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# SUMMATION FORMULAS FOR FOURTH KIND APPELL'S FUNCTION HAVING DIFFERENT ARGUMENTS

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**Abstract:** The objective of this paper is to find the closed form of the summation theorems for Appell's hypergeometric function $F_4$ having the arguments $\frac{-1}{32}, \frac{1}{16}, \frac{\pm i}{6}$, with suitable convergence conditions.

**Keywords and Phrases:** Gauss hypergeometric function; Appell's double hypergeometric function of Fourth kind.

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## 1. Introduction and Preliminaries

In the usual notation, let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. Also let

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \quad , \qquad \mathbb{N} := \{1, 2, 3, \dots\} = \mathbb{N}_0 \setminus \{0\} \quad ,$$

$$\mathbb{Z}_0^- := \{0, -1, -2, \dots\} = \mathbb{Z}^- \cup \{0\} \quad , \qquad \mathbb{Z}^- := \{-1, -2, -3, \dots\}$$

and $\mathbb{Z} = \mathbb{Z}_0^- \cup \mathbb{N}$ being the sets of integers.

In terms of Gamma function, the widely-used Pochhammer symbol $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ is defined, in general, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 \,, & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \,, \\ \lambda(\lambda+1)\dots(\lambda+n-1) \,, & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}) \,, \end{cases} \tag{1.1}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the $\Gamma$ quotient exists

$$\int_0^\infty e^{-st} t^{\alpha-1} \mathrm{d}t = \frac{\Gamma(\alpha)}{s^\alpha}, \tag{1.2}$$

$$\Big(\Re(s) > 0,\ 0 < \Re(\alpha) < \infty \ \text{ or } \ \Re(s) = 0,\ 0 < \Re(\alpha) < 1\Big).$$

Gauss ordinary hypergeometric function is defined as

$${}_2F_1\left[\begin{array}{c} \lambda, \mu; \\ \\ \sigma; \end{array} z\right] = \sum_{n=0}^{\infty} \frac{(\lambda)_n (\mu)_n z^n}{(\sigma)_n \ n!} \tag{1.3}$$

**(1.3a)** The infinite series (1.3) is always convergent when $|z| < 1$ and $\lambda, \mu, \sigma \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

**(1.3b)** The infinite series (1.3) is absolutely convergent when $|z| = 1$, $\Re(\sigma - \mu - \lambda) > 0$ and $\lambda, \mu, \sigma \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

**(1.3c)** The infinite series (1.3) is conditionally convergent when $|z| = 1$, $z \neq 1$, $-1 < \Re(\sigma - \mu - \lambda) \leq 0$ and $\lambda, \mu, \sigma \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

**(1.3d)** The infinite series (1.3) is divergent when $|z| = 1$, $\Re(\sigma - \mu - \lambda) \leq -1$ and $\lambda, \mu, \sigma \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

**(1.3e)** When any one or both numerator parameters of Gauss series (1.3) is zero or a negative integer and denominator parameter is neither zero nor a negative integer then series (1.3) terminates and the questions of convergence does not enter the discussion.

**Appell's double hypergeometric series**

In the year 1880 Appell's defined the following double hypergeometric series ([3, p. 73, Equation 4] see also [9, p. 211. Equation 8.1.6])

$$F_4\left[\alpha, \beta; \gamma, \delta; x, y\right] = F^{2:0;0}_{0:1;1}\left[\begin{array}{c} \alpha,\ \beta : \text{—}; \text{—}; \\ \text{—} : \gamma\ ;\ \delta\ ; \end{array} x, y\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m(\delta)_n} \frac{x^m y^n}{m!n!}. \tag{1.4}$$

**Convergence conditions of Appell's double series $F_4$**

**(1.17a)** Appell's series $F_4$ is convergent when $\sqrt{|x|}+\sqrt{|y|} < 1$ ; $\alpha, \beta, \gamma, \delta \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

**(1.17b)** Appell's series $F_4$ is absolutely convergent when $\sqrt{|x|} + \sqrt{|y|} = 1$ ; $x \neq 0, y \neq 0$ ; $\alpha, \beta, \gamma, \delta \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $\Re[(\alpha + \beta) - (\gamma + \delta)] < -1$.

**(1.17c)** When $\alpha$ or $\beta$ or $\alpha, \beta$ both are negative integers and $\gamma, \delta \in \mathbb{C}\backslash\mathbb{Z}_0^-$, then Appell's series $F_4$ will be a polynomial.

For absolute convergence (1.17b) of above series see a paper of Hái *et al.* [5, p. 105, Equation 1.3 and p. 107, Theorem 3 (Equations 1.7, 1.8 and 1.9) ].

**Some Summation theorems**

**(i)** Andrews *et al.* [2, p. 131, Entry (3.1.17)]; Abramowitz [1, p. 557, Entry (15.1.30)]; Kummer [7, v-4]; Per W. Karlsson [6, p. 330, Equation (1.1)]

$${}_2F_1\left[\begin{array}{c} \frac{a}{2}, \frac{a+1}{2}; \\ \\ \frac{2a+5}{6}; \end{array} \frac{1}{9}\right] = \left(\frac{3}{4}\right)^a \frac{\sqrt{(\pi)}\Gamma(\frac{2a+2}{3})}{\Gamma(\frac{a+4}{6})\Gamma(\frac{a+1}{2})} = \left(\frac{3}{4}\right)^{\frac{a}{2}} \frac{\sqrt{(\pi)}\Gamma(\frac{2a+5}{6})}{\Gamma(\frac{a+3}{6})\Gamma(\frac{a+5}{6})}, \tag{1.5}$$

provided $\left(\dfrac{2a+5}{6} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)$.

**(ii)** Prudnikov *et al.* [8, p. 495, Equation (22)]

$${}_2F_1\left[\begin{array}{c} a, \frac{2-a}{3}; \\ \frac{2a+5}{6}; \end{array} \frac{4-3\sqrt{2}}{8}\right] = \left(\frac{2}{3}\right)^{\frac{a}{2}} \sqrt{\pi} \frac{\Gamma(\frac{2a+5}{6})}{\Gamma(\frac{a+3}{6})\Gamma(\frac{a+5}{6})}, \tag{1.6}$$

provided $\left(\dfrac{2a+5}{6} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)$.

**(iii)** Prudnikov *et al.* [8, p. 495, Equation (25)]

$${}_2F_1\left[\begin{array}{c} a, 2-3a; \\ \frac{3-2a}{2}; \end{array} \frac{2-\sqrt{3}}{4}\right] = \frac{3^{\frac{3a}{2}}}{2^{2a-1}\sqrt{\pi}} \frac{\Gamma(\frac{4}{3})\Gamma(\frac{3-2a}{2})}{\Gamma(\frac{4-3a}{3})}, \tag{1.7}$$

provided $\left(\dfrac{3-2a}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)$.

In each section of this paper, any values of parameters and variables leading to the results which do not make sense, are tacitly excluded.

## 2. Application in $F_4[A, B; C, 1 + A + B - C; x(1 - y), y(1 - x)]$

Consider the following transformation ([3, p. 81, Equation(1)] see also [4, p. 238(5), Aritcle 5.10])

$$F_4[A, B; C, 1+A+B-C; x(1-y), y(1-x)] = {}_2F_1\left[\begin{array}{r} A, B; \\ C; \end{array} x\right] \times {}_2F_1\left[\begin{array}{r} A, B; \\ 1 + A + B - C; \end{array} y\right], \tag{2.1}$$

$$\left(|x| < 1, |y| < 1, \sqrt{|x(1 - y)|} + \sqrt{|y(1 - x)|} < 1, \; \textit{and } A, B, C, 1 + A + B - C \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

Now put $C = \frac{1+A+B}{2}$ and $y = x$ in (2.1), we get

$$F_4[A, B; \frac{1 + A + B}{2}, \frac{1 + A + B}{2}; x(1-x), x(1-x)] = \left\{{}_2F_1\left[\begin{array}{r} A, B; \\ \frac{1+A+B}{2}; \end{array} x\right]\right\}^2, \tag{2.2}$$

$$\left(|x| < 1, \; |x(1 - x)| < \frac{1}{4} \; ; \; A, \; B, \; \frac{1 + A + B}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

Put $A = a$, $B = \frac{2-a}{3}$, $x = \frac{4-3\sqrt{2}}{8}$ in the equation (2.2) and using summation theorem (1.6), we get

$$F_4\left[a, \frac{2 - a}{3}; \frac{2a + 5}{6}, \frac{2a + 5}{6}; \frac{-1}{32}, \frac{-1}{32}\right] = \pi\left(\frac{2}{3}\right)^a \left[\frac{\Gamma(\frac{2a+5}{6})}{\Gamma(\frac{a+3}{6})\Gamma(\frac{a+5}{6})}\right]^2, \tag{2.3}$$

$$\left(a, \; \frac{2 - a}{3}, \; \frac{2a + 5}{6} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

Put $A = a$, $B = 2 - 3a$, $x = \frac{2-\sqrt{3}}{4}$ in the equation (2.2) and using summation theorem (1.7), we get

$$F_4\left[a, 2 - 3a; \frac{3 - 2a}{2}, \frac{3 - 2a}{2}; \frac{1}{16}, \frac{1}{16}\right] = \frac{3^{3a}}{\pi 2^{4a-2}} \left[\frac{\Gamma(\frac{4}{3})\Gamma(\frac{3-2a}{2})}{\Gamma(\frac{4-3a}{3})}\right]^2, \tag{2.4}$$

$$\left(a, \; 2 - 3a, \; \frac{3 - 2a}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

## 3. Application in $F_4[A, B; C, C; x, -x]$

Consider the following reduction formula of Srivastava [10 pp. 296-299, 1973]

$$F_4[A, B; C, C; x, -x] = {}_4F_3\left[\begin{array}{c} \frac{A}{2}, \frac{A+1}{2}, \frac{B}{2}, \frac{B+1}{2}; \\ C, \frac{C}{2}, \frac{C+1}{2}; \end{array} -4x^2\right], \tag{3.1}$$

$$\left(|x| < \frac{1}{4}\ ;\ A,\ B,\ C \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

Put C=A in equation (3.1), we get

$$F_4[A, B; A, A; x, -x] = {}_2F_1\left[\begin{array}{c} \frac{B}{2}, \frac{B+1}{2}; \\ A; \end{array} -4x^2\right], \tag{3.2}$$

$$\left(|x| < \frac{1}{4}\ ;\ A,\ B \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

Put $A = \frac{2a+5}{6}$, $B = a$, $x = \frac{i}{6}$ in the equation (3.2) and using summation theorem (1.5), we get

$$F_4\left[\frac{2a+5}{6}, a; \frac{2a+5}{6}, \frac{2a+5}{6}; \frac{i}{6}, \frac{-i}{6}\right] = \left(\frac{3}{4}\right)^{\frac{a}{2}} \frac{\sqrt(\pi)\Gamma(\frac{2a+5}{6})}{\Gamma(\frac{a+3}{6})\Gamma(\frac{a+5}{6})}, \tag{3.3}$$

$$\left(a, \frac{2a+5}{6} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

We conclude our present investigation by observing that several summation formulas can be derived in an analogous manner.

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## References

[1] Abramowitz, M. and Stegun, I. A., Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards Applied Mathematics Series, vol. 55, U. S. Government Printing Office, Washington, (1964).

[2] Andrews, G. E., Askey, R. and Roy, R., Special Functions, Encyclopedia of Mathematics and its Applications, Vol. 71, Cambridge University Press, Cambridge, (1999).

[3] Bailey, W. N., Generalized Hypergeometric Series, Cambridge Math. Tract No. 32, Cambridge University Press, Cambridge, 1935; Reprinted by Stechert-Hafner, New York, (1964).

[4] Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. G., Higher Transcendental Functions, Vol. I, McGraw -Hill Book Company, New York, Toronto and London, (1953).

[5] Hái, N. T., Marichev, O. I. and Srivastava, H. M., A note on the convergence of certain families of multiple hypergeometric series, Journal Math. Anal. Appl., 164 (1992), 104-115.

[6] Karlsson, Per W., On two Hypergeometric Summation Formulas Conjectured by Gosper, Simon Stevin, 60(4), (1986), 329-337.

[7] Kümmer, E. E., Über die hypergeometrische Reihe

$$1+\frac{\alpha.\beta}{1.\gamma}x+\frac{\alpha(\alpha+1).\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2+\frac{\alpha(\alpha+1)(\alpha+2).\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3+\ldots,$$

J. Reine Angew. Math. 15 (1836), 39-83 and 127-172; see also Collected papers, Vol. II: Function Theory, Geometry and Miscellaneous (Edited and with a Foreword by André Weil), Springer-Verlag, Berlin, Heidelberg and New York, (1975).

[8] Prudnikov, A. P., Brychkov, Yu. A. and Marichev, O. I., Integrals and Series, Volume III : More Special Functions, Nauka, Moscow, 1986 (In Russian); (Translated from the Russian by G. G. Gould) Gordon and Breach Science Publishers, New York, (1990).

[9] Slater, L. J., Generalized Hypergeometric Functions, Cambridge University Press, Cambridge, (1966).

[10] Srivastava, H. M., On the Reducibility of Appell's Function $F_4$, Canad. Math Bull., Vol. 16 (1973), 296-299.