)} - 1 \leq r(\frac{1}{M(Tx,Ty,t)} - 1)$ for $x, y \in K$, where $r : (0, \infty) \to (0, \infty)$ is a continuous function such that $r(s) < s$ for each $s > 0$, $r(0) = 0$ and $S(K) \subseteq T(K)$, $S(K)$ is complete and either S or T is continuous then S and T have unique common fixed point in X.*

Proof. Let $x_0 \in K$ be arbitrary. Choose $x_1 \in K$ such that $Sx_0 = Tx_1$.In general choose $x_{n+1} \in K$ such that $Sx_n = Tx_{n+1}$ for $n = 0, 1, 2....$

$$\begin{aligned} \frac{1}{M(Sx_n, Sx_{n+1}, t)} - 1 &\leq r(\frac{1}{M(Tx_n, Tx_{n+1}, t)} - 1) \\ &< \frac{1}{M(Tx_n, Tx_{n+1}, t)} - 1 \\ &= \frac{1}{M(Sx_{n-1}, Sx_n, t)} - 1 \end{aligned}$$

Therefore $\frac{1}{M(Sx_n,Sx_{n+1},t)} - 1 < \frac{1}{M(Sx_{n-1},Sx_n,t)} - 1$ and
$M(Sx_n, Sx_{n+1}, t) > M(Sx_{n-1}, Sx_n, t)$
Thus $\{M(Sx_n, Sx_{n+1}, t) : n \geq 0\}$ is an increasing sequence of positive real numbers in $[0, 1]$ and it tends to limit $L \leq 1$. Our claim is to show that $L = 1$. For if $L < 1$, $\frac{1}{M(Sx_n,Sx_{n+1},t)} - 1 \leq \frac{1}{M(Tx_n,Tx_{n+1},t)} - 1 < \frac{1}{M(Sx_{n-1},Sx_n,t)} - 1$.
Taking limit as $n \to \infty$ implies that $\frac{1}{L} - 1 \leq r(\frac{1}{L} - 1) < \frac{1}{L} - 1$, a contradiction. Hence $L = 1$ and $M(Sx_n, Sx_{n+1}, t) \to 1$ as $n \to \infty$. Next we prove that $\{Sx_n\}$ is

a Cauchy sequence in $S(K)$.
For each $t > 0$ and for positive integer m,

$$\begin{aligned} M(Sx_n, Sx_{n+m}, t) &\geq M(Sx_n, Sx_{n+1}, t/m) * \ldots * M(Sx_{n+m-1}, Sx_{n+m}, t/m) \\ &\geq 1 * 1 * 1 \ldots * 1 \geq 1 \end{aligned}$$

Hence $\{Sx_n\}$ is a Cauchy sequence in $S(K)$ and it converges to z in $S(K)$. By the definition T, $T(x_n) \to z$ as $n \to \infty$. we assume that S is continuous, then $SSx_n \to Sz$ as $n \to \infty$. Since the pair $\{S, T\}$ is R-weakly commuting of type(P), it follows that $M(TTx, SSx, t) \geq M(Tx, Sx, t/R)$. Letting $n \to \infty, TTx_n \to Sz$. our claim is to show that $z = Sz$.

$$\begin{aligned} \frac{1}{M(z, Sz, t)} - 1 &= \frac{1}{lim_{n\to\infty} M(Sx_n, SSx_n, t)} - 1 \\ &\leq lim_{n\to\infty} r(\frac{1}{M(Tx_n, TSx_n, t)} - 1) \\ &< lim_{n\to\infty} \frac{1}{M(Tx_n, TSx_n, t)} - 1 \\ &= lim_{n\to\infty} \frac{1}{M(Tx_n, TTx_{n+1}, t)} - 1 \\ &= \frac{1}{M(z, Sz, t)} - 1 \quad \text{as} \quad n \to \infty \end{aligned}$$

which is a contradiction and hence $z = Sz$
Since $S(K) \subseteq T(K)$, there exists $z_1 \in K$ such that $z = Sz = Tz_1$

$$\begin{aligned} \frac{1}{M(SSx_n, Sz_1, t)} - 1 &\leq \frac{1}{M(TSx_n, Tz_1, t)} - 1 \\ &< \frac{1}{M(TSx_n, Tz_1, t)} - 1 \\ &= \frac{1}{M(TTx_{n+1}, Tz_1, t)} - 1 \end{aligned}$$

Letting $n \to \infty$ we get $M(Sz, Sz_1, t) > M(Sz, Sz_1, t)$, a contradiction.
Hence $Sz = Sz_1$ and it follows that $z = Sz = Sz_1 = Tz_1$.
Now for any $t > 0$,

$$\begin{aligned} \frac{1}{M(Sz, Tz, t)} - 1 &= \frac{1}{M(SSz_1, TTz_1, t)} - 1 \\ &\leq \frac{1}{M(Sz_1, Tz_1, t/R)} - 1 \end{aligned}$$

This implies that $z = Sz = Tz$.
Thus z is the common fixed point of S and T.

Example 2.7. Let $X = [-1, 2]$. Mappings S and T are defined by

$$S(x) = \begin{cases} 1, & -1 \leq x \leq 1, \\ \frac{2-x}{4}, & 1 < x \leq 2 \end{cases}$$

$$T(x) = \begin{cases} \frac{3}{4}, & -1 \leq x < 1, \\ 1, & x = 1 \\ \frac{3-x}{2}, & 1 < x \leq 2 \end{cases}$$

Define $M(x, y, t) = \frac{t}{t+|x-y|}$ and take $K = [-1, 1]$
For $-1 < x < 1$, $Sx = 1, Tx = \frac{3}{4}$, $SSx = 1, TTx = \frac{3}{4}$.
Then $M(SSx, TTx, t) \geq M(Sx, Tx, \frac{t}{R})$ for all $R \geq 1$.
For $x = 1$, $Sx = 1, Tx = 1, SSx = 1, TTx = 1$.
Then $M(SSx, TTx, t) \geq M(Sx, Tx, \frac{t}{R})$ for all $R > 0$.
For $1 < x \leq 2$,$Sx = \frac{2-x}{4}, T(x) = \frac{3-x}{2}, SSx = 1, TTx = \frac{3}{4}$.
Then $M(SSx, TTx, t) \geq M(Sx, Tx, \frac{t}{R})$ for all $R \geq 1$.
Hence S and T are R-weakly commuting of type (P) with $R = 1$.
Here $T(K) = [\frac{3}{4}, 1] \supset \{1\} = S(K)$. Further

$$\frac{1}{M(Tx, Ty, t)} - 1 = 0 \leq r(\frac{1}{M(Sx, Sy, t)} - 1) \qquad (1)$$

and $T(K)$ is complete.Thus all the conditions of Theorem 2.6 are satisfied and 1 is the unique common fixed point of S and T in X.

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