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FRACTIONAL SHEHU TRANSFORM AND ITS APPLICATIONS

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Abstract: In these study, we proposed new generalized Shehu transform of fractional order called "fractional Shehu transform" of order $0 < \alpha < 1$. This transform is applicable for functions which are differentiable but by fractional order. By using the definition of fractional order Shehu transform we prove fundamental properties of these integral transform. Finally, we have obtained convolution and inversion.

Keywords and Phrases: Shehu Transform, Laplace Transform, Mittag-Leffler function, Generalized function (Dirac's Distribution), Fractional Derivative, Fractional Integration.

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1. Introduction

We all are familiar about the applications of integral transform for finding the solution of different differential and integral equations [4, 16]. It is best tool to find the solutions of many of these problems. Shehu transform is the Laplace type integral transform moreover it is generalization of Laplace and Sumudu transform [2, 10] which is widely used for solving differential equations with efficient and more convenient way. If p(z) is continuous and continuously differentiable then by using regular definitions of different integral transform, we solve differential equations of function p(z) but if p(z) is continues and differentiable by fractional order α, then these definitions doesn't work, in that case, we use the definition of fractional order Shehu transform to find the solution of differential equations in particular fractional order differential equation of function p(z).

Authors worked in these area, to develop some integral transform [12, 15], use that generalized concepts to derive new methods [13, 14] to find solution of fractional differential equations.

In these paper their are three main sections, particularly in second section we list basic definitions, like definition of Shehu transform, Mittag-Leffler function, fractional derivative and so on. In the third section we define fractional order Shehu transform and prove some important results and properties. Further in the final section we have included derivation of convolution theorem and inversion formula and solve some examples of fractional differential equations using fractional Shehu transform.

2. Background of Shehu Transform and Fractional Derivatives

First, we list definitions of Shehu transform, fractional order derivative in the finite difference form and other related definitions.

Definition 2.1. Shehu transform [2, 6, 7, 10].
The Shehu transform of function p(z) of exponential order define over the set,

$$A = \{p(z)/\exists K, \tau_1, \tau_2 > 0, |p(z)| \leq Ke^{|z|/\tau_i}, if \quad z \in (-1)^i \in [0, \infty)\}$$

where K is finite constant and τ_1, τ_2 may be finite or infinite.
is the integral equation,

$$\mathcal{S}[p(z)] = P(v, w) = \int_0^\infty e^{-vz/w} p(z) dz \tag{1}$$

where $v > 0$ and $w > 0$.
The Shehu inverse integral transform [6, 7, 10] is define as,

$$\mathcal{S}^{-1}[P(v, w)] = p(z) = \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} \frac{1}{w} e^{(vz/w)} P(v, w) dv. \tag{2}$$

Definition 2.2. Derivative of Shehu transform [2, 6, 10].
If the function $p^{(n)}(z)$ is the n^{th} derivative of the function $p(z) \in A$ with respect to z then it's Shehu transform is defined as,

$$\mathcal{S}[p^{(n)}(z)] = \frac{v^n}{w^n} P(v, w) - \sum_{k=0}^{n-1} (\frac{v}{w})^{n-(k+1)} p^{(k)}(0)$$

where $n \in \mathbb{N}$.

Definition 2.3. Shehu transform of Mittag-Leffler function [2],

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

$\alpha,\ \beta\ \in\ \mathbb{C}$ *and* $Re(\alpha),\ \ Re(\beta) > 0$ *is given by,*

$$\mathcal{S}[z^{\tau-1} E_{\alpha,\beta}(\zeta z^{\alpha})] = \frac{v^{\beta-1}}{w^{\beta-1}}(1 - \zeta(\frac{v}{w})^{\alpha})^{-\tau} \tag{3}$$

Where $Re(\alpha)$, $Re(\beta)$, $Re(\tau) > 0$ *and* $\zeta \in \mathbb{C}$.

Definition 2.4. Definition of Fractional order derivative in finite Difference form[5]. *Let* $p : \mathbb{R} \to \mathbb{R}$ *denotes a continuous function and* $h > 0$ *denote content discretization span then,*
define the forward operator $FW(h)$ *by the equality,*

$$FW(h)(p(z)) = p(z + h).$$

Then fractional order derivative of order α*,where* $0 < \alpha < 1$ *of p(z) is,*

$$\Delta^{\alpha} p(z) = (FW - 1)^{\alpha} = \sum_{k=0}^{\infty} (-1)^k C_k^{\alpha} p(z + (\alpha - k)h). \tag{4}$$

Fractional derivative of order α *is the limit,*

$$p^{(\alpha)}(z) = \lim_{h \to 0} \frac{\Delta^{\alpha} p(z)}{h^{\alpha}}. \tag{5}$$

Definition 2.5. *Let* a, α *and* $z \in \mathbb{R}$ *with* $\alpha>0$, $z>a$ *then the operator,*

$$D^{\alpha}[p(z)] = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dz^n} \int_a^t \frac{p(t)}{(z-t)^{\alpha+1-n}} dt, \quad n-1<\alpha<n$$

is called Riemann-Liouville fractional derivative of order α [11]*, and*

$$D_*^{\alpha}[p(z)] = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{p^{(n)}(t)}{(z-t)^{\alpha+1-n}} dt, \quad n-1<\alpha<n$$

is called Caputo fractional derivative of order α [11].

Definition 2.6. Fractional derivative of Shehu transform [10, 11].
If the function p(z) $\in A$, is of exponential order then Shehu transform of Riemann-Liouville fractional derivative of p(z) with order α is defined as,

$$\mathcal{S}[p^{(\alpha)}(z)] = \frac{v^\alpha}{w^\alpha}P(v,w) - \sum_{k=0}^{n-1}(\frac{v}{w})^{\alpha-(k+1)}[D^k I^{(n-\alpha)}p(z)]_{z=0} \tag{6}$$

where $\alpha \in \mathbb{R}$.
Shehu transform of Caputo fractional derivative of p(z) with order α is defined as,

$$\mathcal{S}[p_*^{(\alpha)}(z)] = \frac{v^\alpha}{w^\alpha}P(v,w) - \sum_{k=0}^{n-1}(\frac{v}{w})^{\alpha-(k+1)}p^{(k)}(0) \tag{7}$$

where $\alpha \in \mathbb{R}$.

3. Main Result

Definition 3.1. *Shehu transform of fractional order α of non-negative function p(z), denoted by $\mathcal{S}_\alpha[p(z)]$, and defined as,*

$$\mathcal{S}_\alpha[p(z)] = P_\alpha(v,w) = \int_0^\infty E_\alpha(\frac{-v^\alpha z^\alpha}{w^\alpha})p(z)(dz)^\alpha \tag{8}$$

where $v \in \mathbb{C}$, *and* $E_\alpha(z)$ *is Mittag-Leffler function* $E_\alpha p(z) = \sum_{k=0}^\infty \frac{z^\alpha}{\Gamma(\alpha k+1)}$.

Theorem 3.1. Existence Theorem of Fractional order Shehu Transform. *If function p(z) is non-negative piecewise continuous in interval $0 \leq z \leq \zeta$ and it is of exponential order α then its fractional Shehu transform $P_\alpha(v,w)$ exist.*
Proof. Suppose that,

$$\int_0^\infty E_\alpha(\frac{-v^\alpha z^\alpha}{w^\alpha})p(z)(dz)^\alpha = \int_0^\zeta E_\alpha(\frac{-v^\alpha z^\alpha}{w^\alpha})p(z)(dz)^\alpha + \int_\zeta^\infty E_\alpha(\frac{-v^\alpha z^\alpha}{w^\alpha})p(z)(dz)^\alpha. \tag{9}$$

Since p(z) is pricewise continuous in interval $[0,\zeta]$ with $0 \leq z \leq \zeta$ then first integral of RHS of (9) exist, since p(z) is of exponential order α for $\zeta < z$, to check the existence we concentrate on second term of RHS of (9) then,

$$\begin{aligned}
|\int_{\zeta}^{\infty} E_{\alpha}(\frac{-v^{\alpha}z^{\alpha}}{w^{\alpha}})p(z)(dz)^{\alpha}| &\leq \int_{\zeta}^{\infty} |E_{\alpha}(\frac{-v^{\alpha}z^{\alpha}}{w^{\alpha}})p(z)|(dz)^{\alpha} \\
&\leq \int_{\zeta}^{\infty} E_{\alpha}(\frac{-v^{\alpha}z^{\alpha}}{w^{\alpha}})|p(z)|(dz)^{\alpha} \\
&\leq \int_{\zeta}^{\infty} E_{\alpha}(\frac{-v^{\alpha}z^{\alpha}}{w^{\alpha}})CE_{\alpha}(u^{\alpha}z^{\alpha})(dz)^{\alpha} \\
&= C\int_{\zeta}^{\infty} E_{\alpha}(\frac{-(v-uw)^{\alpha}z^{\alpha}}{w^{\alpha}})(dz)^{\alpha} \\
&= C\lim_{n\to\infty}\int_{\zeta}^{n} E_{\alpha}(\frac{-(v-uw)^{\alpha}z^{\alpha}}{w^{\alpha}})(dz)^{\alpha}
\end{aligned}$$

$$\begin{aligned}
&= \frac{-Cw^{\alpha}}{(v-uw)^{\alpha}}\lim_{n\to\infty} E_{\alpha}(\frac{-(v-uw)^{\alpha}z^{\alpha}}{w^{\alpha}})]_{\zeta}^{n} \\
&= \frac{-Cw^{\alpha}}{(v-uw)^{\alpha}}[0-E_{\alpha}(\frac{-(v-uw)^{\alpha}\zeta^{\alpha}}{w^{\alpha}})]
\end{aligned}$$

But as $\zeta \to 0$ then we get the existence of Second term of RHS also,

$$|\int_{\zeta}^{\infty} E_{\alpha}(\frac{-v^{\alpha}z^{\alpha}}{w^{\alpha}})p(z)(dz)^{\alpha}| \leq \frac{Cw^{\alpha}}{(v-uw)^{\alpha}}. \tag{10}$$

This completes the proof.
Now we prove basic properties related to fractional order Shehu transform,

Theorem 3.2. Linearity property.
Let functions $ap(z), bq(z) \in A$ then $ap(z)+bq(z) \in A$ where a and b are nonzero arbitrary constants and,

$$\mathcal{S}_{\alpha}[ap(z)+bq(z)] = a\mathcal{S}_{\alpha}[p(z)] + b\mathcal{S}_{\alpha}[q(z)]. \tag{11}$$

Proof. By using definition in equation (8) for LHS in Equation (11) we get,

$$\begin{aligned}
\mathcal{S}_{\alpha}[ap(z)+bq(z)] &= \int_{0}^{\infty} E_{\alpha}(\frac{-v^{\alpha}z^{\alpha}}{w^{\alpha}})[ap(z)+bq(z)](dz)^{\alpha} \\
&= a\int_{0}^{\infty} E_{\alpha}(\frac{-v^{\alpha}z^{\alpha}}{w^{\alpha}})p(z)(dz)^{\alpha} + b\int_{0}^{\infty} E_{\alpha}(\frac{-v^{\alpha}z^{\alpha}}{w^{\alpha}})q(z)(dz)^{\alpha} \\
&= a\mathcal{S}_{\alpha}[p(z)] + b\mathcal{S}_{\alpha}[q(z)]
\end{aligned}$$

This is the complete proof.

Theorem 3.3. *If $p(z) \in A$ and D_v^α is the derivative of a function with respect to v of order α then,*

$$\mathcal{S}_\alpha[\frac{z^\alpha}{w^\alpha}p(z)] = -D_v^\alpha \mathcal{S}_\alpha[p(z)]. \tag{12}$$

Proof. By using definition of fractional Shehu transform in (8) then,

$$\begin{aligned} D_v^\alpha \mathcal{S}_\alpha[p(z)] = D_v^\alpha P_\alpha(v,w) &= D_v^\alpha \int_0^\infty E_\alpha(\frac{-v^\alpha z^\alpha}{w^\alpha})[p(z)](dz)^\alpha \\ &= \int_0^\infty D_v^\alpha E_\alpha(\frac{-v^\alpha z^\alpha}{w^\alpha})[p(z)](dz)^\alpha \\ &= \int_0^\infty E_\alpha(\frac{-v^\alpha z^\alpha}{w^\alpha})p(z)(\frac{-z^\alpha}{w^\alpha})(dz)^\alpha \\ &= -\int_0^\infty E_\alpha(\frac{-v^\alpha z^\alpha}{w^\alpha})(\frac{z^\alpha}{w^\alpha})p(z)(dz)^\alpha \\ &= -\mathcal{S}_\alpha[\frac{z^\alpha}{w^\alpha}p(z)] \end{aligned}$$

This is the final proof of above equation.

Theorem 3.4. Change of scale property of fractional order Shehu transform.
Let $p(az) \in A$, where a be any constant then,

$$\mathcal{S}_\alpha[p(az)]_v = (\frac{1}{a})^\alpha \mathcal{S}_\alpha[p(z)]_{v/a}. \tag{13}$$

Proof. By using definition of fractional Shehu transform,

$$\mathcal{S}_\alpha[p(az)]_v = \int_0^\infty E_\alpha(\frac{-v^\alpha z^\alpha}{w^\alpha})[p(az)](dz)^\alpha$$

Put $az = x$ then $z = \frac{x}{a}$ then,

$$\begin{aligned} &= \int_0^\infty E_\alpha(\frac{-v^\alpha x^\alpha}{w^\alpha a^\alpha})[p(x)]\frac{(dx)^\alpha}{a^\alpha} \\ &= (\frac{1}{a})^\alpha \int_0^\infty E_\alpha(\frac{-v^\alpha x^\alpha}{w^\alpha a^\alpha})[p(x)](dx)^\alpha \\ &= (\frac{1}{a})^\alpha \int_0^\infty E_\alpha(\frac{-v^\alpha z^\alpha}{w^\alpha a^\alpha})[p(z)](dz)^\alpha \\ &= (\frac{1}{a})^\alpha \mathcal{S}_\alpha[p(z)]_{v/a} \end{aligned}$$

is the complete proof.

Theorem 3.5. Shifting property.
Let $p(z) \in A$ then for $p(z-b) \in A$ where b is constant, following holds,

$$\mathcal{S}_\alpha[p(z-b)] = E_\alpha(\frac{-v^\alpha b^\alpha}{w^\alpha})\mathcal{S}_\alpha[p(z)]. \tag{14}$$

Proof. By using definition of fractional Shehu transform,

$$L.H.S. = \mathcal{S}_\alpha[p(z-b)] = \int_0^\infty E_\alpha(\frac{-v^\alpha z^\alpha}{w^\alpha})p(z-b)(dz)^\alpha$$

Put $z-b=x$ then $z=x+b$ then,

$$\begin{aligned}
&= \int_0^\infty E_\alpha(\frac{-v^\alpha (x+b)^\alpha}{w^\alpha})p(x)(dx)^\alpha \\
&= \int_0^\infty E_\alpha(\frac{-v^\alpha x^\alpha}{w^\alpha})E_\alpha(\frac{-v^\alpha b^\alpha}{w^\alpha})p(x)(dx)^\alpha \\
&= E_\alpha(\frac{-v^\alpha b^\alpha}{w^\alpha})\int_0^\infty E_\alpha(\frac{-v^\alpha x^\alpha}{w^\alpha})p(x)(dx)^\alpha \\
&= E_\alpha(\frac{-v^\alpha b^\alpha}{w^\alpha})\int_0^\infty E_\alpha(\frac{-v^\alpha z^\alpha}{w^\alpha})p(z)(dz)^\alpha \\
&= E_\alpha(\frac{-v^\alpha b^\alpha}{w^\alpha})\mathcal{S}_\alpha[p(z)] = R.H.S.
\end{aligned}$$

is the complete proof.

Theorem 3.6. *Let $E_\alpha(z)$ be the Mittag-Leffler function and $p(z) \in A$ then,*

$$\mathcal{S}_\alpha[E_\alpha(\frac{-c^\alpha z^\alpha}{w^\alpha})p(z)] = \mathcal{S}_\alpha[p(z)]_{v+c}. \tag{15}$$

Proof. By using definition of fractional Shehu transform,

$$\begin{aligned}
L.H.S. &= \mathcal{S}_\alpha[E_\alpha(\frac{-c^\alpha z^\alpha}{w^\alpha})p(z)] \\
&= \int_0^\infty E_\alpha(\frac{-v^\alpha z^\alpha}{w^\alpha})E_\alpha(\frac{-c^\alpha z^\alpha}{w^\alpha})p(z)](dz)^\alpha \\
&= \int_0^\infty E_\alpha(-[\frac{v^\alpha z^\alpha}{w^\alpha} + \frac{c^\alpha z^\alpha}{w^\alpha}])p(z)(dz)^\alpha \\
&= \int_0^\infty E_\alpha(\frac{-(v+c)^\alpha z^\alpha}{w^\alpha})[p(z)](dz)^\alpha \\
&= \mathcal{S}_\alpha[p(z)]_{v+c}.
\end{aligned}$$

Theorem 3.7. *Let $p(z) \in A$ then,*

$$\mathcal{S}_\alpha[D_z^\alpha p(z)] = \frac{v^\alpha}{w^\alpha}\mathcal{S}_\alpha[p(z)] - \Gamma(1+\alpha)p(0). \tag{16}$$

Proof. Using the definition of fractional order Shehu transform,

$$\mathcal{S}_\alpha[D_z^\alpha p(z)] = \int_0^\infty E_\alpha(\frac{-v^\alpha z^\alpha}{w^\alpha}) D_z^\alpha p(z)(dz)^\alpha.$$

By using the definition of fractional integration by part formula we get,

$$\begin{aligned} L.H.S. &= \Gamma(1+\alpha)p(z)E_\alpha(\frac{-v^\alpha z^\alpha}{w^\alpha})|_0^\infty - \int_0^\infty D_z^\alpha E_\alpha(\frac{-v^\alpha z^\alpha}{w^\alpha})p(z)(dz)^\alpha \\ &= -\Gamma(1+\alpha)p(0) - (\frac{-v^\alpha}{w^\alpha})\int_0^\infty E_\alpha(\frac{-v^\alpha z^\alpha}{w^\alpha})p(z)(dz)^\alpha \\ &= -\Gamma(1+\alpha)p(0) + (\frac{v^\alpha}{w^\alpha})\int_0^\infty E_\alpha(\frac{-v^\alpha z^\alpha}{w^\alpha})p(z)(dz)^\alpha \\ &= -\Gamma(1+\alpha)p(0) + (\frac{v^\alpha}{w^\alpha})\mathcal{S}_\alpha[p(z)] \\ &= (\frac{v^\alpha}{w^\alpha})\mathcal{S}_\alpha[p(z)] - \Gamma(1+\alpha)p(0). \end{aligned}$$

Theorem 3.8. *Let $p(z) \in A$ then,*

$$\mathcal{S}_\alpha[\int_0^z p(t)dt^\alpha] = \Gamma(1+\alpha)^{-1}(\frac{v}{w})^{-\alpha}\mathcal{S}_\alpha[p(z)]. \tag{17}$$

Proof. By using the definition of fractional order integral transform of Laplace transform [5],

$$\mathcal{L}_\alpha[\int_0^z p(t)dt^\alpha] = \Gamma(1+\alpha)^{-1}(s)^{-\alpha}\mathcal{L}_\alpha[p(z)].$$

Using the duality of Laplace-Shehu transform,

$$\mathcal{S}_\alpha[\int_0^z p(t)dt^\alpha] = \Gamma(1+\alpha)^{-1}(\frac{v}{w})^{-\alpha}\mathcal{S}_\alpha[p(z)].$$

4. Convolution theorem of fractional order Shehu transform

Theorem 4.1. *If the convolution of order α of two functions p(z) and q(z) is define by the integral of the form,*

$$(p(z) * q(z))_\alpha = \int_0^z p(z-u)q(u)(du)^\alpha.$$

Then we can write,

$$\mathcal{S}_\alpha[[p(z) * q(z)]_\alpha] = \mathcal{S}_\alpha[p(z)]\mathcal{S}_\alpha[q(z)]. \tag{18}$$

Proof. We starts from definition,

$$\mathcal{S}_\alpha[[p(z) * q(z)]_\alpha] = \int_0^\infty E_\alpha(\frac{-v^\alpha z^\alpha}{w^\alpha}) \int_0^z [p(z-u)q(u)](du)^\alpha (dz)^\alpha$$

By substituting $t = z - u$, and taking limits from zero to infinite we get,

$$\begin{aligned}
\mathcal{S}_\alpha[[p(z) * q(z)]_\alpha] &= \int_0^\infty E_\alpha(\frac{-v^\alpha (t+u)^\alpha}{w^\alpha}) \int_0^\infty [p(t)q(u)](du)^\alpha (dt)^\alpha \\
&= [\int_0^\infty E_\alpha(\frac{-v^\alpha t^\alpha}{w^\alpha}) p(t)(dt)^\alpha][\int_0^\infty E_\alpha(\frac{-v^\alpha u^\alpha}{w^\alpha}) q(u)(du)^\alpha] \\
&= [\int_0^\infty E_\alpha(\frac{-v^\alpha z^\alpha}{w^\alpha}) p(z)(dz)^\alpha][\int_0^\infty E_\alpha(\frac{-v^\alpha z^\alpha}{w^\alpha}) q(z)(dz)^\alpha] \\
&= \mathcal{S}_\alpha[p(z)]\mathcal{S}_\alpha[q(z)]
\end{aligned}$$

Is the final proof.

4.2. Inversion formula for fractional Shehu Transform

Definition 4.1. *The Dirac's distribution* [8] *also known as generalized function, $\delta_\alpha(z)$ of order α, where $\alpha \in (0,1)$, is define as,*

$$\int_{\mathcal{R}} p(z)\delta_\alpha(z) dz^\alpha = \alpha p(0). \tag{19}$$

Example 1. The fractional Shehu transform of $\delta_\alpha(z-a)$ can be written as follows,

$$\mathcal{S}_\alpha[\delta_\alpha(z-a)] = \int_0^\infty E_\alpha(\frac{-v^\alpha z^\alpha}{w^\alpha}) \delta_\alpha(z-a)(dz)^\alpha$$

$$= \alpha E_\alpha(\frac{-v^\alpha a^\alpha}{w^\alpha})$$

Particularly for the value of a=0 then, $\mathcal{S}_\alpha[\delta_\alpha(z)] = \alpha$.

Lemma 4.2. *The equality,*

$$\frac{\alpha}{(\vartheta_\alpha)^\alpha} \int_{-\infty}^{+\infty} E_\alpha(i(-\rho z)^\alpha)(d\rho)^\alpha = \delta_\alpha(z). \tag{20}$$

[8] *holds where* ϑ_α *is the period of complexed value Mittag-Leffler function defined by the equality,* $E_\alpha(i(\vartheta_\alpha)^\alpha) = 1$.

Proof. From equation (19) we can write,

$$\alpha = \int_{\mathcal{R}} E_\alpha(i(\rho z)^\alpha)\delta_\alpha(z)dz^\alpha.$$

By using the value $\delta_\alpha(z)$ to LHS of the above equation we get,

$$\begin{aligned}
&= \int_{\mathcal{R}} E_\alpha(i(\rho z)^\alpha)\frac{\alpha}{(\vartheta_\alpha)^\alpha}\int_{-\infty}^{+\infty} E_\alpha(i(-uz)^\alpha)(du)^\alpha dz^\alpha \\
&= \int_{\mathcal{R}} \frac{\alpha}{(\vartheta_\alpha)^\alpha}\int_{\mathcal{R}} E_\alpha(i(\rho z)^\alpha)E_\alpha(i(-uz)^\alpha)(du)^\alpha dz^\alpha \\
&= \int_{\mathcal{R}}\int_{\mathcal{R}} \frac{\alpha}{(\vartheta_\alpha)^\alpha} E_\alpha(i((\rho-u)z)^\alpha)(du)^\alpha dz^\alpha \\
&= \int_{\mathcal{R}}\int_{\mathcal{R}} \frac{\alpha}{(\vartheta_\alpha)^\alpha} E_\alpha(i(\varphi z)^\alpha)(d\varphi)^\alpha dz^\alpha \\
&= \int_{\mathcal{R}} \delta_\alpha(z)dz^\alpha.
\end{aligned}$$

4.3. Inversion Theorem of Fractional Order Shehu Transform

Lemma 4.3. *Fractional Shehu transform define in definition (7),*

$$\mathcal{S}_\alpha[p(z)] = P_\alpha(v,w) = \int_0^\infty E_\alpha(\frac{-v^\alpha z^\alpha}{w^\alpha})p(z)(dz)^\alpha. \tag{21}$$

then its inversion formula is,

$$p(z) = \frac{1}{(\vartheta_\alpha)^\alpha}\int_{+i\infty}^{-i\infty} E_\alpha(\frac{v^\alpha z^\alpha}{w^\alpha})P_\alpha(v)(dv)^\alpha. \tag{22}$$

Proof. By substituting equation (8) in (22) and using (19) in (20) we get,

$$\begin{aligned}
p(z) &= \frac{1}{(\vartheta_\alpha)^\alpha}\int_{+i\infty}^{-i\infty} E_\alpha(\frac{v^\alpha z^\alpha}{w^\alpha})\int_0^\infty E_\alpha(\frac{-v^\alpha t^\alpha}{w^\alpha})p(t)(dt)^\alpha(dv)^\alpha \\
&= \frac{1}{(\vartheta_\alpha)^\alpha}\int_0^\infty p(t)(dt)^\alpha\int_{+i\infty}^{-i\infty} E_\alpha(\frac{v^\alpha z^\alpha}{w^\alpha})E_\alpha(\frac{-v^\alpha t^\alpha}{w^\alpha})(dv)^\alpha \\
&= \frac{1}{(\vartheta_\alpha)^\alpha}\int_0^\infty p(t)(dt)^\alpha\int_{+i\infty}^{-i\infty} E_\alpha(\frac{-v^\alpha(t-z)^\alpha}{w^\alpha})(dv)^\alpha \\
&= \frac{1}{(\vartheta_\alpha)^\alpha}\int_0^\infty \frac{(\vartheta_\alpha)^\alpha}{\alpha}p(t)\delta_\alpha(z-t)(dt)^\alpha \\
&= \frac{1}{\alpha}\int_{\mathcal{R}} p(t)\delta_\alpha(z-t)(dt)^\alpha \\
&= \frac{1}{\alpha}\alpha p(z) = p(z).
\end{aligned}$$

Example 4.4. Fractional differential equation,

$$D_*^\alpha[p(z)] = 0 \; with \; p^{(k)}(0) = T_k, \quad where \;\; k = 0, 1, 2, 3... \tag{23}$$

D_*^α shows Caputo derivative of order α.

Solution. Appying fractional Shehu transform on both sides to the equation (23),

$$\frac{v^\alpha}{w^\alpha}P(v,w) - \sum_{k=0}^{n-1}(\frac{v}{w})^{\alpha-(k+1)}p^{(k)}(0) = 0$$

By rearranging,

$$P(v,w) = \frac{\sum_{k=0}^{n-1}(\frac{v}{w})^{\alpha-(k+1)}p^{(k)}(0)}{(\frac{v}{w})^\alpha}$$

$$P(v,w) = \sum_{k=0}^{n-1}(\frac{v}{w})^{-(k+1)}p^{(k)}(0)$$

using initial conditions,

$$P(v,w) = \sum_{k=0}^{n-1}(\frac{v}{w})^{-(k+1)}T_k$$

Taking inverse fractional Shehu transform of about equation gives,

$$p(z) = \sum_{k=0}^{n-1} T_k z^k E_{\alpha,k+1}(z^\alpha)$$

is the final solution.

Example 4.5. Fractional differential equation [11],

$$D_*^\alpha[p(z)] = f(z) \; with \; p^{(k)}(0) = T_k, \quad where \;\; k = 0, 1, 2, 3... \tag{24}$$

D_*^α shows Caputo derivative of order α.
Solution. Using fractional Shehu transform on both sides to the equation (24),

$$\frac{v^\alpha}{w^\alpha}P(v,w) - \sum_{k=0}^{n-1}(\frac{v}{w})^{\alpha-(k+1)}p^{(k)}(0) = F(z)$$

By rearranging,

$$P(v,w) = \frac{F(z) + \sum_{k=0}^{n-1}(\frac{v}{w})^{\alpha-(k+1)}p^{(k)}(0)}{(\frac{v}{w})^\alpha}$$

$$P(v,w) = \frac{F(z)}{(\frac{v}{w})^\alpha} + \frac{\sum_{k=0}^{n-1}(\frac{v}{w})^{-(k+1)}p^{(k)}(0)}{(\frac{v}{w})^\alpha}$$

using initial conditions,

$$P(v,w) = \frac{F(z)}{(\frac{v}{w})^\alpha} + \frac{\sum_{k=0}^{n-1}(\frac{v}{w})^{-(k+1)}}{(\frac{v}{w})^\alpha}T_k$$

Taking inverse fractional Shehu transform,

$$p(z) = z^{\alpha-1}E_{\alpha,\alpha}(z^\alpha)f(z) + \sum_{k=0}^{n-1}T_k z^k E_{\alpha,k+1}(z^\alpha)$$

is the final solution.

5. Conclusion

From the above study we have developed fractional Shehu transform. Fractional Shehu transform satisfy properties of integral transform with main results like convolution and inversion theorem. Even they gives analytic solution of linear fractional differential equations, for further study one can try to find solution of non-linear fractional differential equations using fractional Shehu transform.

References

[1] Alfaqeih S. and Ozis T., Note on double Aboodh transform of fractional order and its properties, online mathematics, (01) (2019), 2672-7501.

[2] Belgacem R., Baleanu D., Bokhari A., Shehu transform and applications to caputo-fractional differential equations, international journal of analysis and applications, (17) (November 2019), 917-927.

[3] Demirci E., Zalp N. O., A methods for solving differential equation of fractional order, Journal of computational and applied mathematics, (236) (2012).

[4] Eltayeb H., Kilicman A., Jleli M. B., Fractional integral transform and its applications, Hindawi publication Corporation, 2(1) (2014), 1-15.

[5] Jumarie G., Laplace transform of fractional order via the Mittag-Leffler function and modified Riemann-Liouville derivative, Elsevier applied mathematics letters, (22) (2019) 1659-1664.

[6] Khalouta A. and Kadem A., A new method to solve fractional differential equations: Inverse fractional Shehu transform method, Appl. Appl. Math., 14(2) (2019), 926-941.

[7] Khalouta A. and Kadem A., A Comparative Study of Shehu Variational Iteration Method and Shehu Decomposition Method for Solving Nonlinear Caputo Time-Fractional Wave-like Equations with Variable Coefficients, Appl. Appl. Math., 15(1) (2020), 430-445.

[8] Kimeu J. M., Fractional calculus definitions and application, top scholar master's thesis and specialist project.

[9] Lakshmikantam V., Vatsala A. S., Basic theory of fractional differential equation, Science Direct, nonlinear analysis, (69) (2008).

[10] Maitama S., Zhao W., New integral transform: Shehu transform a generalization of Sumudu and Laplace transform for solving differential equations, international journal of analysis and applications, (17) (November 2019), 167-190.

[11] Shaikh A. S. and Timol M. G., On the solution of fractional differential equation using Laplace and Sumudu transform method, Int. Journal of eng. Res. and Tech., 23 (3) (2015), 1-4.

[12] Thange T. G. and Gade A. R., On the Aboodh transform of fractional differential operator, Malaya Journal of Matematik, 8(01) (2020), 225-229.

[13] Thange T. G. and Gade A. R., Analytical solution for time fractional diffusion equation by Laplace-Carson Decomposition method, Our heritage, 68 (6) (2020), 45-50.

[14] Thange T. G. and Gade A. R., A new approach for solving a system of fractional partial differential Equation Advances in Mathematics Scientific Journal, 9 (4) (2020), 1933-1944.

[15] Thange T. G. and Gade A. R., Laplace-Carson Transform of Fractional Order Malaya Journal of Matematik, 8 (04) (2020), 2253-2258.

[16] Yang X. J., Beleanu D., Shrivastava H. M., Local fractional integral transform and their applications, Ebook (1) (2015), 0-262.