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A NEW SUBCLASS OF p-VALENT ANALYTIC FUNCTIONS ASSOCIATED WITH HILBERT SPACE OPERATOR

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Abstract: In this paper, we define a new subclass of $M(\alpha, A, B, p, \mathbb{T})$ of p-valent analytic functions associated with Hilbert Space operator. For this new subclass of functions, we determine the coefficient estimate, growth and distortion bounds along with extreme points. Furthermore, we consider applications of fractional calculus on functions in this subclass.

Keywords and Phrases: p-valent functions, Hilbert space, coefficient estimates, distortion bounds, extreme points.

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1. Introduction and Preliminaries

Let $\mathcal{A}_p$ denote the class of p-valent analytic functions defined on the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ having the Taylor series expansion

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p},\ z \in \Delta$$

about the origin. Denote by $\mathcal{TA}_p$ the subclass of functions in $\mathcal{A}_p$ of the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p}, \quad a_{n+p} \geq 0. \tag{1}$$

Let $\mathcal{H}$ be a Hilbert space over the complex field $\mathbb{C}$ and $\mathbb{T}$ be a bounded linear transformation on $\mathcal{H}$. For a complex analytic function f on the unit disk Δ, let $f(\mathbb{T})$ denote the operator on $\mathcal{H}$ defined by the Riesz-Dunford integral [2]

$$f(\mathbb{T}) = \frac{1}{2\pi i}\int_C (z\mathbb{I} - \mathbb{T})^{-1} f(z)dz$$

where $\mathbb{I}$ stands for the identity operator on $\mathcal{H}$ and C is a positively-oriented simple closed rectifiable contour containing the spectrum of $\mathbb{T}$ in its interior domain [3]. The operator $f(\mathbb{T})$ can also be defined by the following series [4]:

$$f(\mathbb{T}) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\mathbb{T}^n \tag{2}$$

which converges in the norm topology.

Several researchers in the past have defined and studied various subclasses of analytic functions defined on the unit disk Δ using the Hilbert space operator. For more details, one may refer to $[4-11]$. Xiaopei [13] defined and studied p-valent analytic functions using Hilbert space operator.

In this paper, we define a new subclass of p-valent functions using Hilbert space operator and prove several interesting results for functions in this class.

We begin with the following lemma which is used to define our subclass.

Lemma 1.1. *For f in $\mathcal{TA}_p$ given by* (1), $0 \le \mu \le 1$, $0 \le \theta \le 1$, *if the operator $\mathcal{M}^{\theta}_{p,\mu} : \mathcal{TA}_p \to \mathcal{TA}_p$ defined by*

$$\mathcal{M}^{\theta}_{p,\mu} f(z) = \frac{1}{\Gamma(\theta+p)(1-\mu)^{\theta+p}} \int_0^{\infty} e^{-(\frac{t}{1-\mu})} f(zt)dt.$$

then

$$\mathcal{M}^{\theta}_{p,\mu} f(z) = z^p - \Sigma_{n=1}^{\infty} E(n,p,\mu,\theta) a_{n+p} z^{n+p}$$

where $E(n,p,\mu,\theta) = \frac{\Gamma(\theta+p+n)}{\Gamma(\theta+p)}(1-\mu)^n$.

Remark 1.2. *For p=1, the operator $\mathcal{M}^{\theta}_{p,\mu}$ reduces to the Rafid operator R^{θ}_{μ} introduced by Atshan and Rafid* [1].

We now define a subclass of $\mathcal{TA}_p$, using the Hilbert space operator.

Definition 1.3. *A function f of the form* (1) *is said to be in the class* $\mathcal{M}(\alpha, A, B, p, \mathbb{T})$ *if*

$$\frac{\mathbb{T}(M^{\theta}_{p,\mu} f(\mathbb{T}))' + \alpha\mathbb{T}^2(M^{\theta}_{p,\mu} f(\mathbb{T}))''}{(1-\alpha)(M^{\theta}_{p,\mu} f(\mathbb{T})) + \alpha\mathbb{T}(M^{\theta}_{p,\mu} f(\mathbb{T}))'} \prec \frac{1+A\mathbb{T}}{1+B\mathbb{T}} \tag{3}$$

for $0 \leq \alpha \leq 1$, $0 \leq B \leq 1$, $-B \leq A \leq B$, *and all operators* $\mathbb{T}$ *with* $\|\mathbb{T}\| < 1$ *and* $\mathbb{T} \neq \mathbb{O}$, *where* $\mathbb{O}$ *denotes the zero operator on* $\mathcal{H}$.

2. Main Results

Theorem 2.1. *An analytic* $p-$ *valent function* f *of the form* (1) *is in the class* $\mathcal{M}(\alpha, A, B, p, \mathbb{T})$ *if and only if*

$$\sum_{n=1}^{\infty}[\alpha(B+1)(n+p)^2 + (1+(1-\alpha)B - (A+2)\alpha)(n+p) - (A+1)(1-\alpha)]E(n,p,\mu,\theta)a_{n+p} \leq \alpha(B+1)p^2 + (1+(1-\alpha)B-(A+2)\alpha)p - (A+1)(1-\alpha). \quad (4)$$

Proof. Assume that (3) holds. Then

$$\frac{\mathbb{T}(M_{p,\mu}^{\theta} f(\mathbb{T}))' + \alpha\mathbb{T}^2(M_{p,\mu}^{\theta} f(\mathbb{T}))''}{(1-\alpha)(M_{p,\mu}^{\theta} f(\mathbb{T})) + \alpha\mathbb{T}(M_{p,\mu}^{\theta} f(\mathbb{T}))'} = \frac{1+Aw(\mathbb{T})}{1+Bw(\mathbb{T})}$$

where $w(\mathbb{O}) = \mathbb{O}$ where $\mathbb{O}$ is as in Definition 1.3 and $\|w(\mathbb{T})\| < 1$ for all operators $\mathbb{T} \neq \mathbb{O}$. Thus we have

$$\|[-\alpha p^2+(2\alpha-1)p+1-\alpha]\mathbb{T}^p + \sum_{n=1}^{\infty}[\alpha(n+p)^2+(1-2\alpha)(n+p)+\alpha-1]E(n,p,\mu,\theta)\mathbb{T}^{n+p}a_{n+p}\|$$

$$< \|[\alpha Bp^2 + (B(1-\alpha) - \alpha A)p + A(\alpha-1)]\mathbb{T}^p + \sum_{n=1}^{\infty}[-\alpha B(n+p)^2 + (A\alpha - B + \alpha B)(n+p) + (1-\alpha)A]E(n,p,\mu,\theta)a_{n+p}\mathbb{T}^{n+p}\|$$

Choosing $\mathbb{T} = e\mathbb{I}$, $(0 < e < 1)$ we get

$$[-\alpha p^2 + (2\alpha-1)p + 1 - \alpha]e^p + \sum_{n=1}^{\infty}[\alpha(n+p)^2 + (1-2\alpha)(n+p) + \alpha - 1]E(n,p,\mu,\theta)e^{n+p}a_{n+p}$$

$$< [\alpha Bp^2 + (B(1-\alpha) - \alpha A)p + A(\alpha-1)]e^p + \sum_{n=1}^{\infty}[-\alpha B(n+p)^2 + (A\alpha - B + \alpha B)(n+p) + (1-\alpha)A] \times E(n,p,\mu,\theta)a_{n+p}e^{n+p}$$

letting $e \to 1$ we get (4).
Conversely, assume that the inequality (4) holds. Then
$\|(1-\alpha)[(M^{\theta}_{p,\mu}f(\mathbb{T}) - \mathbb{T}(M^{\theta}_{p,\mu}f(\mathbb{T}))'] - \alpha\mathbb{T}^2(M^{\theta}_{p,\mu}f(\mathbb{T}))''\|$
$-\|\alpha B\mathbb{T}^2(M^{\theta}_{p,\mu}f(\mathbb{T}))'' + (B-\alpha A)\mathbb{T}(M^{\theta}_{p,\mu}f(\mathbb{T}))' - A(1-\alpha)(M^{\theta}_{p,\mu}f(\mathbb{T}))\|$
$= \|(1-\alpha)[(1-p)\mathbb{T}^p + \sum_{n=1}^{\infty}(n+p-1)E(n,p,\mu,\theta)a_{n+p}\mathbb{T}^{n+p} - \alpha p(p-1)\mathbb{T}^p$

$$+\sum_{n=1}^{\infty}\alpha(n+p)(n+p-1)E(n,p,\mu,\theta)a_{n+p}\mathbb{T}^{n+p}\|$$

$-\|(\alpha Bp^2 - \alpha Bp + Bp - \alpha Ap - A + \alpha A)\mathbb{T}^p$

$$-\sum_{n=1}^{\infty}[\alpha B(n+p)(n+p-1) + (B-\alpha A)(n+p) - A(1-\alpha)]E(n,p,\mu,\theta)a_{n+p}\mathbb{T}^{n+p}\|$$

$$\le \sum_{n=1}^{\infty}[\alpha(B+1)(n+p)^2 + (1+(1-\alpha)B-(A+2)\alpha)(n+p) - (A+1)(1-\alpha)]E(n,p,\mu,\theta)a_{n+p}$$

$-\alpha(B+1)p^2 + (1+(1-\alpha)B-(A+2)\alpha)p - (A+1)(1-\alpha)$
≤ 0
Hence $f(z) \in \mathcal{M}(\alpha, A, B, p, \mathbb{T})$.

Corollary 2.2. *If f of the form* (1) *is in the class $\mathcal{M}(\alpha, A, B, p, \mathbb{T})$ then*

$$a_{n+p} \le \frac{\alpha(B+1)p^2 + [(1+(1-\alpha)B-(A+2)\alpha]p - (A+1)(1-\alpha)}{(\alpha(B+1)(n+p)^2 + [(1+(1-\alpha)B-(A+2)\alpha](n+p) - (A+1)(1-\alpha))E(n,p,\mu,\theta)}.$$

The result is sharp for the function
$f(z) = z^p -$

$$\frac{\alpha(B+1)p^2 + [(1+(1-\alpha)B-(A+2)\alpha]p - (A+1)(1-\alpha)}{(\alpha(B+1)(n+p)^2 + [(1+(1-\alpha)B-(A+2)\alpha](n+p) - (A+1)(1-\alpha))E(n,p,\mu,\theta)}z^{n+p}.$$

Theorem 2.3. *If f of the form* (1) *is in the class $\mathcal{M}(\alpha, A, B, p, \mathbb{T})$ then*
$\|f(\mathbb{T})\| \ge \|\mathbb{T}\|^p -$

$$\frac{\alpha(B+1)p^2 + [1+(1-\alpha)B-(A+2)\alpha]p - (A+1)(1-\alpha)}{[\alpha(B+1)(p+1)^2 + [1+(1-\alpha)B-(A+2)\alpha](p+1) - (A+1)(1-\alpha)]E(1,p,\mu,\theta)}\|\mathbb{T}\|^{p+1}$$

and
$\|f(\mathbb{T})\| \le \|\mathbb{T}\|^p +$

$$\frac{\alpha(B+1)p^2 + [1+(1-\alpha)B-(A+2)\alpha]p - (A+1)(1-\alpha)}{[\alpha(B+1)(p+1)^2 + [1+(1-\alpha)B-(A+2)\alpha](p+1) - (A+1)(1-\alpha)]E(1,p,\mu,\theta)}\|\mathbb{T}\|^{p+1}.$$

The result is sharp for
$f(z) = z^p -$

$$\frac{\alpha(B+1)p^2 + [1+(1-\alpha)B-(A+2)\alpha]p-(A+1)(1-\alpha)}{[\alpha(B+1)(p+1)^2 + [1+(1-\alpha)B-(A+2)\alpha](p+1)-(A+1)(1-\alpha)]E(1,p,\mu,\theta)} z^{p+1}.$$

Proof. By Theorem 2.1

$$[\alpha(B+1)(p+1)^2 + [1+(1-\alpha)B-(A+2)\alpha](p+1)-(A+1)(1-\alpha)]E(1,p,\mu,\theta)\sum_{n=1}^{\infty} a_{n+p}$$

$$\leq \alpha(B+1)p^2 + [1+(1-\alpha)B-(A+2)\alpha]p-(A+1)(1-\alpha)$$

which gives
$\sum_{n=1}^{\infty} a_{n+p} \leq$

$$\frac{\alpha(B+1)p^2 + [1+(1-\alpha)B-(A+2)\alpha]p-(A+1)(1-\alpha)}{[\alpha(B+1)(p+1)^2 + [1+(1-\alpha)B-(A+2)\alpha](p+1)-(A+1)(1-\alpha)]E(1,p,\mu,\theta)}. \tag{5}$$

Now,

$$\|f(\mathbb{T})\| \geq \|\mathbb{T}\|^p - \sum_{n=1}^{\infty} a_{n+p}\|\mathbb{T}\|^{n+p} \geq \|\mathbb{T}\|^p - \|\mathbb{T}\|^{p+1}\sum_{n=1}^{\infty} a_{n+p}$$

and

$$\|f(\mathbb{T})\| \leq \|\mathbb{T}\|^p + \sum_{n=1}^{\infty} a_{n+p}\|\mathbb{T}\|^{n+p} \leq \|\mathbb{T}\|^p + \|\mathbb{T}\|^{p+1}\sum_{n=1}^{\infty} a_{n+p}.$$

Using (5), in the above two inequalities we get the result.

Theorem 2.4. *If f of the form* (1) *is in the class $\mathcal{M}(\alpha, A, B, p, \mathbb{T})$ then*
$\|f'(\mathbb{T})\| \geq \quad p\|\mathbb{T}\|^{p-1} -$

$$\frac{(p+1)[\alpha(B+1)p^2 + [1+(1-\alpha)B-(A+2)\alpha]p-(A+1)(1-\alpha)]E(1,p,\mu,\theta)}{\alpha(B+1)(p+1)^2 + [1+(1-\alpha)B-(A+2)\alpha](p+1)-(A+1)(1-\alpha)}\|\mathbb{T}\|^p$$

and

$\|f'(\mathbb{T})\| \leq \quad p\|\mathbb{T}\|^{p-1} +$

$$\frac{(p+1)[\alpha(B+1)p^2 + [1+(1-\alpha)B-(A+2)\alpha]p-(A+1)(1-\alpha)]E(1,p,\mu,\theta)}{\alpha(B+1)(p+1)^2 + [1+(1-\alpha)B-(A+2)\alpha](p+1)-(A+1)(1-\alpha)}\|\mathbb{T}\|^p.$$

The result is sharp for $f(z) = z^p -$

$$\frac{\alpha(B+1)p^2 + [1+(1-\alpha)B-(A+2)\alpha]p-(A+1)(1-\alpha)}{[\alpha(B+1)(p+1)^2 + [1+(1-\alpha)B-(A+2)\alpha](p+1)-(A+1)(1-\alpha)]E(1,p,\mu,\theta)} z^{p+1}.$$

Proof. Note that

$$\frac{n+p}{p+1}[\alpha(B+1)(1+p)^2+(1+(1-\alpha)B-(A+2)\alpha)(1+p)-(A+1)(1-\alpha)]$$

$\leq [\alpha(B+1)(n+p)^2+(1+(1-\alpha)B-(A+2)\alpha)(n+p)-(A+1)(1-\alpha)]E(1,p,\mu,\theta).$
Using Theorem 2.1
$\sum_{n=1}^{\infty}(n+p)a_{n+p}$

$$\leq \frac{(p+1)[\alpha(B+1)(n+p)^2+(1+(1-\alpha)B-(A+2)\alpha)(n+p)-(A+1)(1-\alpha)]E(1,p,\mu,\theta)}{\alpha(B+1)(1+p)^2+(1+(1-\alpha)B-(A+2)\alpha)(1+p)-(A+1)(1-\alpha)}. \tag{6}$$

By (1)

$$\|f'(\mathbb{T})\| \leq p\|\mathbb{T}\|^{p-1}+\|\mathbb{T}\|^p\sum_{n=1}^{\infty}(n+p)a_{n+p}$$

and

$$\|f'(\mathbb{T})\| \geq p\|\mathbb{T}\|^{p-1}-\|\mathbb{T}\|^p\sum_{n=1}^{\infty}(n+p)a_{n+p}.$$

Using (6), we get the result.

Theorem 2.5. *Let* $f_0(z)=z^p$ *and*
$f_n(z)=z^p-$

$$\frac{\alpha(B+1)p^2+[1+(1-\alpha)B-(A+2)\alpha]p-(A+1)(1-\alpha)}{[\alpha(B+1)(n+p)^2+[1+(1-\alpha)B-(A+2)\alpha](n+p)-(A+1)(1-\alpha)]E(n,p,\mu,\theta)}z^{n+p},$$

for $n\geq 1$. *Then* $f\in\mathcal{M}(\alpha,A,B,p,\mathbb{T})$ *if and only if it can be expressed as*

$$f(z)=\sum_{n=0}^{\infty}\lambda_n f_n(z)$$

where $\lambda_n\geq 0, (n=0,1,2,...)$ *and* $\sum_{n=0}^{\infty}\lambda_n=1$.

Proof. Suppose that
$f(z)=\sum_{n=0}^{\infty}\lambda_n f_n(z)=z^p-$

$$\sum_{n=1}^{\infty}\lambda_n\frac{\alpha(B+1)p^2+[1+(1-\alpha)B-(A+2)\alpha]p-(A+1)(1-\alpha)}{[\alpha(B+1)(n+p)^2+[1+(1-\alpha)B-(A+2)\alpha](n+p)-(A+1)(1-\alpha)]E(n,p,\mu,\theta)}z^{n+p}.$$

Then

$$\sum_{n=1}^{\infty}\frac{[\alpha(B+1)(n+p)^2+[1+(1-\alpha)B-(A+2)\alpha](n+p)-(A+1)(1-\alpha)]E(n,p,\mu,\theta)}{\alpha(B+1)p^2+(1+(1-\alpha)B-(A+2)\alpha)p-(A+1)(1-\alpha)}$$

$$\times \frac{\lambda_n[\alpha(B+1)p^2+(1+(1-\alpha)B-(A+2)\alpha)p-(A+1)(1-\alpha)]}{[\alpha(B+1)(n+p)^2+[1+(1-\alpha)B-(A+2)\alpha](n+p)-(A+1)(1-\alpha)]E(n,p,\mu,\theta)}$$

$= \sum_{n=1}^{\infty}\lambda_n = 1-\lambda_0 \leq 1.$

Conversely, let $f \in \mathcal{M}(\alpha, A, B, p, \mathbb{T})$.
By Corollary 2.2,

$$a_{n+p} \leq \frac{\alpha(B+1)p^2+[(1+(1-\alpha)B-(A+2)\alpha]p-(A+1)(1-\alpha)}{[\alpha(B+1)(n+p)^2+[(1+(1-\alpha)B-(A+2)\alpha](n+p)-(A+1)(1-\alpha)]E(n,p,\mu,\theta)}$$

Setting

$$\lambda_n = \frac{[\alpha(B+1)(n+p)^2+[(1+(1-\alpha)B-(A+2)\alpha](n+p)-(A+1)(1-\alpha)]E(n,p,\mu,\theta)}{\alpha(B+1)p^2+[(1+(1-\alpha)B-(A+2)\alpha]p-(A+1)(1-\alpha)}a_{n+p}$$

and $\lambda_0 = 1-\sum_{n=1}^{\infty}\lambda_n \geq 0$, the result follows.

Theorem 2.6. *Let $f \in \mathcal{M}(\alpha, A, B, p, \mathbb{T})$. Then f is close-to-convex of order δ, $(0 \leq \delta < 1)$ in the disc $|z| < r_1$, where*
$r_1 =$
$$inf_{n\geq 2}\left[\frac{(p-\delta)[\alpha(B+1)(n+p)^2+(1+(1-\alpha)B-(A+2)\alpha)(n+p)-(A+1)(1-\alpha)]E(n,p,\mu,\theta)}{(n+p)[\alpha(B+1)p^2+(1+(1-\alpha)B-(A+2)\alpha)p-(A+1)(1-\alpha)}\right]^{\frac{1}{n}}.$$

Proof. Let $f \in \mathcal{M}(\alpha, A, B, p, \mathbb{T})$ be close-to-convex of order δ, then

$$\|\mathbb{T}^{1-p}f'(\mathbb{T})-p\| \leq p-\delta. \tag{7}$$

Using (1)

$$\|\mathbb{T}^{1-p}f'(\mathbb{T})-p\| \leq \sum_{n=1}^{\infty}(n+p)a_{n+p}\|\mathbb{T}\|^n.$$

This expression is less than $p-\delta$ if

$$\sum_{n=1}^{\infty}\frac{n+p}{p-\delta}a_{n+p}\|\mathbb{T}\|^n \leq 1. \tag{8}$$

By Theorem 2.1

$$\sum_{n=1}^{\infty}\frac{[\alpha(B+1)(n+p)^2+(1+(1-\alpha)B-(A+2)\alpha)(n+p)-(A+1)(1-\alpha)]E(n,p,\mu,\theta)a_{n+p}}{\alpha(B+1)p^2+(1+(1-\alpha)B-(A+2)\alpha)p-(A+1)(1-\alpha)} \tag{9}$$

$< 1.$
Thus (8) is true if

$\frac{n+p}{p-\delta}\|\mathbb{T}\|^n \leq$

$$\frac{[\alpha(B+1)(n+p)^2+(1+(1-\alpha)B-(A+2)\alpha)(n+p)-(A+1)(1-\alpha)]E(n,p,\mu,\theta)}{\alpha(B+1)p^2+(1+(1-\alpha)B-(A+2)\alpha)p-(A+1)(1-\alpha)}$$

which gives

$\|\mathbb{T}\| \leq \left[\frac{(p-\delta)[\alpha(B+1)(n+p)^2+(1+(1-\alpha)B-(A+2)\alpha)(n+p)-(A+1)(1-\alpha)]E(n,p,\mu,\theta)}{(n+p)[\alpha(B+1)p^2+(1+(1-\alpha)B-(A+2)\alpha)p-(A+1)(1-\alpha)}\right]^{\frac{1}{n}}$ and hence the result follows.

Theorem 2.7. *Let $f \in \mathcal{M}(\alpha, A, B, p, \mathbb{T})$. Then f is starlike of order $\delta, (0 \leq \delta < 1)$ in the disc $|z| < r_2$, where*

$r_2 =$

$inf_{n\geq 2}\left[\frac{(p-\delta)[\alpha(B+1)(n+p)^2+(1+(1-\alpha)B-(A+2)\alpha)(n+p)-(A+1)(1-\alpha)]E(n,p,\mu,\theta)}{(n+p-\delta)[\alpha(B+1)p^2+(1+(1-\alpha)B-(A+2)\alpha)p-(A+1)(1-\alpha)]}\right]^{\frac{1}{n}}.$

Proof. Since $f \in \mathcal{M}(\alpha, A, B, p, \mathbb{T})$ is starlike of order δ, we have

$$\left\|\frac{\mathbb{T}f'(\mathbb{T})}{f(\mathbb{T})} - p\right\| \leq p - \delta. \tag{10}$$

Using (1)

$$\left\|\frac{\mathbb{T}f'(\mathbb{T})}{f(\mathbb{T})} - p\right\| \leq \sum_{n=1}^{\infty}(n+p)a_{n+p}\|\mathbb{T}\|^n.$$

This expression is less than $p - \delta$ if

$$\sum_{n=1}^{\infty}\frac{n+p-\delta}{p-\delta}a_{n+p}\|\mathbb{T}\|^n \leq 1. \tag{11}$$

By Theorem 2.1

$$\sum_{n=1}^{\infty}\frac{[\alpha(B+1)(n+p)^2+(1+(1-\alpha)B-(A+2)\alpha)(n+p)-(A+1)(1-\alpha)]E(n,p,\mu,\theta)a_{n+p}}{\alpha(B+1)p^2+(1+(1-\alpha)B-(A+2)\alpha)p-(A+1)(1-\alpha)} \tag{12}$$

$< 1.$

Thus (11) is true if

$\left(\frac{n+p-\delta}{p-\delta}\right)\|\mathbb{T}\|^n$

$$\leq \frac{[\alpha(B+1)(n+p)^2+(1+(1-\alpha)B-(A+2)\alpha)(n+p)-(A+1)(1-\alpha)]E(n,p,\mu,\theta)}{\alpha(B+1)p^2+(1+(1-\alpha)B-(A+2)\alpha)p-(A+1)(1-\alpha)}$$

which gives

$$\|\mathbb{T}\| \leq \left[\frac{(p-\delta)[\alpha(B+1)(n+p)^2+(1+(1-\alpha)B-(A+2)\alpha)(n+p)-(A+1)(1-\alpha)]E(n,p,\mu,\theta)}{(n+p-\delta)[\alpha(B+1)p^2+(1+(1-\alpha)B-(A+2)\alpha)p-(A+1)(1-\alpha)}\right]^{\frac{1}{n}}$$

and hence the result follows.

Theorem 2.8. *Let $f \in \mathcal{M}(\alpha, A, B, p, \mathbb{T})$. Then f is convex of order δ $(0 \leq \delta < 1)$ in the disc $|z| < r_3$, where*

$r_3 =$

$$inf_{n\geq 2}\left[\frac{p(p-\delta)[\alpha(B+1)(n+p)^2+(1+(1-\alpha)B-(A+2)\alpha)(n+p)-(A+1)(1-\alpha)]E(n,p,\mu,\theta)}{(n+p-\delta)(n+p)[\alpha(B+1)p^2+(1+(1-\alpha)B-(A+2)\alpha)p-(A+1)(1-\alpha)}\right]^{\frac{1}{n}}.$$

Proof. Since f in $\mathcal{M}(\alpha, A, B, p, \mathbb{T})$ is convex of order δ we have

$$\left\|1+\frac{\mathbb{T}f''(\mathbb{T})}{f'(\mathbb{T})}-p\right\| \leq p-\delta. \tag{13}$$

Using (1)

$$\left\|1+\frac{\mathbb{T}f''(\mathbb{T})}{f'(\mathbb{T})}-p\right\| \leq \frac{\sum_{n=1}^{\infty} n(n+p)a_{n+p}\|\mathbb{T}\|^n}{p-\sum_{n=1}^{\infty}(n+p)a_{n+p}\|T\|^n}.$$

This expression is less than $p-\delta$ if

$$\sum_{n=1}^{\infty}\frac{(n+p)(n+p-\delta)}{p(p-\delta)}a_{n+p}\|\mathbb{T}\|^n \leq 1. \tag{14}$$

By Theorem 2.1

$$\sum_{n=1}^{\infty}\frac{[\alpha(B+1)(n+p)^2+(1+(1-\alpha)B-(A+2)\alpha)(n+p)-(A+1)(1-\alpha)]E(n,p,\mu,\theta)a_{n+p}}{\alpha(B+1)p^2+(1+(1-\alpha)B-(A+2)\alpha)p-(A+1)(1-\alpha)} \tag{15}$$

$< 1.$

Thus (14) is true if

$\frac{(n+p)(n+p-\delta)}{p(p-\delta)}\|\mathbb{T}\|^n$

$$\leq \frac{[\alpha(B+1)(n+p)^2+(1+(1-\alpha)B-(A+2)\alpha)(n+p)-(A+1)(1-\alpha)]E(n,p,\mu,\theta)}{\alpha(B+1)p^2+(1+(1-\alpha)B-(A+2)\alpha)p-(A+1)(1-\alpha)}$$

which gives

$$\|\mathbb{T}\| \leq \left[\frac{p(p-\delta)[\alpha(B+1)(n+p)^2+(1+(1-\alpha)B-(A+2)\alpha)(n+p)-(A+1)(1-\alpha)]E(n,p,\mu,\theta)}{(n+p-\delta)(n+p)[\alpha(B+1)p^2+(1+(1-\alpha)B-(A+2)\alpha)p-(A+1)(1-\alpha)}\right]^{\frac{1}{n}}$$

and hence the result follows.

3. Application of Fractional Calculus

In this section we use the definition of fractional calculus operators defined in [8].

Definition 3.1. *The fractional integral operator of order k of an analytic function f defined in a simply connected domain containing the origin is defined by*

$$D_{\mathbb{T}}^{-k} f(\mathbb{T}) = \frac{1}{\Gamma(k)} \int_0^1 \mathbb{T}^k f(t\mathbb{T})(1-t)^{k-1} dt$$

Definition 3.2. *The fractional derivative operator of order k of an analytic function f defined in a simply connected domain containing the origin is defined by*

$$D_{\mathbb{T}}^{k} f(\mathbb{T}) = \frac{1}{\Gamma(1-k)} g'(\mathbb{T})$$

where $g(z) = \int_0^1 z^{1-k} f(tz)(1-t)^{-k} dt,\ (0 < k < 1).$

Theorem 3.3. *Let f defined by* (1) *be in the class* $\mathcal{M}(\alpha, A, B, p, \mathbb{T})$. *Then*

$\|D_{\mathbb{T}}^{-k} f(\mathbb{T})\| \geq \frac{\Gamma(p+1)}{\Gamma(p+k+1)} \|\mathbb{T}\|^{p+k} -$

$\frac{\Gamma(p+1)[\alpha(B+1)p^2+[1+(1-\alpha)B-(A+2)\alpha]p-(A+1)(1-\alpha)]}{\Gamma(p+k+1)[\alpha(B+1)(p+1)^2+[1+(1-\alpha)B-(A+2)\alpha](p+1)-(A+1)(1-\alpha)]E(1,p,\mu,\theta)} \|\mathbb{T}\|^{p+k+1}$

and

$\|D_{\mathbb{T}}^{-k} f(\mathbb{T})\| \leq \frac{\Gamma(p+1)}{\Gamma(p+k+1)} \|\mathbb{T}\|^{p+k} +$

$\frac{\Gamma(p+1)[\alpha(B+1)p^2+[1+(1-\alpha)B-(A+2)\alpha]p-(A+1)(1-\alpha)]}{\Gamma(p+k+1)[\alpha(B+1)(p+1)^2+[1+(1-\alpha)B-(A+2)\alpha](p+1)-(A+1)(1-\alpha)]E(1,p,\mu,\theta)} \|\mathbb{T}\|^{p+k+1}$

for $k > 0$ *and all invertible operators* $\mathbb{T}$ *with* $(\mathbb{T}^{\frac{1}{q}})^* \ (\mathbb{T}^{\frac{1}{q}}) = (\mathbb{T}^{\frac{1}{q}})(\mathbb{T}^{\frac{1}{q}})^*$,
$(q \in \mathbf{N}), \|\mathbb{T}\| < 1$ *and* $r_{sp}(\mathbb{T}) r_{sp}(\mathbb{T}^{-1}) \leq 1$, *where* $r_{sp}(\mathbb{T})$ *is the radius of spectrum of* $\mathbb{T}$.

Proof. Using (1),

$$D_{\mathbb{T}}^{-k} f(\mathbb{T}) = \frac{\Gamma(p+1)}{\Gamma(p+k+1)} \mathbb{T}^{p+k} - \sum_{n=1}^{\infty} \frac{\Gamma(n+p+1)}{\Gamma(n+p+k+1)} a_{n+p} \mathbb{T}^{n+p+k}$$

which gives

$$F(\mathbb{T}) = \mathbb{T}^p - \sum_{n=1}^{\infty} \frac{\Gamma(n+p+1)\Gamma(p+k+1)}{\Gamma(n+p+k+1)\Gamma(p+1)} a_{n+p} \mathbb{T}^{n+p}$$

where $F(\mathbb{T}) = \frac{\Gamma(p+k+1)}{\Gamma(p+1)}\mathbb{T}^{-k}D_{\mathbb{T}}^{-k}f(\mathbb{T})$.

Since $0 < \frac{\Gamma(n+p+1)\Gamma(p+k+1)}{\Gamma(n+p+k+1)\Gamma(p+1)} < 1$,

$\sum_{n=1}^{\infty}[\alpha(B+1)(n+p)^2 + (1+(1-\alpha)B - (A+2)\alpha)(n+p) - (A+1)(1-\alpha)]E(n,p,\mu,\theta)\frac{\Gamma(n+p+1)\Gamma(p+k+1)}{\Gamma(n+p+k+1)\Gamma(p+1)}a_{n+p}$

$\leq \alpha(B+1)p^2 + (1+(1-\alpha)B - (A+2)\alpha)p - (A+1)(1-\alpha)$

$\Rightarrow F(\mathbb{T}) \in \mathcal{M}(\alpha, A, B, p, \mathbb{T})$.

By Theorem 2.3

$\|D_{\mathbb{T}}^{-k}f(\mathbb{T})\| \geq \frac{\Gamma(p+1)}{\Gamma(p+k+1)}\|\mathbb{T}^k\|\|\mathbb{T}\|^p -$

$\frac{\Gamma(p+1)[\alpha(B+1)p^2+[1+(1-\alpha)B-(A+2)\alpha]p-(A+1)(1-\alpha)]}{\Gamma(p+k+1)[\alpha(B+1)(p+1)^2+[1+(1-\alpha)B-(A+2)\alpha](p+1)-(A+1)(1-\alpha)]E(1,p,\mu,\theta)}\|\mathbb{T}^k\|\|\mathbb{T}\|^{p+1}$

and

$\|D_{\mathbb{T}}^{-k}f(\mathbb{T})\| \leq \frac{\Gamma(p+1)}{\Gamma(p+k+1)}\|\mathbb{T}^k\|\|\mathbb{T}\|^p +$

$\frac{\Gamma(p+1)[\alpha(B+1)p^2+[1+(1-\alpha)B-(A+2)\alpha]p-(A+1)(1-\alpha)]}{\Gamma(p+k+1)[\alpha(B+1)(p+1)^2+[1+(1-\alpha)B-(A+2)\alpha](p+1)-(A+1)(1-\alpha)]E(1,p,\mu,\theta)}\|\mathbb{T}^k\|\|\mathbb{T}\|^{p+1}$

Using the argument in (13) we get the result.

Theorem 3.4. *Let $f \in \mathcal{M}(\alpha, A, B, p, \mathbb{T})$. Then*
$\|D_{\mathbb{T}}^{k}f(\mathbb{T})\| \geq \frac{\Gamma(p+1)}{\Gamma(p+1-k)}\|\mathbb{T}\|^{p-k} -$

$\frac{(p+1)[\alpha(B+1)p^2+[1+(1-\alpha)B-(A+2)\alpha]p-(A+1)(1-\alpha)]\Gamma(p+1)}{[\alpha(B+1)(p+1)^2+[1+(1-\alpha)B-(A+2)\alpha](p+1)-(A+1)(1-\alpha)]E(1,p,\mu,\theta)\Gamma(p+1-k)}\|\mathbb{T}\|^{p+1-k}$

and

$\|D_{\mathbb{T}}^{k}f(\mathbb{T})\| \leq \frac{\Gamma(p+1)}{\Gamma(p+1-k)}\|\mathbb{T}\|^{p-k} +$

$\frac{(p+1)[\alpha(B+1)p^2+[1+(1-\alpha)B-(A+2)\alpha]p-(A+1)(1-\alpha)]\Gamma(p+1)}{[\alpha(B+1)(p+1)^2+[1+(1-\alpha)B-(A+2)\alpha](p+1)-(A+1)(1-\alpha)]E(1,p,\mu,\theta)\Gamma(p+1-k)}\|\mathbb{T}\|^{p+1-k}.$

Proof. Using (1)

$$D_{\mathbb{T}}^{k}f(\mathbb{T}) = \frac{\Gamma(p+1)}{\Gamma(p+1-k)}\mathbb{T}^{p-k} - \sum_{n=1}^{\infty}\frac{\Gamma(n+p+1)}{\Gamma(n+p+1-k)}\mathbb{T}^{n+p-k}a_{n+p}$$

which gives

$$G(\mathbb{T}) = \mathbb{T}^p - \sum_{n=1}^{\infty} \frac{\Gamma(n+p+1)\Gamma(p+1-k)}{\Gamma(n+p+1-k)\Gamma(p+1)} a_{n+p}\mathbb{T}^{n+p}$$

where $G(\mathbb{T}) = \frac{\Gamma(p+1-k)}{\Gamma(p+1)}\mathbb{T}^k D_{\mathbb{T}}^k f(\mathbb{T})$.
This gives

$$\|G(\mathbb{T})\| \ \leq \|\mathbb{T}\|^p + \sum_{n=2}^{\infty} \frac{\Gamma(n+p+1)\Gamma(p+1-k)}{\Gamma(n+p+1+k)\Gamma(p+1)} a_{n+p}\|\mathbb{T}\|^{n+p}$$

and

$$\|G(\mathbb{T})\| \ \geq \|\mathbb{T}\|^p - \sum_{n=2}^{\infty} \frac{\Gamma(n+p+1)\Gamma(p+1-k)}{\Gamma(n+p+1+k)\Gamma(p+1)} a_{n+p}\|\mathbb{T}\|^{n+p}$$

Since $0 < \frac{\Gamma(n+p+1)\Gamma(p+1-k)}{\Gamma(n+p+1-k)\Gamma(p+1)} < n+p$, the above inequalities becomes

$$\|G(\mathbb{T})\| \ \leq \|\mathbb{T}\|^p + \|\mathbb{T}\|^{p+1} \sum_{n=2}^{\infty}(n+p)a_{n+p}$$

and

$$\|G(\mathbb{T})\| \ \geq \|\mathbb{T}\|^p - \|\mathbb{T}\|^{p+1} \sum_{n=2}^{\infty}(n+p)a_{n+p}.$$

Using (6), we get the result.

4. Conclusion

In the present work, we have defined a new subclass of $p-$ valent analytic functions on the unit disk using Hilbert space operator and certain properties of the functions in this class has been studied.

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