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SOME RESULTS ON CERTAIN SUBCLASS OF MEROMORPHIC FUNCTIONS ASSOCIATED WITH (p, q)-DERIVATIVE

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Abstract: In the present work, we define a new subclass of meromorphic functions by using newly defined (p, q)-differential operator and some geometrical properties such as Sufficiency criteria, coefficient estimates , distortion bounds, radius of starlikeness, radius of convexity and partial sums are discussed for these subclass.

Keywords and Phrases: Regular and Meromorphic functions, Ruscheweyh and Salagean derivative, Janwoski function and (p, q)-differential operator.

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1. Introduction

Let $\mathcal{C}$ be a complex plane and $\mathcal{M}$ denote the collection of all meromorphic functions f of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \; z \in \mathcal{D}^* \tag{1.1}$$

which are regular in punctured open unit disc $\mathcal{D}^* = {}^{\mathcal{D}}/_{\{0\}} = \left\{z \in \mathcal{C} : 0 < |z| < 1\right\}$.

Also let $\mathcal{MS}^*(\beta)$ and $\mathcal{MC}(\beta)$ denote the well known families of meromorphic starlike and meromorphic convex functions of order $\beta(0 \leq \beta < 1)$ respectively.

We first recall some basic notations and definitions which are used to prove our main result.

Definition 1.1. *(Convolution)* [1] The convolution or Hadamard product of any two meromorphic functions f and g of the form (1.1), that are regular in $\mathcal{D}^*$ is given by

$$f(z) * g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n, \quad z \in \mathcal{D}^*.$$

Definition 1.2. *(Subordination)* [6] For any two regular functions f and g in $\mathcal{D}$, we say that f is subordinate to g in $\mathcal{D}$, i.e $f \prec g$, if there exists a Schwarz function $w(z)$, wich is regular in $\mathcal{D}$ with $w(0) = 1$ and $|w(z)| < 1$ such that $f(z) = g(w(z)), z \in \mathcal{D}$.

Definition 1.3. [6] *For a non negative integer n and $0 < q < p \leq 1$, the (p,q)-integer number $[n]_{p,q}$ is defined as*

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}$$

and (p,q)-number shift factorial is given by

$$[n]_{p,q}! = \begin{cases} [n]_{p,q}[n-1]_{p,q}...[2]_{p,q}[1]_{p,q}, & if \quad n \geq 1 \\ 1 \quad if \quad n = 0 \end{cases}.$$

Definition 1.4. [6] *The (p,q)- analogue of Jackson derivative of the function f is given by*

$$D_{p,q}(f(z)) = \frac{f(pz) - f(qz)}{(p-q)z}, \quad z \neq 0, p \neq q \quad 0 < q < 1 \tag{1.2}$$

.

Thus, the (p,q)- derivative of the function $f \in \mathcal{M}$ of the form (1.1) is given by

$$D_{p,q}(f(z)) = -\frac{1}{pqz^2} + \sum_{n=1}^{\infty} [n]_{p,q} a_n z^{n-1},$$

In the present work , we use a new differential operator $\mathcal{L}^{\delta,m}_{p,q} f(z)$ introduced by Nandini and Latha [5] as

$$\mathcal{L}^{\delta,m}_{p,q} f(z) = \mathcal{R}^{\delta}_{p,q} f(z) * \mu^m_{p,q} f(z) = \frac{(-1)^m}{z} + \sum_{n=1}^{\infty} \frac{p^m q^m [n]^m_{p,q} [\delta + n + 1]_{p,q}!}{[\delta]_{p,q}! [n+1]_{p,q}!} a_n z^n, \tag{1.3}$$

$$\delta, m \in \mathcal{N}U\{0\}, \quad 0 < q < 1 \; and \quad p \neq q$$

$$= \frac{(-1)^m}{z} + \sum_{n=1}^{\infty} \psi_n a_n z^n$$

Where $\psi_n = \frac{p^m q^m [n]_{p,q}^m [\delta + n + 1]_{p,q}!}{[\delta]_{p,q}! [n+1]_{p,q}!}$, $\mathcal{R}_{p,q}^{\delta}$ be the (p, q)-Ruschweyh derivative operator and is given by [1]

$$\mathcal{R}_{p,q}^{\delta}(f(z)) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{[\delta + n + 1]_{p,q}!}{[\delta]_{p,q}! [n+1]_{p,q}!} a_n z^n, \quad \delta \in \mathcal{N}$$

and $\mu_{p,q}^m f(z)$ is given by

$$\mu_{p,q}^m f(z) = p^m q^m z D_{p,q}(\mathcal{S}_{p,q}^{m-1} f(z)) = \frac{(-1)^m}{z} + \sum_{n=1}^{\infty} p^m q^m [n]_{p,q}^m a_n z^n, \quad (m \in \mathcal{N}),$$

where $\mathcal{S}_{p,q}^{m-1}$ denote the (p, q)-Salagean differential operator [6] and is given by

$$\mathcal{S}_{p,q}^{m-1} f(z) = \frac{(-1)^{m-1}}{p^{m-1} q^{m-1} z} + \sum_{n=1}^{\infty} [n]_{p,q}^{m-1} a_n z^n.$$

Now we define a subfamily $\mathcal{MC}_{p,q}(A, B, \delta, m)$ of $\mathcal{M}$ by using the operator $\mathcal{L}_{p,q}^{\delta,m}$ as follows.

Definition 1.5. *Let $-1 \leq B < A \leq 1$, $\delta, m \in \mathcal{N} \cup \{0\}$ and $0 < q < p \leq 1$. Then a function $f \in \mathcal{M}$ as in (1.1) is said to be in the class $\mathcal{MC}_{p,q}(A, B, \delta, m)$ if it obeys the condition*

$$\frac{-pq D_{p,q}(z D_{p,q}(\mathcal{L}_{p,q}^{\delta,m} f(z)))}{D_{p,q}(\mathcal{L}_{p,q}^{\delta,m} f(z))} \prec \frac{1 + Az}{1 + Bz} \tag{1.4}$$

Equivalently , a function $f \in \mathcal{M}$ is said to belongs to the class $\mathcal{MC}_{p,q}(A, B, \delta, m)$ if and only if

$$\left| \frac{pq D_{p,q}(z D_{p,q}(\mathcal{L}_{p,q}^{\delta,m} f(z))) + D_{p,q}(\mathcal{L}_{p,q}^{\delta,m} f(z))}{A(D_{p,q} \mathcal{L}_{p,q}^{\delta,m} f(z)) + B(pq D_{p,q}(z D_{p,q}(\mathcal{L}_{p,q}^{\delta,m} f(z))))} \right| < 1 \tag{1.5}$$

2. The Main Results and Their Consequences

Theorem 2.1. *Let $f \in \mathcal{M}$ be given by (1,1) and obeys the inequality*

$$\sum_{n=1}^{\infty} pq[n]_{p,q}\left(pq[n]_{p,q}(1+B)+(1+A)\right)\psi_n|a_n| \leq A-B. \tag{2.1}$$

Then $f \in \mathcal{MC}_{p,q}(A,B,\delta,m)$.

Proof. To prove that $f \in \mathcal{MC}_{p,q}(A,B,\delta,m)$, it is enough to show that

$$\left|\frac{pqD_{p,q}(zD_{p,q}(\mathcal{L}_{p,q}^{\delta,m}f(z)))+D_{p,q}(\mathcal{L}_{p,q}^{\delta,m}f(z))}{AD_{p,q}(\mathcal{L}_{p,q}^{\delta,m}f(z))+BpqD_{p,q}(zD_{p,q}(\mathcal{L}_{p,q}^{\delta,m}f(z)))}\right| < 1$$

consider

$$\left|\frac{pqD_{p,q}(zD_{p,q}(\mathcal{L}_{p,q}^{\delta,m}f(z)))+D_{p,q}(\mathcal{L}_{p,q}^{\delta,m}f(z))}{AD_{p,q}(\mathcal{L}_{p,q}^{\delta,m}f(z))+BpqD_{p,q}(zD_{p,q}(\mathcal{L}_{p,q}^{\delta,m}f(z)))}\right|$$

$$= \left|\frac{pq\left[\frac{(-1)^m}{p^2q^2z^2}+\sum_{n=1}^{\infty}[n]_{p,q}^2\psi_n a_n z^{n-1}\right]+\frac{(-1)^{m+1}}{pqz^2}+\sum_{n=1}^{\infty}[n]_{p,q}\psi_n a_n z^{n-1}}{A\left[\frac{(-1)^{m+1}}{pqz^2}+\sum_{n=1}^{\infty}[n]_{p,q}\psi_n a_n z^{n-1}\right]+Bpq\left[\frac{(-1)^m}{p^2q^2z^2}+\sum_{n=1}^{\infty}[n]_{p,q}^2\psi_n a_n z^{n-1}\right]}\right|$$

$$= \left|\frac{\sum_{n=1}^{\infty}(pq[n]_{p,q}+1)[n]_{p,q}\psi_n a_n z^{n-1}}{\frac{-(A-B)}{pqz^2}(-1)^m+\sum_{n=1}^{\infty}(A+Bpq[n]_{p,q})[n]_{p,q}\psi_n a_n z^{n-1}}\right|$$

$$\leq \frac{pq[n]_{p,q}(pq[n]_{p,q}+1)\psi_n|a_n|}{(A-B)-\sum_{n=1}^{\infty}pq[n]_{p,q}(A+Bpq[n]_{p,q})\psi_n|a_n|} < 1, \quad \text{by (2.1)}$$

which completes the proof.

Theorem 2.2. *Let $f \in \mathcal{MC}_{p,q}(A,B,\delta,m)$ and has the form (1.1). Then for $|z|=r$*

$$\frac{1}{r}-rr_1 \leq |f(z)| \leq \frac{1}{r}+rr_1$$

where $r_1 = \dfrac{[2]_{p,q}!(A-B)}{p^{m+1}q^{m+1}[\delta+1]_{p,q}[\delta+2]_{p,q}(pq(1+B)+(1+A))}.$

Proof. Consider

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$$

$$|f(z)| = \left|\frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n\right|$$

$$\leq \frac{1}{|z|} + \sum_{n=1}^{\infty} |a_n||z^n|$$

$$= \frac{1}{r} + \sum_{n=1}^{\infty} |a_n| r^n$$

since $|z| = r < 1$ we have $r^n < r$

$$|f(z)| \leq \frac{1}{r} + r\sum_{n=1}^{\infty} |a_n| \tag{2.2}$$

similarly

$$|f(z)| \geq \frac{1}{r} - r\sum_{n=1}^{\infty} |a_n| \tag{2.3}$$

from(2.1), we have

$$\sum_{n=1}^{\infty} pq[n]_{p,q}\left(pq[n]_{p,q}(1+B)+(1+A)\right)\psi_n|a_n| \leq A-B.$$

But

$$\frac{p^{m+1}q^{m+1}[\delta+2]_{p,q}!}{[\delta]_{p,q}![2]_{p,q}!}\left[pq(1+B)+(1+A)\right]\sum_{n=1}^{\infty}|a_n|$$

$$\leq \sum_{n=1}^{\infty} pq[n]_{p,q}\left(pq[n]_{p,q}(1+B)+(1+A)\right)\psi_n|a_n| \leq A-B.$$

thus

$$\frac{p^{m+1}q^{m+1}[\delta+2]_{p,q}!}{[\delta]_{p,q}![2]_{p,q}!}\left[pq(1+B)+(1+A)\right]\sum_{n=1}^{\infty}|a_n| \leq A-B$$

therefore

$$\sum_{n=1}^{\infty}|a_n| \leq \frac{[2]_{p,q}!(A-B)}{p^{m+1}q^{m+1}[\delta+1]_{p,q}[\delta+2]_{p,q}(pq(1+B)+(1+A))}.$$

So, we get the required result by substituting this in (2.2) and (2.3).

Theorem 2.3. *If $f \in \mathcal{MC}_{p,q}(A,B,\delta,m)$ and has the form (1.1).Then for $|z|=r$*

$$\frac{[\lambda]_{p,q}!}{(pq)^{\gamma}r^{\lambda+1}} - rr_2 \leq |D^{\lambda}_{p,q}f(z)| \leq \frac{[\lambda]_{p,q}!}{(pq)^{\gamma}r^{\lambda+1}} + rr_2$$

where

$$r_2 = \frac{(A-B)}{p^{m+1}q^{m+1}(pq(1+B)+(1+A))} \quad \textit{and} \quad \gamma = \sum_{j=1}^{\lambda} j.$$

Proof. On application of (1.2)to the equation (1.1), we get

$$D^{\lambda}_{p,q}f(z) = \frac{(-1)^{\lambda}[\lambda]_{p,q}!}{(pq)^{\gamma}z^{\lambda+1}} + \sum_{n=1}^{\infty}\frac{[n]_{p,q}!}{[n-\lambda]_{p,q}!}a_n z^{n-\lambda}, \quad \textit{where} \quad \gamma = \sum_{j=1}^{\lambda} j.$$

Since $|z| = r < 1$, we have $r^{n-\lambda} < r$ for $\lambda \leq n$. Thus

$$|D^{\lambda}_{p,q}f(z)| \leq \frac{[\lambda]_{p,q}!}{(pq)^{\gamma}r^{\lambda+1}} + r\sum_{n=1}^{\infty}\frac{[n]_{p,q}!}{[n-\lambda]_{p,q}!}|a_n| \tag{2.4}$$

and similarly

$$|D^{\lambda}_{p,q}f(z)| \geq \frac{[\lambda]_{p,q}!}{(pq)^{\gamma}r^{\lambda+1}} - r\sum_{n=1}^{\infty}\frac{[n]_{p,q}!}{[n-\lambda]_{p,q}!}|a_n| \tag{2.5}$$

from theorem (2.1), we have

$$pq\left(pq(1+B)+(1+A)\right)\sum_{n=1}^{\infty}\psi_n|a_n|$$

$$\leq \sum_{n=1}^{\infty} pq[n]_{p,q}\left(pq[n]_{p,q}(1+B)+(1+A)\right)\psi_n|a_n| \leq A-B$$

Therefore

$$\sum_{n=1}^{\infty}\psi_n|a_n| \leq \frac{A-B}{pq\left(pq(1+B)+(1+A)\right)},$$

but certainly

$$\sum_{n=1}^{\infty} \frac{[n]_{p,q}!}{[n-\lambda]_{p,q}!}|a_n| \leq \frac{1}{p^m q^m} \sum_{n=1}^{\infty} \psi_n |a_n|,$$

which implies that

$$\sum_{n=1}^{\infty} \frac{[n]_{p,q}!}{[n-\lambda]_{p,q}!}|a_n| \leq \frac{A-B}{p^{m+1}q^{m+1}\left(pq(1+B)+(1+A)\right)}.$$

Finally, using this in (2.4)and (2.5), we get the required result.

Theorem 2.4. *Let* $f \in \mathcal{MC}_{p,q}(A,B,\delta,m)$*. Then* $f \in \mathcal{MC}(\beta)$ *for* $|z| < r^*$*, where*

$$r^* = \left(\frac{(1-\beta)[n]_{p,q}(pq[n]_{p,q}(1+B)+(1+A))}{n(n+\beta)(A-B)} \psi_n \right)^{\frac{1}{n+1}}$$

Proof. We know that f is meromorphically convex of order β if it obeys the inequality

$$-\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \beta$$

or equivalently

$$\left|\frac{2f'(z)+zf''(z)}{zf''(z)+2\beta f'(z)}\right| \leq 1.$$

Using (1.1) along with simple computation yields,

$$\sum_{n=1}^{\infty} \frac{n(n+\beta)}{(1-\beta)}|a_n||z|^{n+1} \leq 1. \tag{2.6}$$

from (2.1) we have

$$\frac{\sum_{n=1}^{\infty} pq[n]_{p,q}\left(pq[n]_{p,q}(1+B)+(1+A)\right)}{A-B}\psi_n|a_n| < 1$$

thus, the inequality (2.6) will be true if

$$\sum_{n=1}^{\infty} \frac{n(n+\beta)}{1-\beta}|a_n||z|^{n+1} < \frac{\sum_{n=1}^{\infty} pq[n]_{p,q}\left(pq[n]_{p,q}(1+B)+(1+A)\right)}{A-B}\psi_n|a_n|,$$

which implies that

$$|z|^{n+1} < \frac{(1-\beta)pq[n]_{p,q}\left(pq[n]_{p,q}(1+B)+(1+A)\right)}{n(n+\beta)(A-B)}\psi_n$$

therefore,

$$|z| < \left(\frac{(1-\beta)pq[n]_{p,q}\left(pq[n]_{p,q}(1+B)+(1+A)\right)}{n(n+\beta)(A-B)}\psi_n\right)^{\frac{1}{n+1}}$$

which completes the proof.

Theorem 2.5. *Let $f \in \mathcal{MC}_{p,q}(A,B,\delta,m)$. Then $f \in \mathcal{MS}^*(\beta)$ for $|z| < r^{**}$, where*

$$r^{**} = \left(\frac{(1-\beta)pq[n]_{p,q}\left(pq[n]_{p,q}(1+B)+(1+A)\right)}{(n+1+B)(A-B)}\right)^{\frac{1}{n+1}}.$$

Proof. Let $f \in \mathcal{MC}_{p,q}(A,B,\delta,m)$. To prove that $f \in \mathcal{MS}^*(\beta)$, we need to prove that

$$\Re\left\{\frac{-zf'(z)}{f(z)}\right\} > \beta$$

or equivalently

$$\left|\frac{zf'(z)+f(z)}{zf'(z)-(1-2\beta)f(z)}\right| \leq 1$$

using (1.1) and simplification yields

$$\sum_{n=1}^{\infty}\frac{(n+1+\beta)}{(1-\beta)}|a_n||z|^{n+1} \leq 1. \tag{2.7}$$

From (2.1), we have

$$\sum_{n=1}^{\infty}\frac{pq[n]_{p,q}\left(pq[n]_{p,q}(1+B)+(1+A)\right)}{A-B}\psi_n|a_n| < 1.$$

Thus (2.7) will be true only if

$$\sum_{n=1}^{\infty}\frac{(n+1+\beta)}{(1-\beta)}|a_n||z|^{n+1} < \sum_{n=1}^{\infty}\frac{pq[n]_{p,q}\left(pq[n]_{p,q}(1+B)+(1+A)\right)}{A-B}\psi_n|a_n| < 1.$$

Therefore

$$|z|^{n+1} < \frac{(1-\beta)pq[n]_{p,q}\left(pq[n]_{p,q}(1+B)+(1+A)\right)}{(n+1+\beta)(A-B)}\psi_n$$

Hence

$$|z| < \left(\frac{(1-\beta)pq[n]_{p,q}\,(pq[n]_{p,q}(1+B)+(1+A))}{(n+1+\beta)(A-B)} \psi_n \right)^{\frac{1}{n+1}} = r^{**}.$$

The proof is completed.

The partial sum for $f(z) \in \mathcal{MC}_{p,q}(A, B, \delta, m)$ is discussed in the following theorem.

Let

$$g(z) = \mathcal{L}_{p,q}^{\delta,m} f(z) = \frac{(-1)^m}{z} + \sum_{n=1}^{\infty} \psi_n a_n z^n$$

and its sequence of partial sum is given by

$$g_k(z) = \frac{(-1)^m}{z} + \sum_{n=1}^{k} \psi_n a_n z^n$$

we obtain the lower bound for the ratio of real part

$$\Re\left(\frac{g(z)}{g_k(z)}\right), \Re\left(\frac{g_k(z)}{g(z)}\right), \Re\left(\frac{D_{p,q}(g(z))}{D_{p,q}(g_k(z))}\right) \textit{and } \Re\left(\frac{D_{p,q}(g_k(z))}{D_{p,q}(g(z))}\right).$$

Theorem 2.6. *If $g(z) \in \mathcal{MC}_{p,q}(A, B, \delta, m)$, then $\forall z \in \mathcal{D}$*

$$\Re\left(\frac{g(z)}{g_k(z)}\right) \geq 1 - \frac{1}{\xi_{k+1}} \tag{2.8}$$

and

$$\Re\left(\frac{g_k(z)}{g(z)}\right) \geq \frac{\xi_{k+1}}{1+\xi_{k+1}} \tag{2.9}$$

where $\xi_k = \dfrac{pq[k]_{p,q}\,(pq[k]_{p,q}(1+B)+(1+A))}{A-B}$.

Proof. For (2.8), consider

$$\xi_{k+1}\left(\frac{g(z)}{g_k(z)} - \left(1 - \frac{1}{\xi_{k+1}}\right)\right)$$

$$= \frac{(-1)^m + \sum_{n=1}^{k} \psi_n a_n z^{n+1} + \xi_{k+1}\left(\sum_{n=k+1}^{\infty} \psi_n a_n z^{n+1}\right)}{(-1)^m + \sum_{n=1}^{k} \psi_n a^n z^{n+1}}$$

$$= \frac{(-1)^m + u_1(z)}{(-1)^m + u_2(z)}$$

if we put

$$\frac{(-1)^m + u_1(z)}{(-1)^m + u_2(z)} = \frac{(-1)^m + v(z)}{(-1)^m - v(z)}$$

then

$$v(z) = \frac{(-1)^m(u_1(z) - u_2(z))}{2(-1)^m + u_1(z) + u_2(z)}$$

$$|v(z)| \leq \frac{\xi_{k+1} \sum_{n=k+1}^{\infty} \psi_n |a_n|}{2 - 2\sum_{n=1}^{k} \psi_n |a_n| - \xi_{k+1} \sum_{n=k+1}^{\infty} \psi_n |a_n|}$$

$$|v(z)| \leq 1 \iff 2\xi_{k+1} \sum_{n=k+1}^{\infty} \psi_n |a_n| \leq 2 - 2\sum_{n=1}^{k} \psi_n |a_n|$$

$$\left[\sum_{n=1}^{k} \psi_n |a_n| + \xi_{k+1} \sum_{n=k+1}^{\infty} \psi_n |a_n|\right] \leq 1$$

to obtain (2.8), it is enough to prove that the left hand side of this expression is bounded above by $\sum_{n=1}^{\infty} \xi_n \psi_n |a_n|$

that is,

$$\sum_{n=1}^{k}(1 - \xi_n)\psi_n |a_n| + \sum_{n=k+1}^{\infty} (\xi_{k+1} - \xi_n)\psi_n |a_n| \geq 0.$$

To prove (2.9), consider

$$(1 + \xi_{k+1}) \left[\frac{g_k(z)}{g(z)} - \frac{\xi_{k+1}}{1 + \xi_{k+1}}\right]$$

$$= \frac{(-1)^m + \sum_{n=1}^{k} \psi_n a_n z^{n+1} - \xi_{k+1} \sum_{n=k+1}^{\infty} \psi_n a_n z^{n+1}}{(-1)^m + \sum_{n=1}^{\infty} \psi_n a_n z^{n+1}}$$

$$= \frac{(-1)^m + v(z)}{(-1)^m - v(z)}$$

$$|v(z)| \leq \frac{(1+\xi_{k+1}) \displaystyle\sum_{n=k+1}^{\infty} \psi_n |a_n|}{2 - 2\displaystyle\sum_{n=1}^{k} \psi_n |a_n| - (\xi_{k+1} - 1) \displaystyle\sum_{n=k+1}^{\infty} \psi_n |a_n|} \leq 1$$

if and only if

$$\sum_{n=1}^{k} \psi_n |a_n| + \xi_{k+1} \sum_{n=k+1}^{\infty} \psi_n |a_n| \leq 1$$

to prove (2.9), it is enough to prove that the left hand side of this expression is bounded by $\displaystyle\sum_{n=1}^{\infty} \xi_n \psi_n |a_n|$. that is,

$$\sum_{n=1}^{k} (1-\xi_n)\psi_n |a_n| + \sum_{n=k+1}^{\infty} (\xi_{k+1} - \xi_n)\psi_n |a_n| \geq 0.$$

Theorem 2.7. *If $g(z) \in \mathcal{MC}_{p,q}(A, B, \delta, m)$, then $\forall z \in \mathcal{D}$*

$$\Re\left(\frac{D_{p,q}g(z)}{D_{p,q}g_k(z)}\right) \geq 1 - \frac{[k+1]_{p,q}}{\xi_{k+1}} \tag{2.10}$$

and

$$\Re\left(\frac{D_{p,q}g_k(z)}{D_{p,q}g(z)}\right) \geq \frac{\xi_{k+1}}{\xi_{k+1} + [k+1]_{p,q}} \tag{2.11}$$

where $\quad \xi_k = \dfrac{pq[k]_{p,q}\left(pq[k]_{p,q}(1+B) + (1+A)\right)}{A-B}.$

Proof. The proof is similar to the previous theorem.

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