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BASIC ANALOGUE OF STIELTJES TRANSFORM AND ITS PROPERTIES

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Abstract: In this paper, basic analogue of Stieltjes transform has been established. Properties of basic Stieltjes transform have been also discussed.

Keywords and Phrases: Stieltjes transform, Gauss's hypergeometric series, basic hypergeometric series, ordinary binomial theorem, basic binomial theorem.

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1. Introduction, Notations and Definitions

Gaussian hypergeometric series is defined as,

$$ {}_2F_1\left[a,b;c;z\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n n!}, \tag{1.1}$$

where $(a)_n = a(a+1)...(a+n-1) = \dfrac{\Gamma(a+n)}{\Gamma(a)}$, and $(a)_0 = 1$.

For the convergence of the series (1.1), $|z| < 1$ is needed.

Confluent hypergeometric series is defined as,

$$ {}_1F_1[a;b;z] = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!}, \tag{1.2}$$

where for convergence $|z| < \infty$ i.e. the series (1.2) is convergent in the whole complex plane. Binomial theorem is defined as,

$$ {}_1F_0[a;-;z] = (1-z)^{-a}, \tag{1.3}$$

provided $|z| < 1$.
Basic hypergeometric series is defined as,

$$\begin{aligned} {}_2\Phi_1[a,b,c;q;z] &= {}_2\Phi_1\left[\begin{array}{l} a,b;q;z \\ c \end{array}\right] \\ &= \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n z^n}{(c;q)_n (q;q)_n}, \qquad |z|<1, \end{aligned} \tag{1.4}$$

where $(a;q)_n = (1-a)(1-aq)...(1-aq^{n-1}),\ n = 1,2,3,...$ and $(a;q)_0 = 1,\ (a;q)_\infty = \prod_{r=0}^{\infty}(1-aq^r)$. ${}_2\Phi_1$ series given in (1.4) is the basic analogue of ${}_2F_1$ series given in (1.1). Taking $b = c$ in (1.4) we have,

$$ {}_1\Phi_0[a;-;q;z] = \sum_{n=0}^{\infty} \frac{(a;q)_n z^n}{(q;q)_n}, \quad |z|<1, \tag{1.5}$$

which is the basic analogue of (1.3).
Basic binomial theorem is given as,

$$ {}_1\Phi_0[a;-;q;z] = \sum_{n=0}^{\infty} \frac{(a;q)_n z^n}{(q;q)_n} = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad \text{where} \quad |z|<1. \qquad (1.6),$$

2. Generalized Stieltjes Transform and its Basic Analogue

The Stieltjes transform comes out naturally by repeated application of Laplace transform. If

$$ g(s) = \int_0^{\infty} e^{-su}\Phi(u)du, \tag{2.1}$$

where

$$ \Phi(u) = \int_0^{\infty} e^{-ut}g(t)dt, \tag{2.2}$$

then

$$g(s) = \int_0^\infty e^{-su} \left\{ \int_0^\infty e^{-ut} g(t) dt \right\} du. \tag{2.3}$$

If the integral are convergent then

$$\begin{aligned} g(s) &= \int_0^\infty g(t) \left\{ \int_0^\infty e^{-(s+t)u} du \right\} dt \\ &= \int_0^\infty \left[-\frac{e^{-(s+t)u}}{(s+t)} \right]_0^\infty g(t) dt \\ &= \int_0^\infty \frac{g(t)}{s+t} dt \end{aligned} \tag{2.4}$$

(2.4) is the special case of the general Stieltjes transform, which is defined as

$$g_\alpha(s) = \int_0^\infty \frac{g(t)}{(s+t)^\alpha} dt, \tag{2.5}$$

provided $g(t)$ is bounded in the interval $[0, \infty]$ and $Re(\alpha) > 0$.
Taking $z = -\frac{t}{s}$ in (1.3) we have

$${}_1F_0 \left[\alpha; -; -\frac{t}{s} \right] = \left(1 + \frac{t}{s} \right)^{-\alpha} = \frac{s^\alpha}{(s+t)^\alpha}, \quad \text{provided} \quad \left| \frac{t}{s} \right| < 1. \tag{2.6}$$

From (2.5) and (2.6) we have

$$g_\alpha(s) = s^{-\alpha} \int_0^\infty {}_1F_0 \left[\alpha; -; -\frac{t}{s} \right] g(t) dt. \tag{2.7}$$

(2.7) can also be expressed as

$$g_\alpha(s) = s^{-\alpha} \int_0^\infty {}_2F_1 \left[\alpha, \beta; \beta; -\frac{t}{s} \right] g(t) dt. \tag{2.8}$$

Replacing ${}_1F_0$ series in (2.7) by its basic analogue (1.5), we get the basic Stieltjes transform as

$$g_{q,\alpha}(s) = s^{-\alpha} \int_0^\infty {}_1\Phi_0 \left[\alpha; -; q; -\frac{t}{s} \right] g(t) d_q(t). \tag{2.9}$$

The integral in (2.9) is $q-$integarl defined as [Gasper and Rahman: 1; (1.11.4), p.10],

$$\int_0^\infty f(t) d_q t = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^n. \tag{2.10}$$

3. Certain Properties of Basic Stieltjes Transform

In this section we shall discuss certain properties of $q-$Stieltjes transform. Making use of (1.6) in (2.9) we find,

$$g_{q,\alpha}(s) = s^{-\alpha} \int_0^\infty \frac{\left(-\frac{\alpha t}{s};q\right)_\infty}{\left(-\frac{t}{s};q\right)_\infty} g(t) d_q t. \tag{3.1}$$

Making use of (2.10) in (3.1) we obtain

$$\begin{aligned} g_{q,\alpha}(s) &= \frac{(1-q)}{s^\alpha} \sum_{n=-\infty}^{\infty} \frac{\left(-\frac{\alpha}{s}q^n;q\right)_\infty}{\left(-\frac{q^n}{s};q\right)_\infty} g(q^n)q^n \\ &= \frac{(1-q)}{s^\alpha} \frac{\left(-\frac{\alpha}{s};q\right)_\infty}{\left(-\frac{1}{s};q\right)_\infty} \sum_{n=-\infty}^{\infty} \frac{\left(-\frac{1}{s};q\right)_n}{\left(-\frac{\alpha}{s};q\right)_n} g(q^n)q^n. \end{aligned} \tag{3.2}$$

For $\alpha = 1$, (3.2) yields

$$g_{q,1}(s) = \frac{(1-q)}{s} \sum_{n=-\infty}^{\infty} g(q^n)q^n. \tag{3.3}$$

For $\alpha = q$, (3.2) gives

$$\begin{aligned} g_{q,q}(s) &= \frac{(1-q)}{s^q\left(1+\frac{1}{s}\right)} \sum_{n=-\infty}^{\infty} \frac{\left(1+\frac{1}{s}\right)g(q^n)}{\left(1+\frac{q^n}{s}\right)} q^n \\ &= \frac{(1-q)}{s^q} \sum_{n=-\infty}^{\infty} \frac{g(q^n)}{\left(1+\frac{q^n}{s}\right)} q^n. \end{aligned} \tag{3.4}$$

Putting $\alpha = 0$ in (3.2) we find,

$$g_{q,0}(s) = \frac{(1-q)}{\left(-\frac{1}{s};q\right)_\infty} \sum_{n=-\infty}^{\infty} \left(-\frac{1}{s};q\right)_n g(q^n)q^n. \tag{3.5}$$

These are certain interesting special cases of basic Stieltjes transform. For details about the Stieltjes transform one is referred [2].

References

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[2] Murrey Rognlie D., Generalized integral transforms, Ph.D. thesis submitted to Lowa State University, Ames, Lowa (1969).