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CERTAIN INTEGRATION INVOLVING HERMITE AND GEGENBAUER POLYNOMIALS

Salahuddin and Anita

Department of Mathematics,
PDM University, Bahadurgarh - 124507, Haryana, INDIA

E-mail : vsludn@gmail.com

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Abstract: The present paper deals with some Definite Integral comprising Hermite along with Gegenbauer polynomials in association with Hypergeometric function.

Keywords and Phrases: Pochhammer symbol, Hermite Polynomial, Hypergeometric Function, Gegenbauer polynomial.

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1. Introduction

In calculation, Pochhammer symbol is stated as [Steffensen p.8]

$$(z)_p = z(z-1)(z-2).......(z-p+1) = \prod_{t=1}^{p}(z-t+1) = \prod_{t=0}^{p-1}(z-t) \qquad (1.1)$$

The Gegenbauer polynomial is defined as[Abramowitz & Stegun p. 561]

$$C_n^{(\lambda)}(z) = \frac{(2\lambda)_n}{n!}\ {}_2F_1\Big(-n, 2\lambda+n; \lambda+\frac{1}{2}; \frac{1-z}{2}\Big). \qquad (1.2)$$

The expansion formula of Hermite polynomial is defined as[Poularikas p.437(22.1.2)]

$$H_m(t) = \sum_{p=0}^{[m/2]} \frac{(-1)^p\ m!}{p!\ (m-2p)!}\ (2t)^{m-2p}, \quad [m/2] = largest\ integer \leq m/2. \qquad (1.3)$$

Philosophized hypergeometric function ${}_rF_s(a_1, ..., a_r; b_1, ..., b_s; z)$ is stated as

$$ {}_rF_s\left[\begin{array}{ccc} a_1, a_2, \cdots, a_r & ; & \\ & & z \\ b_1, b_2, \cdots, b_s & ; & \end{array}\right] = \sum_{t=0}^{\infty} \frac{(a_1)_t (a_2)_t \cdots (a_r)_t z^t}{(b_1)_t (b_2)_t \cdots (b_s)_t t!} \tag{1.5} $$

where the parameters $b_1, b_2, \cdots, b_s$ are positive integers.

2. Main Formulae of the Integration

$$ \int_0^1 \frac{cos^{-1}x \ H_0(ax)}{x^n} dx = \frac{\sqrt{\pi}\ \Gamma(1-\frac{n}{2})}{(n-1)^2\ \Gamma(\frac{1-n}{2})} \quad for\ Re(n) < 1 \tag{2.1} $$

$$ \int_0^1 \frac{cos^{-1}x \ H_1(ax)}{x^n} dx = \frac{2\ \sqrt{\pi}\ a\ \Gamma(\frac{3-n}{2})}{(n-2)^2\ \Gamma(\frac{2-n}{2})} \quad for\ Re(n) < 2 \tag{2.2} $$

$$ \int_0^1 x^n\ cos^{-1}x\ H_0(ax) dx = \frac{\sqrt{\pi}\ \Gamma(1+\frac{n}{2})}{(n+1)^2\ \Gamma(\frac{1+n}{2})} \quad for\ Re(n) > -1 \tag{2.3} $$

$$ \int_0^1 x^n\ cos^{-1}x\ H_1(ax) dx = \frac{2\ \sqrt{\pi}\ a\ \Gamma(\frac{3+n}{2})}{(n+2)^2\ \Gamma(\frac{2+n}{2})} \quad for\ Re(n) > -2 \tag{2.4} $$

$$ \int_0^1 x^n\ cos^{-1}x\ C_0(ax) dx = \tilde{\infty} \tag{2.5} $$

$$ \int_0^1 x^n\ cos^{-1}x\ C_1(ax) dx = \frac{2\ \sqrt{\pi}\ a\ \Gamma(\frac{3+n}{2})}{(n+2)^2\ \Gamma(\frac{2+n}{2})} \quad for\ Re(n) > -2 \tag{2.6} $$

$$ \int_0^1 \frac{cos^{-1}x\ C_0(ax)}{x^n} dx = \tilde{\infty} \tag{2.7} $$

$$ \int_0^1 \frac{cos^{-1}x \ C_1(ax)}{x^n} dx = \frac{2\ \sqrt{\pi}\ a\ \Gamma(\frac{3-n}{2})}{(n-2)^2\ \Gamma(\frac{2-n}{2})} \quad for\ Re(n) < 2 \tag{2.8} $$

3. Derivation of Main formulae

Derivation of(2.3)

$$ \int_0^1 x^n\ cos^{-1}x\ H_0(ax) dx = \Big[\frac{x^{n+1}\ \{x\ {}_2F_1(\frac{1}{2}, \frac{n+2}{2}; \frac{n+4}{2}; x^2) + (n+2)\ cos^{-1}\ x\}}{(n+1)(n+2)}\Big]_0^1 $$

$$ = \Big[\frac{(1)^{n+1}\ \{1 *\ {}_2F_1(\frac{1}{2}, \frac{n+2}{2}; \frac{n+4}{2}; (1)^2) + (n+2)\ cos^{-1}\ (1)\}}{(n+1)(n+2)}\Big] - $$

$$-\Big[\frac{(0)^{n+1}\ \{0*\ {}_2F_1(\frac{1}{2},\frac{n+2}{2};\frac{n+4}{2};(1)^2)+(n+2)\ cos^{-1}\ (0)\}}{(n+1)(n+2)}\Big]$$

$$=\Big[\frac{\{{}_2F_1(\frac{1}{2},\frac{n+2}{2};\frac{n+4}{2};1)+(n+2)*0\}}{(n+1)(n+2)}\Big]=\Big[\frac{\{{}_2F_1(\frac{1}{2},\frac{n+2}{2};\frac{n+4}{2};1)\}}{(n+1)(n+2)}\Big]$$

$$=\frac{\sqrt{\pi}\ \frac{n+2}{2}!}{(n+1)(n+2)\frac{n+1}{2}!}=\frac{\sqrt{\pi}\ \Gamma(\frac{n}{2}+1)}{(n+1)^2\Gamma(\frac{n+1}{2})}$$

On this way remaining formulae can be proved.

4. Conclusion

The formulae established in this paper are open. One can derived many other formulae by using these formulae.

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