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THE ROLE OF MENTORS IN THE LIFE OF THE MATHEMATICAL GENIUS SRINIVASA RAMANUJAN

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Abstract: This article, in this the Ramanujan (22 Dec. 1887 – 26 Apr. 1920) Remembrance Centenary Year, presents the role of mentors in the life of Ramanujan. In Dewan Bahadur S. Narayana Iyer, Ramanujan had a friend, philosopher and guide in India, who helped Ramanujan to go to Trinity College, Cambridge and also after his return to India, after a five year sojourn (1914 – 1919) in Cambridge. From the moment he landed in England, till his return, Professor G.H. Hardy was all-in-all, a genuine, true well-wisher. This article details the roles of Narayana Iyer and Hardy as mentors of the natural mathematical genius Ramanujan, a 'Swayambhu'.

Keywords and Phrases: Biographical, mathematics, mentors, number theory.

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1. Introduction

There is a general impression in the minds of many that G.H. Hardy of the Cambridge University discovered Ramanujan. The facts, on the other hand, clearly reveal that it is Ramanujan who discovered Hardy.

Early years of Ramanujan, the student

Ramanujan completed his Matriculation from the Town High School, with distinction and several Prizes, for proficiency in English and Mathematics, in December 1903. He was unsuccessful in clearing the First Degree in Arts examinations, of the University of Madras, in 1905 and 1906. This marked the end of formal education for Ramanujan.

Ramanujan was on the look out for a job to eke out a livelihood between 1906 and 1911 and ran from pillar to post in search of a benefactor. Very few details are known about how he kept himself focused and occupied in this trying period in his life. On the occasion of the 125th Birth Anniversary of Ramanujan, in The Hindu, dated Dec.25, 2011, the following note appeared:

"A MISSING BOY: To the Editor of the 'Hindu'

SIR, – Kindly insert the following in your widely circulated journal: 'A Brahmin boy of the Vaishnava (Thengalai) sect, named Ramanujan, of fair complexion and aged about 18 years was till recently a student of the Kumbakonam College. He left his home on some misunderstanding. His guardian is very solicitous about the boy's returning home. He stayed at Rajahmundry for about a month, and was last seen there five days back. Those who happen to see him are kindly requested to persuade him to return home, and to communicate his whereabouts to: J. SEENIVASA RAGHAVA AYANGAR, 18, Sarangapani Sannidhi Street, Kumbakonam'."

This is perhaps the only instance where we have any information about the role of Ramanujan's father in his life. No photograph of the father is available till date! It is widely believed and reported that the father was least impressed with Ramanujan's extraordinary prowess in Mathematics, and perhaps felt that mathematics will not enable his son to get even a job.

The features of Ramanujan bear a close resemblance to those of his mother Komalathammal, who was convinced that his very birth, many years after her marriage, was due to her prayers for a son, as a boon granted by the Goddess Namagiri of Namakkal. Very little is known regarding the five year period after his Schooling (1904 - 1909). Surprisingly, there are no dates in his Notebooks. It is perhaps during this five year period that he started noting down his discoveries as Entries in two Notebooks. The first Notebook alone has a title: 'Magic squares' and is three pages long.

During his School days, Ramanujan was entrusted with the task of preparing the conflict-free time tables, for the entire School, which had about 1500 students, by the Senior Mathematics teacher, Ganapathy Subba Aiyar, who was an admirer of Ramanujan and was greatly attached to him. This possibly, was the time, when Ramanujan thought of filling the squares with numbers. The sixth Form in those days was called as the Matriculation class, and the School final examination held by the end of the calendar year, which coincided with the academic year as the British preferred to go home, for the annual Christmas vacation.

Ramanujan was married to Janaki, in 1909, solely due to the efforts of his mother, who identified a sprightly nine year old girl, in 1908, when she went to attend a marriage in Rajendram, a village about 45 km northeast of Tiruchirapalli. She spontaneously negotiated and settled all the details for the marriage, and it is said that the father of Ramanujan did not attend the marriage of 21-year old Ramanujan to 9-year old Janaki, who returned to live with her parents until she came of age, in 1912. Also, soon after the marriage, Ramanujan underwent a surgery for hydrocele, by Dr. Kuppuswamy, who volunteered, at the instance of Komalathammal, to do the surgery free of cost. This left him prostrated for about a year, during which period he recovered.

Janaki lived with Ramanujan for about a year before he went to England, in 1914, and for a year after his return to Madras, in 1919. The author had opportunities to meet this lady, being a resident of Chennai, and had the pleasure of meeting her many times with mathematicians who expressed their desire to meet the spouse of the mathematical genius Ramanujan, including Richard Askey and Bela Bollabos. Among those Ramanujan came into contact with were: first and foremost in importance, though not chronologically, is S. Narayana Iyer, then Treasurer of the Madras Port Trust, who became the primary force behind Ramanujan, from the time they met, when Ramanujan came to Madras till his death in 1920. For from 1920 onwards, there were innumerable occasions when Narayana Iyer was helpful to Janakiammal in every possible way, including Fostering W. Narayanan, providing him all the help for his schooling and collegiate education, getting him a job in the State Bank of India (SBI), getting him married to Vaidehi, also an employee of SBI.

The others are: P.V. Seshu Iyer (Mathematics Professor), V. Ramaswamy Iyer (Founder of the Analytical Club, in 1907, which later became the Indian Mathematical Society of today), Dewan Bahadur Ramachandra Rao (Collector of Nellore). The orchestrated efforts of these four veterans, resulted in Ramanujan's meteoric rise to fame as a world-renowned Mathematician from obscurity. This would not have been possible but for the timely recommendation letters and help of several including: Sir Francis Spring (Chairman, Madras Port Trust), Professors E.W. Middlemast, C.L.T. Griffiths, A.G. Bourne, W. Graham, M.J.M. Hill, R. Littlehailes, G.T. Walker, E.H. Neville, E.W. Barnes, who wrote to Dewsbury (Registrar, Madras University) and to the concerned authorities till eventually Ramanujan was persuaded to write to Godfrey Harold Hardy at Trinity College, University of Cambridge.

Why did Ramanujan approach Hardy at Trinity College, Cambridge? The answer is obvious when the following chance occurrence is recalled: Ramanujan went to meet his Professor Seshu Iyer, who was transferred from Kumbakonam Government Arts College to Presidency College, Madras. While waiting to meet his Professor, Ramanujan saw on the table the book "Orders of Infinity", by G.H. Hardy. As one whose passion was mathematics, Ramanujan started thumbing through Hardy's book and was excited to read: "no definite expression has yet been found for the number of prime numbers less than any given number". So, when he went in to see Seshu Iyer, he burst out with the observation that he had the exact formula in his Notebooks.Asked to fetch the Notebooks from his home, he returned with the same and Seshu Iyer then set the wheels of support in motion.

In this article, we will briefly highlight the roles played by his Mother Komalathammal, V. Ramaswamy Iyer, M.A., the Founder President of the Indian Mathematical Society, Dewan Bahadur S. Narayana Iyer, M.A., the then Treasurer who eventually became the Chairman of Madras Port Trust, and Cambridge Professor of renown G.H. Hardy, FRS, who could be considered the pillars of support who made the transition of Ramanujan from Obscurity to Fame and renown possible, so that Ramanujan returned to India, in 1919, with the B.A. degree of the Cambridge University, by research and the Fellowship of the Royal Society, London, the citation for which read: "Srinivasa Ramanujan, Trinity College, Cambridge. Research student in Mathematics, Distinguished as a pure mathematician, particularly for his investigations in elliptic functions and the theory of numbers."

There were some who played significant roles in the eventual shaping of the future of Ramanujan. What they said and did is recorded for posterity by P.K. Srinivasan, a Mathematics Teacher at Muthialpet High School, Madras and a life-long admirer of Ramanujan, who brought out two Memorial Numbers, in 1967, starting his work at the time of the 75th Birth Anniversary of Ramanujan, in 1962, soon after his return from America after a two-year sojourn, as a Fulbright Scholar. The first of these two volumes has been a significant resource for every biographer, and the author had the pleasure of acquiring for the Institute of Mathematical Sciences Library and for himself, from P.K. Srinivasan, the Editor himself, copies of this two volume compilation: "Ramanujan: Letters and Reminiscences", Memorial

Number Volume-1, and “Ramanujan an Inspiration”, Memorial Number Volume-2, published by The Muthialpet High School, Number Friends Society, Old Boys’ Committee, Madras-1, in 1968.

The other Indian sources about the life and work of Ramanujan of significance are: “*Ramanujan: the Man and the Mathematician*, Asia Publishing House (1967) by Dr. S.R. Ranganathan, who was a Ph.D. in Library Science, from Trinity College, Cambridge. On his return, he became the renowned Librarian of the University of Madras. He introduced the Colon System for classification for books and periodicals (which is no longer in vogue, after the Dewey Decimal Classification, DDC, gained acceptance and became Universal). S.R. Ranganthan revealed his role, in the “Collected Papers of Srinivasa Ramanujan”, edited by G.H. Hardy, P.V. Seshu Aiyar and B.M. Wilson, Chelsea, New York (1962), first published by Cambridge University Press, Cambridge, as follows: “ In 1923, it was decided that a biography of Ramanujan should be given at the beginning of the Collected Papers. The University of Madras appointed a Committee to write the biography. It consisted of E.M. MacPhail, the Vice Chancellor, P.V. Seshu Ayyar, Professor of Applied Matheamtics in the Presidency College and Secretary of the Indian Mathematical Society, and R. Ramachandra Rao, Education Secretary of the Government of Madras and former President of the Society. As a junior member of the staff of the Department of Mathematics of the Presidency College, it fell to my share to prepare a draft of the biography. As approved by the Committee, it was published in the Collected Papers, in 1927.”

Interestingly, Dr S.R. Ranagnathan, after he left the University of Madras, became the Chief Librarian at the Benares Hindu University. It was he who drew the attention of Dr. K.S. Krishnan, a student and collaborator of Sir C.V. Raman, regarding the preservation of Ramanujan’s Notebooks. At that time, Krishnan was a neighbour of Pundit Jawaharlal Nehru, in Allahabad. He induced Ranganathan to write to Dr. Hommi Bhabha at the Tata Institute of Fundamental Research, Bombay. Subsequently, Ranganathan, Krishnan and Bhabha called on Pundit Nehru and as a consequence a Facsimile edition of the Notebooks of Ramanujan was published, in 1957.

In 1987, on the occasion of the Birth Centenary of Ramanujan, with the help of the K-12 group of the National Institute of Information technology, New Delhi, the author produced 2 CD ROMs in a Pilot CD ROM Project and exhibited the same at the Indian Science Congress Exhibition 1999. Amongst the visitors to the exhibition were Dr. R Chidambaram, Dr. A.P.J. Abdul Kalam, and Dr.V.S. Ramamurthy, who felt that this should be an Exhibition on Wheels! On Science Day, Feb. 28, 1999, 'the PIE (or πie) Pavilion and the Ramanujan Gallery", were inaugurated by Ms. C.K. Gariyali, IAS, Vice Chairperson, Science City, and dedicated to the Nation at the Periyar Science and Technology Center (PSTC), Kotturpuram, Chennai. This has been upgraded through projects proposed by one of us (KSR), funded by the State and DST twice, in 2012 and in 2016, into becoming a full-fledged Ramanujan Mathematics Gallery.

At the suggestion of one of us (VSR), as Secretary, DST, Government of India, KSR took up a two and a half-year Project, from Dec. 2002 to Aril 2005, to bring out two CD ROMS on the Life and Work of Srinivasa Ramanujan, containing a multi-media presentation of the life and work of Ramanujan, produced by the National Multimedia Resource Center, of the Center for Development of Advanced Computing, Pune. The Collected Papers of Ramanujan, consisting of his 37 Research Papers (5 in collaboration with Professor Hardy, during his 1914-1919 stay in Cambridge), and 59 Questions and/or Answers to Questions in the Journal of the IMS, scanned copies of the three Original Notebooks of Ramanujan.

The Acting Principal and Professor of Mathematics, Presidency College, Madras, E.W. Middlemast, "Strongly recommend[ed] the applicant,..., a young man of quite exceptional capacity in Mathematics and especially in work relating to numbers. He has a natural aptitude for computation and is very quick at figure work. Though he had no experience of statistical work, I am confident he can pick up details in a very short time." This was written in Sep. 1911.

Professor C.L.T. Griffith was at the Engineering College, Madras and in Dec. 1912, he wrote to Sir Francis Spring the Chairman of the Madras Port Trust, that he was "writing to one of the leading mathematics professors at home about him (Ramanujan) and sending copies of some of Ramanujan's papers and results." But concluded with "until I hear from home, I don't feel sure that it is worthwhile spending time or money on him." He wrote to the skeptic, Prof. M.J.M. Hill at the University of London, as Hill was Griffith's mathematics professor at University College, London. Unfortunately, Hill felt that Ramanujan "got the erroneous results ...:

$$1+2+3+\ \&c = -1/12 \tag{1}$$

$$1^2+2^2+3^2+\ \&c = 0 \tag{2}$$

$$1^3+2^3+3^3+\ \&c = \frac{1}{240} \tag{3}$$

All these 3 series have infinity for their sums. The book which will be most useful to him is Bromwich's Theory of Infinite Series, published by the Cambridge

University Press (of Macmillan). Bruce Berndt, the mathematician who devoted a considerable part of his time to editing the Notebooks of Ramanujan and binging out a definitive five volume investigative research work on them, entitled: ”Ramanujan’s Notebooks”, Springer-Verlag (Parts: I, 1985; II, 1989; III, 1991; IV, 1994; V, 1997) wrote: “Evidently, Hill shared Abel’s view that ‘divergent series are in general deadly’, for, not surprisingly, he failed to discern the origin of the three results quoted...[which] give the values of $\zeta(-n)$, for $n = 1, 2, 3$, respectively.”

Griffith forwarded Hill’s letter to Sir Francis Spring about ‘S. Ramanujan who is a most remarkable mathematician. He may be a poor Accountant, but I hope you will see that he is kept happily employed until something can be done to make use of his extraordinary gifts.’ The Director of Public Instruction, A.G. Bourne, who had not heard of Ramanujan, also felt that Ramanujan may be sent to see Middlemast, since Littlehailes was away on the West Coast.

When Ramanujan went to see W. Graham, the Accountant General of Madras, on Nov.27, 1912, after assessing Ramanujan’s work, Graham also felt that the opinion of Middlemast may be sought. However, Griffith contacted M.J.M. Hill, who felt that Ramanujan “does not understand the precautions which have to be taken in dealing with divergent series, otherwise he could not have obtained the erroneous results”, (1)-(3), “you send me.” ... The sums to n terms of these series are:

$$\frac{1}{2}n(n+1), \qquad \frac{1}{6}n(n+1)(2n+1), \qquad [\frac{1}{2}n(n+1)]^2 \tag{4}$$

and they all tend to ∞ as $n \to \infty$. I do think you can do no better for him than to get him a copy of the book I recommended, Bromwich’s Theory of Infinite Series, published by Macmillan and Co., which has branches in Calcutta and Bombay. Price 15/- net.”
Accordingly, Griffith wrote to Mr. Rmachandra Rao, Collector of Nellore, ‘to buy the book mentioned, and give it to Ramanujan.’

Ramanujan also approached both Sir Gilbert T. Walker, the Director General of Meteorological Observatory, then in Simla, and Sir Francis Spring, Chairman of the Madras Port Trust, for help. These appeals produced the most desirable results: Walker had ‘the honour to draw attention to the case of S. Ramanujan, a clerk in the Accounts Department of the Madras Port Trust’, to the Registrar, Dewsbury, of the University of Madras, in February 1913 and suggested ‘that they should communicate with Mr. G.H. Hardy, Fellow of Trinity College, Cambridge, with whom he is already in correspondence and assure Mr. Hardy of their interest in him’.

Dewan Bahadur S. Narayana Iyer, M.A. – the Indian Mentor

S. Narayana Iyer (1874–1937), the Chief Accountant of Madras Port Trust, was a M.A.(Maths) from St. Joseph's College, Tiruchirapalli, and a good Mathematician, who was a college teacher initially. Sir Francis Spring who was in the Railways, came into contact with Narayana Iyer and was impressed with the mathematician. Hence, when he became the Chairman of the Madras Port Trust, he offered the Office Manager's post to Narayana Iyer. He was later promoted to become the Chief Accountant and eventually retired as the Chairman of the Port Trust.

Narayana Iyer recognized the extraordinary talent and interest of Ramanujan, who lived in a small house in Triplicane, Madras. Narayana Iyer encouraged in every possible way, especially in correspondence with higher authorities, for the welfare and progress of Ramanujan as a mathematician. He was not only instrumental in Ramanujan being offered a Class III, Grade IV, clerical post in the Accounts section of the Madras Port Trust, but also secured for Ramanujan the life-long support of Sir Francis Spring to Ramanujan's career. Narayana Iyer's son N. Subbanarayanan relates the role his father played in the life of Ramanujan:

"To illustrate the depth of his genius, I can quote one of the incidents which happened during the stay of Sri Ramanujan with my father when we were living in No. 580, Pycrofts Road, Triplicane. During that period, every night both Ramanujan and my father used to work Mathematics on two big sized slates sitting on a small parapet upstairs. This used to be carried on till about 11.30 p.m. and was a source of nuisance to other inmates of the house who used to sleep in the adjoining rooms.I distinctly remember the noises of the slate pencils which used to be a background music for my sleep. Several nights I have seen Sri Ramanujan get up at 2 O'clock in the night and note down something in the slate in the dull light of a hurricane lamp. When my father asked him what he was writing, he used to say that he worked out mathematics in his dreams and now he was jotting the results in the slate to remember them. Evidently, this goes to show that he must have had a remarkable power of subconscious working and grasp of intellectual strides in mathematics. Peculiarly enough, my father used to ask Ramanujan some doubts in these jottings. Between one step and another step there used to be a big gap. My father, being a fairly good mathematician himself, was unable to capture the strides of Ramanujan's discoveries. He used to tell him, "When I am not able to understand your steps, I do not know how other mathematicians of a critical nature will accept your genius. You must descend to my level and write at least ten steps between the two steps of yours." Sri Ramanujan used to say, "When it is so simple and clear to me, why should I write more steps?" But somehow my father slowly got him round, cajoled him and made him write some more, though it used to be a mighty task of boredom for him (Ramanujan)."

Mr. Narayana Iyer requested for and obtained the slate used by Ramanujan as a memento (in exchange of his slate) and it is still a heir-loom with the family of Mr. Subbanarayanan. It is interesting to note that Mr. Narayana Iyer used to send his son Subbanarayanan to get a book on Elliptic Functions from the library of the University of Madras [12].

Narayana Iyer being a resident of Triplicane, used to spend the time after dinner with Ramanujan and the two of them used large sized slates to work out the mathematical problems conjured by Ramanujan which were noted down by him from around 1905 in his Notebooks, without proofs, simply as Entries and considered by experts as the statements of about 3254 Theorems, by the reckoning of Richard Askey. The slate used by Ramanujan is no longer there, but the one used by Narayana Iyer is one of the few memorabilia of Ramanujan which are with the family. It is time that the Government acquires the memorabilia – this slate, the invaluable original Notebooks[1].

Narayana Iyer accidentally brought the attention of the Chairman of Madras Port Trust to the plight of Ramanujan, who was running from pillar to post in search of a benefactor. From then on Sir Francis Spring and Narayana Iyer gave Ramanujan every possible encouragement, which eventually led to Ramanujan's discovery of G.H. Hardy, and the beginning of a unique story, from obscurity to international name and fame for Ramanujan.

In the words, of Robert Kanigel [5]: 'The wheels of Ramanujan's career, for ten years barely creaking along, now greased by Hardy's approval, began to whir and whine like a finely tuned race engine.' The story is well told, by Kangiel, of how Hardy and Neville, the Cambridge mathematicians; Sir Francis Spring and Narayana Iyer, of the Madras Port Trust; Ramanujan's mother and her devotion to the Goddess Namagiri of Namakkal; Lord Pentland, the Governor of Madras and his Private Secretary, C.B. Cottrell; Professor Littlehailes and Sir Francis Dewsbury, Registrar, University of Madras; in an orchestrated effort found the funds to send Ramanujan sailing to England, on a unique historical, mathematical adventure of the twentieth Century.

[1]Three leather-bound volumes with the University of Madras. They are now in a highly brittle state, despite repeated efforts by us, that they should not be kept locked-up in a Godrej Bureau, for "safe custody", of the Registrar of the University of Madras – definitely not the way to preserve National Archival material!

It is remarkable that while there was no one in Madras could appreciate the depth of the mathematical capabilities of Ramanujan, the University of Madras awarded its first-ever Research scholarship in Mathematics to Ramanujan, which enabled him to sail to Cambridge, despite the fact he was a failed F.A. of the University of Madras. This is due mainly to the courage of conviction of a team led by Narayana Iyer, consisting of the stalwarts of that era: Dewanbahadur Ramachandra Row, V. Ramaswamy Iyer (the Founder of the Indian Mathematical Society, and the Journal of IMS), Prof. Seshu Iyer; who succeeded in convincing and persuading the British authorities to act post haste on the merits of the candidate.

Ramanujan wrote in a letter to Narayana Iyer from Cambridge: 'I am ever indebted to you and Sir Francis Spring for your zealous interest in my case from the very beginning of acquaintance', (11 Feb. 1915) and in a P.S. wrote 'I am glad to tell you that I may be conferred upon the Research Degree next March.'

Narayana Iyer was a disciple of Bhagavan Ramana Maharshi and played a prominent role in the execution of the Will of Bhagavan. His reminiscences are recorded in his book 'Ramana Periya Puranam'.

Rao Bahadur Narayana Ayyar, retired as Chief Accountant of the Madras Port Trust, till 1934. He was one of the founding members of the Indian Mathematical Club, and was its Assistant Secretary from 1907 - 1910 and later its Treasurer and a member of its Managing Committee from 1914 - 1928. The list of his mathematical contributions to the Journal of the Indian Mathematical Society are:

Note on $\int \frac{\sin\ mx}{\sin\ nx}\ dx$,	Vol. I, p.9
The distribution of primes,	Vol. I, p.60
Some theorem on summation,	Vol. I, p.185
Some infinite determinants,	Vol. VII, p.51
A small theorem,	Vol. VIII, p. 181

He proposed the following Questions in the JIMS for solution: 36 Questions with Q Numbers: 16, 17, 23, 27, 37, 38, 67, 85, 109, 110, 128, 129, 162, 163, 177, 465, 494, 575, 576, 592, 592, 616, 617, 630, 631, 667, 686, 718, 756, 806, 878, 1394, 1405, 1485, 1500, 1548. And the following Eighteen Solutions to questions with Numbers: 2, 4, 7, 12, 18, 19, 21, 22, 31, 33, 39, 49, 68, 69, 104, 111, 113, 114.

Among all those who were approached by Ramanujan, who was in search of a friend, philosopher, guide, and in short, a mentor, the one individual in India, who played the role selflessly was Dewan Bahadur Sri S. Narayana Iyer.

Rise of Ramanujan from Kumbakonam to Cambridge

Before we proceed to the other mentor in the life of Ramanujan, it is but natural for us to discuss the precocious nature of Ramanujan which made all those who could appreciate his mathematical prowess as extraordinary since they were bright students of mathematics at the University level with a Masters Degree in Arts in the subject – Griffiths, Seshu Iyer, Ramachandra Rao, Ramaswamy Iyer, Narayana Iyer and Walker, who was also an F.R.S.

The Senior Mathematics School teacher at Town High School, Kumbakonam, Ganapathy Subbiar, who entrusted the work of preparing the conflict-free Time Tables for the School, which had about 1400 students studying in Forms I to VI – an exercise which led to the Chapter 1 in the Notebooks of Ramanujan. Chapter 1, in Notebook 1, is the only chapter which has the title *Magic Squares* and it is only 3 pages long. While the Chapter 1 in Notebook 2 is 8 pages long and ends with 8×8 magic squares and ends with the following algorithm for the construction of a 7×7 magic square:
"To construct a square of odd rows & columns

A,B,C,D,E,F,G,H & c	P,Q,R,S,T,U,V,W & c
A,B,C,D,E,F & c	R,S,T,T,V,W & c
A,B,C,D & c	T,U,V,W & c
A,B & c	V,W & c

Thus arranging the letters and adding the two sets a square of any numbers of odd rows can be found and we can find many ways of constructing a square and the peculiarities are common to all the odd square."

To augment the meager family income, two college students from Tiruchi and Tirunelveli were taken in as boarders by Ramanujan's mother. While waiting on the pial, they were discussing their collegiate mathematics, when the School boy Ramanujan intervened on occasions to teach them shorter, better methods. The students in turn shared the insights with the mathematics teacher, who was also impressed with the methods. While in III Form in School, in 1900, Ramanujan studied by himself arithmetic and geometric series and trigonometry from S.L. Loney's, book with the same title: 'Trigonometry'. While in his VI Form he obtained from his Goverment college library, through the college students who were boarders, George Shoobridge Carr's (1884) book, "A Synopsis of Elementary Results, a book on Pure Mathematics", which was a compilation of 4865 formulas (and not 6165 which is the last numbered entry in the book, as about 1300 numbers were skipped, due to each chapter starting with a multiple of 100, as pointed out by Richard Askey) without proofs. This book greatly influenced Ramanujan, and gave him the model to record his Entries in his Notebooks. The Notebooks have no dates and the Entries were written down by Ramanujan during his School days and were full of Entries by 1907, when the Analytical Club was formed by V. Ramaswamy Iyer, which later became the Indian Mathematical Society (IMS) in 1907. Possibly the Journal of IMS was started to provide a venue for the publications of the torrent of results with which Ramanujan approached the mathematicians he was in contact with for suggestions as to where he could get his results published. S.L. Loney's Trigonometry and G.S. Carr's Synopsis were the two books which were mastered by Ramanujan, before he wrote his first paper on "Some properties of Bernoulli Numbers", which was published in the Journal of the IMS in 1911 (Vol. III, p. 219 - 234.). In this paper he writes down the first 20 values of Bernoulli numbers, the 21st being

$$B_{40} = \frac{261082718496449122051}{13530}, \cdots, B_{\infty} = \infty. \tag{5}$$

Also, "we see that

$$\frac{B_n}{B_{n-2}} \text{ approaches } \frac{n(n-1)}{4\pi^2} \text{ very rapidly} \tag{6}$$

as n becomes greater and greater. From the values of π, viz.

$$3.14159, 26535, 89793, 23846, 26433, 83279, 50288, 41971, 69399, \cdots, \tag{7}$$

[45-digits without a calculator] the integral part of any B can be found from the previous B; and from the integral part the exact value can at once be written ..." In this paper, Ramanujan stated eight theorems embodying the arithmetical properties of the Bernoulli numbers, indicating the proofs for three of them; two theorems are stated as conjectures. Professor Seshu Iyer, who was indifferent at that time to Ramanujan, stated soon after Ramanujan died – as a celebrity with a B.A. degree by research from the Cambridge University, an F.R.S., and a six year Trinity College Fellowship – that "His (Ramanujan's) first article on 'Some Properties of Bernoulli's Numbers' was published in the December issue of the same volume [JIMS]. Mr. Ramanujan's methods were so terse and novel and his presentation was so lacking in clearness and precision, that the ordinary reader, unaccustomed to such intellectual gymnastics, could hardly follow him. This particular article was returned more than once by the Editor before he gave it a form suitable for publication."

Compare this with what Hardy had stated while communicating Ramanujan's longest paper on Highly Composite Numbers – Ramanujan's definition: 'A number N may be said to be a highly composite number, if $d(N') < d(N)$ for all values of N' less than N, where $d(N)$ denotes the Number of divisors of every number N which can be written as:

$$N = p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_n^{a_n}, \tag{8}$$

where $p_1, p_2, \cdots, p_n$ are distinct prime numbers and $a_1, a_2, \cdots, a_n$ are non-negative integers. Ramannujan defines "a highly composite number as a number whose number of divisors exceeds that of all its predecessors": and

$$d(N) \ = \ (1 + a_1)(1 + a_2)(1 + a_3) \cdots (1 + a_n), \text{and that } a_n = 1. \tag{9}$$

Ramanujan was awarded the Bachelor's Degree in Arts, B.A. degree by research, of the Cambridge University, in March 1916, for his work on *Highly Composite Numbers* and published as a long paper in the JLMS. Ramanujan's dissertation bore the same title and included in addition six other published papers of his.

Ramanujan registered as a research student in June 1914 and the prerequisite of a diploma or a certificate, as well as the domiciliary requirement of six terms was waived in his extraordinary case. It is unfortunate that a copy of this dissertation is not to be found in the records [11] of the University. According to Hardy [3], this long memoir of Ramanujan on Highly Composite Numbers, that it "*represents, perhaps, in a backwater of mathematics, and is somewhat overloaded with detail; but the elementary analysis of "highly composite" numbers – numbers which have more divisors than any preceding number – is most remarkable, and shows very clearly Ramanujan's extraordinary mastery over the algebra of inequalities.*

And also, that it is *a very peculiar one, standing somewhat apart from the main channels of mathematical research. But there can be no question as to the extraordinary insight and ingenuity which he has shown in treating it, nor any doubt that the memoir is one of the most remarkable published in England for many years.*

Godfrey Harold Hardy (1877 - 1947) – THE Mentor

Needless to say, the one person who changed the course of the life of Ramanujan, and did all that he, as a benevolent, generous, kind-hearted human being could do to Ramanujan, with only the desire to keep him happy so that his creativity would not be hampered by the thoughts of the climatically and otherwise too, biting cold conditions, with the exception of himself – Professor Godfrey Harold Hardy (1877 - 1947) – as the all-in-one, mentor for Ramanujan, from the time he analysed the contents of the first unsolicited letter from Ramanujan to Hardy, dated January 16, 1913, with his confidante, friend and collaborator, Professor John Edensor Littlewood, (9 June 1885 – 6 September 1977), also an FRS.

This historic letter of Ramanujan had a a short note of introduction about himself:

"Dear Sir, I beg to introduce myself to you as a clerk in the Accounts Department of the Port Trust Office at Madras on a salaray of only £20 per annum. I am now about 23 years of age. I have had no University education but I have undergone the ordinary school course. After leaving school I have been employing the spare time at my disposal to work at Mathematics. I have not trodden through the conventional regular course which is followed in a University course, but I am striking out a new path for myself. I have made a special investigation of divergent series in general and the results I get are termed by the local mathematicians as 'startling'...My friends who have gone through the regular course of University education tell me that

$$\int_0^\infty x^{n-1}\, e^{-x}\, dx \;=\; \Gamma(n), \text{true only when } n \text{ is positive,} \tag{10}$$

"They say that this integral relation is not true when n is negative. Supposing this is not true when n is negative. Supposing this is true only for positive values of n and also supposing the definition $n\Gamma(n) \;=\; \Gamma(n+1)$ to be universally true, I have given meanings to these integrals and under the conditions I state the integral is true for all values of n negative and fractional. My whole investigations are based upon this and I have been developing this to a remarkable extent so much so that the local mathematicians are not able to understand me in my higher flights.

"Very recently I came across a tract published by you styled *Orders of Infinity* in p.36 of which I find a statement that no definite expression has been as yet found for the number of prime numbers less than any given number. I have found an expression which very nearly approximates to the real result, the error being negligible. I would request you to go through the enclosed papers [11 pages, containing choice results from his Notebooks]. Being poor, if you are convinced that there is anything of value I would like to have my theorems published."

Hardy stated in his book *Orders of Infinity* (p.36) that "the number of prime numbers less than

$$x = \int_2^x \frac{dt}{\log t} + \rho(x) \tag{11}$$

where the precise order of $\rho(x)$ has not been determined." For ever $x > 0$, let $\rho(x)$ denote the prime counting function giving the number of primes that are less than or equal to x asymptotically, that is as $x \to \infty$:

$$\rho(x) = \lim_{x\to\infty} \frac{x}{\log x}. \tag{12}$$

Carl Friedrich Gauss (1777 - 1855) is perhaps the first mathematician who had this result as an unproved hypothesis and more than a hundred years later, the Belgian

Charles de la Vallée - Poussin and the Frenchman Jacques Salomon Hadamard proved independently that

$$\rho(x) = O(xe^{-A\sqrt{\log x}}), \tag{13}$$

where $O(f(x))$ represents an unspecified function of x which grows not faster than some constant times $f(x)$.

Let us briefly look at the life and work of Hardy:

Godfrey Harold Hardy was born on February 7, 1877, at Cranleigh, Surrey, in a gifted, mathematically inclined family. His father, Isaac Hardy, was an Art Master, Bursar, and House Master of the preparatory branch of Cranleigh School and his mother, Sophie Hardy, was a senior mistress at the Lincoln Training College for teachers. It is said that he was writing down numbers up to millions – his earliest sign of mathematical precociousness – when he was just two years old. He and his younger sister were sent to a Victorian nursery and later to the Cranleigh School.

At Cranleigh School, Hardy had private coaching in Mathematics, and not in a class, because of his precocity. He was first in his class in all subjects, but extremely shy to receive the prizes due to his self-consciousness. In Hardy's words, '*I thought of mathematics in terms of examinations and scholarships: I wanted to beat other boys and this seemed to be the way in which I could do so most decisively.*'

Though frail and good looking, he did not like to see himself in a mirror or be photographed. At the age of twelve, Hardy won a scholarship at Winchester public school, the best mathematical school in England then. He left the school with honours and was offered an open scholarship to Trinity College, Cambridge University, Cambridge. He was not only good at studies but was also a natural Tennis player and a batsman in Cricket – a game he could enjoy playing or watching till almost the end of his life.

By the time he finished his schooling, he was an atheist though he came from an orthodox religious family and he stopped going into any chapel. This created problems for him at Trinity College, since attending the chapel was compulsory. He referred to God as his personal enemy all through his life. His strong feelings about the church and orthodoxy are clearly reflected in his statement: '*It's rather unfortunate that some of the happiest hours of my life should have been spent within sound of a Roman Catholic Church.*' The church referred to by Hardy is the one close to Fenner's, the cricket ground of Cambridge university.

Hardy, who graduated in 1899, managed to come as fourth Wrangler in Part I of the Mathematical Tripos examination of Cambridge University, in 1898, a superb performance, which determined the Wrangler standing. The top candidates were ranked as Wranglers, which meant in practical terms, a First Class. An excellent account of the Tripos and the Wrangler ranking can be found in Kanigel's acclaimed biography [6] of Ramanujan: '*The Man Who Knew Infinity: A Life of the Genius Ramanujan.*' Hardy appeared for Part II of the Tripos, the more provocative and more challenging part, in 1900, and secured the first division of the first class and was elected 1900 to a Prize Fellowship at Trinity, at the age of 22.

Hardy won the Smith's Prize in 1901. During his classes for competitive Mathematical Tripos examination his coach Professor A.E.H. Love, advised him to read Jordan's famous *Cours d' Analyse* (the first volume of which was published in 1882 and the third volume in 1887) about which Hardy said: '*I shall never forget the astonishment with which I read that remarkable work, the first inspiration for so many mathematicians of my generation, and learned for the first time as I read it, what mathematics really meant. From that time onward I was in a way a real mathematician, with sound mathematical ambitions and a genuine passion for mathematics.*

Hardy is the second hero in Kanigel's '*The Man Who Knew Infinity: a life of the Genius Ramanujan*'. Kanigel portrays Hardy as scrupulously honest, fastidious about giving others their due; ever so kind to the weak ones; and one in whom '*coexisted a stern, demanding streak with an indulgent liberal-mindedness, a formidable and forbidding exterior with a soft and fragile core.*'

Though Hardy was the '*best and truest*' friend Ramanujan ever had, Kanigel expresses his view that Hardy never really knew Ramanujan. The aloof, formal nature of their friendship is best illustrated by the fact that in one note written by Ramanujan to Hardy from a sanatorium, nearly three years after working together, Ramanujan explains that Srinivasa in his name is not his first name but a part of his father's name. The richness and extent of Ramanujan's inner spiritual life was, perhaps, something he never discussed or revealed to Hardy.

In what could be called as a larger than life portrayal, in Kanigel's biography of Ramanujan, Hardy emerges as an aristocrat of the intellect; busy being brilliant; effectively inculcating in Ramanujan the Gospel of proof and rigour in mathematics; the embodiment of all that was highest and best in the Western mathematical tradition; the most stern taskmaster; the most generous man with not even a wisp of envy tainting his relationship with Ramanujan; who championed and hailed the gifts of Ramanujan; recognizing Ramanujan's genius and pushing it towards its limits.

Hardy on Ramanujan

In Mathematics, George Harold Hardy (G.H. Hardy) on Ramanujan is akin to James Boswell on Samuel Johnson, in literature. The role of Hardy, Professor of Mathematics at the Trinity College, of Cambridge University, in Cambridge city, in the life and uplift of the career of Ramanujan is beyond praise. Hardy considered Ramanujan was in a way, his discovery, though he has stated that '*like all other great men, he invented himself*'. Ramanujan was considered a treasure and in Hardy's words, '*I still know more about Ramanujan than any one else, am still the first authority on this particular subject. ... I owe more to him than to anyone else in the world with one exception, and my own association with him is the one romantic incident in my life*'.

Ramanujan discovers Hardy

This association started with the discovery of G.H. Hardy's book on '*Orders of Infinity*', by Ramanujan. A copy of the book – due to the British ruling over India at that time, 1909 – appeared almost immediately in the Library of Presidency College, where Ramanujan chanced to browse through it when he went to meet the Professor of Mathematics, P.V. Seshu Iyer, who was transferred from the Government Arts College, Kumbakonam, to the prestigious Presidency College's Department of Mathematics. Hardy was thus in a unique position to observe and understand Ramanujan at close quarters. He has also been very generous in doing his very best to propagate the mathematics of Ramanujan through lectures and articles, after Ramanujan died on April 26, 1920, (at 10 A.M.) which actually came as a surprise and shock to Hardy, when he first heard about it from a letter written to him by Ramanujan's brother, Subbanarayanan. Here, we wish to draw heavily on Hardy's views of Ramanujan and quotation of Hardy are from either his NOTICE in the '*Collected Papers of Srinivasa Ramanujan*' or from his book entitled '*Ramanujan: Twelve lectures suggested by his life and work*' (1940).

Hardy was neither the first, nor the only eminent British mathematician to whom Ramanujan wrote or appealed. In Chapter 1, the response of Prof. M.J.M. Hill to Ramanujan's work on divergent series has been pointed out.

In the words of the eminent writer C.P. Snow, '*Hardy was not the first eminent mathematician to be sent the Ramanujan manuscripts. There had been two before him, both English, both of the highest professional standard. They had each returned the manuscripts without comment. I don't think history related what they said, if anything, when Ramanujan became famous.*' As for their identity, Snow adds that '*out of chivalry Hardy concealed this in all that he said or wrote about Ramanujan.*' However, the names are given by Ashish Nandy [9], as H.F. Baker and W.W. Hobson.

It is hard to imagine what would have happened if Hardy, like Hobson and Baker, also returned the manuscript of Ramanujan without comment. In view of the pivotal role Hardy played in the career of Ramanujan, it is necessary to give a biographical account of Hardy's life and work.

Hardy's collaborations

Collaboration is rare in the area of mathematics research. Hardy had two fruitful collaborations in mathematics in his life. He always considered J.E. Littlewood as a better mathematician than himself – and as quoted earlier on, in his self-assessment, he gave himself 25, Littlewood 30, Hilbert 80 and Ramanujan 100 out of 100 and Ramanujan as a natural mathematical genius. Unfortunately, the Hardy-Ramanujan collaboration overlapped with the first world war period, which continued uninterrupted for five years from 1914, until Ramanujan's return to India, in March 1919. During those five years, they wrote five papers of the highest class together, which revealed that: '*Ramanujan intuition incarnate, had run smack into Hardy the Apostle of Proof*' (observes Kanigel). In the words of Littlewood, the collaboration was between two mathematicians of '*quite unlike gifts, in which each contributed the best, most characteristic, and most fortunate work that was in him.*'

The Hardy-Littlewood collaboration started in 1911 and barring the interference due to the war years when Littlewood was away (for four years), doing ballistics as a Second Lieutenant in the Royal Artillery, this partnership lasted 35 years. Together they published nearly a hundred papers and in the words of C.P. Snow, '*a good many of them 'in the Bradman class' and it is "the most famous collaboration in the history of mathematics . . . The Hardy - Littlewood researches dominated English pure mathematics, and much of world pure mathematics, for a generation. . . Of its enduring value there is no question . . . But no one knows how they did it . . . No one has any evidence how they set about it", since during their most productive period, they were not even at the same University*'. Hardy with Littlewood and G. Polya published a book on '*Inequalities*', in 1934. Littlewood and Hardy conducted a seminar or '*conversational class*' at Cambridge which attracted the mathematical community due to its informal nature and openness for all discussions.

Hardy was a confirmed bachelor, in whose life no woman other than his mother and sister played the slightest substantive role. Hardy was a member of a secret intellectual society called the Apostles along with some of the most brilliant Cambridge men of that period – Bertrand Russell, Alfred North Whitehead and James Clerk Maxwell, to name a few. A homosexual current ran through the Society. Kanigel in his book, 'The Man who knew Infinity' asserts that the Hardy-Ramanujan relationship was marked by distance, not comradely intimacy and that Mathematics was the common ground of their relationship. A fact which indirectly corroborates this view is that after his return to India, in April 1919, during the entire period of almost 13 months before he died, Ramanujan wrote only one letter to Hardy, in January 1920. In that letter he did not mention about his deteriorating health. However, he wrote about the new results he had on what he created and named as 'mock theta functions'. This led Hardy to feel, as he stated later on, that Ramanujan was recovering with treatment for his illness diagnosed as Tuberculosis by the British doctors and confirmed by their Indian counterparts. So, the announcement of Ramanujan's death in a letter by Ramanujan's brother to Hardy, came as a shock to the latter.

The Hardy-Ramanujan collaboration led to five papers together and the 'astonishing theorem' on the number of partitions below a given number n, denoted by $p(n)$. The best in Ramanujan was brought out by Hardy, "The great bulk of Ramanujan's published work was done in England. His mind was hardened to some extent, and he never became at all an 'orthodox' mathematician, but he could still learn to do new things, and do them extremely well. It was impossible to teach him systematically, but he gradually absorbed new points of view. In particular, he learnt what it is to provide a proof, and his later papers, while in some ways as odd and individual as ever, read like the works of a well-informed mathematician. His methods and weapons, however, remained essentially the same". ... Hardy did perhaps everything which one human being could have ever done to another to keep Ramanujan in the best of spirits – the Bachelors Degree in Arts, the Fellowship of the Royal Society, the Fellowship of the Trinity College, conferred on Ramanujan were mainly due to the untiring efforts of Hardy. Together Hardy and Ramanujan were a formidable mathematical pair who left indelible footprints in the sands of pure mathematics – a unique pair in the realm of Mathematics.

Pros and cons of the Hardy-Ramanujan collaboration

Hardy has been eloquent to state that Ramanujan 'was by far the greatest formalist of his time'; that Ramanujan's work in the fields of identities and continued fractions is a masterpiece and he wrote that his 'extremely strong' language, in the following paragraph, 'is not extravagant':

"'It was his insight into algebraical formulae, transformations of infinite series, and so forth, that was most amazing. On this side most certainly I have never met his equal, and I can compare him with only Euler or Jacobi. He worked, far more than the majority of modern mathematicians, by induction from numerical examples; all his congruence properties of partitions, for example, were discovered in this way. But with his memory, his patience, and his power of calculation, he combined a power of generalization, a feeling for form, and a capacity for rapid modification of his hypothesis, that were often really startling, and made him, in his own peculiar field, without a rival in his day".

On Ramanujan's work on the Prime Number Theorem, Hardy stated: '*All that is plain is that Ramanujan found the form of the Prime Number Theorem by himself. This was a considerable achievement; for the men who had found the form of the theorem before him, like Legendre, Gauss, and Dirichlet, had all been very great mathematicians, and Ramanujan found other formulae which lie still further below the surface.*'

On the importance of Ramanujan's work, Hardy wrote: '*Opinions may differ about the importance of Ramanujan's work, the kind of standard by which it should be judged, and the influence which it is likely to have on the mathematics of the future. It has not the simplicity and the inevitableness of the very greatest work; it would be greater if it were less strange. One gift it shows which no one can deny, profound and invincible originality. He would probably have been a greater mathematician if he could have been caught and tamed a little in his youth; he would have discovered more that was new, and that, no doubt, of greater importance*'.

In a report submitted to the University of Madras on Ramanujan's mathematical work in England, Hardy wrote: '*In one respect, Mr. Ramanujan has been most unfortunate. The war has naturally distracted three-quarters of the interest that would otherwise have been taken in his work, and has made it almost impossible to bring his results to the notice of the continental mathematicians most certain to appreciate it.* It has moreover deprived him of the teaching of Mr. Littlewood, one of the great benefits which his visit to England was intended to secure. All this will pass; and, in spite of it, it is already safe to say that Mr. Ramanujan has justified abundantly all the hopes that were based upon his work in India and has shown that he possesses powers as remarkable in their way as those of any living mathematician. His work is only the more valuable because his abilities and methods are of so unusual a kind, and so unlike those of a European mathematician trained in the orthodox school.

Hardy concludes this report to the University with the following comments: '*My account of Ramanujan's work has been necessarily fragmentary and incomplete. I have said enough, I hope, to give some idea of its astonishing individuality and power. India has produced many talented mathematicians in recent years, a number of whom have come to Cambridge and attained high academical distinction. They will be the first to recognize that Mr. Ramanujan's work is of a different category. In him India now possesses a pure mathematician of the first order, whose achievements suggest the brightest hopes for its scientific future.*

The above statements of Hardy are all positive and strong and bordering on adulation. To show that Hardy was objective in his assessment of Ramanujan, a few of the drawbacks of Ramanujan pointed out by Hardy are reproduced here:
'*One would have thought that such a formalist as Ramanujan would have reveled in Cauchy's theorem, but he practically never used it, and the most astonishing testimony to his formal genius is that he never seemed to feel the want of it in the least*'.

"... in the analytic theory of numbers, he had, in a sense, discovered a great deal, but he was a very long way from understanding the real difficulties of the subject ... in analytic theory of numbers, even Ramanujan's imagination led him seriously astray ... he showed, as always, astonishing imaginative power, but he proved next to nothing, and a great deal even of what he imagined was false".

Ramanujan "had no knowledge at all, at any time, of the general theory of arithmetical forms."

"In analysis proper, Ramanujan's work is inevitably less impressive, since he knew no theory of functions, and you cannot do real analysis without it ... Still, Ramanujan rediscovered an astonishing number of the most beautiful analytic identities. Thus the functional equation for the Riemann Zeta-function ... stand (in an almost unrecognizable notation) in the notebooks. So does Poisson's summation formula ... and so also does Abel's functional equation...

"Ramanujan knew nothing at all about the theory of analytic functions... Ramanujan's theory of primes was vitiated by his ignorance of functions of a complex variable. It was (so to say) what the theory might be if the Zeta function had no complex zeros. His method depends upon a wholesale use of divergent series... That his proofs should have been invalid was only to be expected. But the mistakes went deeper than that, and many of the actual results were false. He had obtained the dominant terms of the classical formulae, although by invalid methods; but none of them are such close approximations as he supposed... This may be said to have been Ramanujan's one great failure."

Hardy was thus not only lavish in his appreciation of the originality and the mathematical genius of Ramanujan, but also harsh and severe in his criticism of the shortcomings of Ramanujan. While he himself edited only one chapter of Ramanujan's second Notebook on hypergeometric series, he estimated that "about two thirds of Ramanujan's best Indian work was rediscovery, and comparatively little of it was published in his lifetime, though Watson, who has worked systematically through his notebooks, has since disinterested a good deal more".

This estimate of Hardy's has been refuted to be too high by Bruce C. Berndt, who studied each and every one of the 3254 Entries of Ramanujan in his Notebooks, over a period of more than two decades (1974 to 1997) and to understand this one has to study the five-part magnum opus of Prof. Bruce C. Berndt.

Hardy in his 'A Mathematician's Apology' has stated that all his best work, after he began his life long collaboration with Littlewood, in 1911, and after he 'discovered Ramanujan', in 1913, "has been bound up with theirs, and it is obvious that my association with them was the decisive event in my life. I still say to myself when forced to listen to pompous and tiresome people, "Well, I have done one thing you could never have done, and that is to have collaborated with both Littlewood and Ramanujan on something like equal terms".

Hardy's lectures on Ramanujan's work

Hardy who devoted several years of his lifetime to the study of the work of Ramanujan, gave many lectures on Ramanujan's work at a number of Universities and Societies in America and England as well as courses of lectures at Princeton and Cambridge. The notes of his lectures at Princeton were written by Marshall Hall and these were printed as Lecture Notes. A copy of the same is in the Library of the University of Wisconsin, Madison, Wisconsin - a fact I learnt from a private communication with Professor Richard A. Askey, renowned for the Askey-Wilson polynomial scheme.

Hardy, in his 'A Mathematician's Apology' categorically states : "I hate 'teaching' and have had to do very little, such teaching as I have done having been almost entirely supervision of research; I love lecturing, and lectured a great deal to extremely able classes; and I have always had plenty of leisure for the researches which have been the one great permanent happiness of my life. The matter, delivery and hand-writing (a specimen of which appears on the dust cover of 'A Mathematician's Apology') were alike fascinating. (This passage is also quoted by Titchmarsh in the Collected Papers of G.H. Hardy).

Hardy has done yeoman service to Mathematics by publishing these lecture courses of his as a book entitled: 'Ramanujan: Twelve lectures on subjects suggested by his life and work', in 1940. Hardy was a Visiting Professor at Princeton and at the California Institute of Technology during 1928 – 1929. He delivered two lectures at the Harvard Tercentenary Conference of Arts and Sciences in the fall of 1936 – an occasion which 'brought to Harvard more than twenty-five hundred scholars for lectures under broad rubrics of knowledge'. On September 16, before an august audience consisting of 'no fewer than eleven Nobel Prize winners' Hardy gave his lectures on Ramanujan ... "the most romantic figure in the recent history of mathematics; a man whose career seems full of paradoxes and contradictions, who defies almost all the canons by which we are accustomed to judge one another, and about whom all of us will probably agree in one judgment only, that he was in some sense a very great mathematician."

This Harvard lecture is reprinted from the American Mathematical Monthly, where it was first published, in the above said book on 'Ramanujan'. As pointed out earlier, Hardy had lectured at a 'number of Universities and Societies in America and England and connected courses in Princeton and Cambridge' on Ramanujan's work and the remaining 11 chapters in 'Ramanujan' are based on these lectures of Hardy and 'Ramanujan is the thread which holds the whole together'. Hardy's aim in these lectures is stated clearly in his own words, in the Harvard lecture: "... the reputation of a mathematician cannot be made by failures or by rediscoveries; it must rest primarily, and rightly, on actual and original achievement. I have to justify Ramanujan on this ground, and I hope to do so in my later lectures."

The chapter titles given below indicates the range of problems on which Ramanujan had worked during his short productive period, as a professional researcher, after he came into contact with Hardy:

I. The Indian mathematician Ramanujan
II. Ramanujan and the theory of numbers
III. Round numbers
IV. Some more problems of the analytic theory of numbers
V. A lattice-point problem
VI. Ramanujan's work on partitions
VII. Hypergeometric series
VIII. Asymptotic theory of partitions
IX. The representation of numbers as sums of squares
X. Ramanujan's function $\tau(n)$
XI. Definite integrals
XII. Elliptic and modular functions

Apart from these lectures, Hardy has submitted a report on Ramanujan's work in England, for publication, at the request of the University of Madras. This report is with the Wren Library of Trinity College and has the Card Catalogue number Add.Ms.a.94[12(79)] (see Appendix 2 for details about the catalogued articles). The report gives summaries of twelve (out of 21) papers published by Ramanujan from England, including one (of the seven) he wrote with Hardy. Ten of these are in the *Collected papers*, enumerated as 11, 12, 6, 14, 16, 12, 15, 17, 18 and 20 (see Appendix 1 for the titles of these papers). One of the 12 corresponds to 19 (of Appendix 1) though with a slightly different title: *Some series for Euler's constant* and another (with Hardy) has the title: *A problem in the analytic theory of numbers*, which was unpublished at that time but communicated to the London Mathematical Society. Obviously, since this title is not one of the seven joint papers listed as 31 to 37 in the *Collected papers*, it was written subsequently under a different title.

Ramanujan's work on Hypergeometric series

Hardy remarks that "any student of the notebooks can see that Ramanujan's ideal if presentation had been copied from Carr's" Synopsis. Three years after Ramanujan's death, in 1923, Hardy spent about four months on Chapter XII – which is Chapter X in G.N. Watson's copy of the 'second edition', as stated in a Note at the end of his lecture VII in his book 'Ramanujan: Twelve Lectures on topics suggested by his life and work' – of the second Notebook, on Hypergeometric series, and in the introduction to the paper he published on this topic, Hardy wrote that 'a systematic verification of the results would be a very heavy undertaking'. A task taken up by G.N. Watson and B.M. Wilson, which ended abruptly with the premature death of B.M. Wilson, and awaited the devoted continuing efforts of

Bruce C. Berndt, from mid-1970s.

This section is intended for the mathematically inclined readers. This is also the area of research interest to this author from 1970s. Preceding the birth centenary of Ramanujan, the Director of the Institute of Mathematical Sciences, Professor E.C.G. Sudarshan, suggested my going to Mt. Abu to participate in a two week workshop on Special Functions, convened and conducted by the then Vice Chancellor of Rajasthan University, Professor Ratan P. Agarwal, who became my guru and for whose lectures I volunteered to take lecture Notes and the same were brought out as a Report. This was my formal introduction to the work of the group in India, led by Ratan P. Agarwal, on hypergeometric series and the author is thankful to his association from 1995–1996, especially with Professor Ratan P. Agarwal himself and with Professors Arun Verma, Remy Y. Denis, Satya Narayan Singh, Ashok K. Agarwal, Ahmad Ali and S.P. Singh. The British mathematician John Wallis, in his book, 'Arithmetica Infinitorum' (London, 1655), studied the series

$$1 + a + a(a+1) + a(a+1)(a+2) + \cdots \tag{14}$$

as a generalization of the ordinary geometric series:

$$1 + a + a^2 + a^3 + a^4 + \cdots + a^n + \cdots \;=\; (1-a)^{-1} \tag{15}$$

and called (14) as the 'hypergeometric series' (Note: the Greek work 'hyper' means above or beyond). Historically, the term 'hypergeometric', from the Greek $\nu\pi\epsilon\rho$, was first used by John Wallis to denote any series which is beyond the ordinary binomial series $(1-a)^{-1}$ is called a geometric series.

The infinite series

$$1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots , \tag{16}$$

usually represented by the later day symbol, ${}_2F_1(a,b;c;z)$ was defined and introduced by Carl Friedrich Gauss, in 1812, in his famous thesis '*Disquisitiones generales circa seriem infinitam*,' before the Royal Society in Göttingen, Germany. Gauss stated explicitly, that the ${}_2F_1(a,b;c;z)$, a solution of the 2nd order ODE, should be considered as a function of four variables and not as a function of the variable z, with a, b as the numerator and c as the denominator parameters of the series. On January 20, 1812, Gauss also announced the discovery of the series at the Paris International Congress of Mathematicians (ICM 1812). He proved the famous summation theorem, since named after him as the Gauss summation theorem:

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{17}$$

where $\Gamma(n)$ is the ordinary gamma function

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\cdots 2\cdot 1 = (n-1)! \text{ and } \Gamma(0) = 1. \tag{18}$$

Gauss gave a remarkably comprehensive discussion on the convergence properties of the series –

(i) the series is convergent for $|z| < 1$, (19a)

(ii) divergent for $|z| > 1$ and for (19b)

(iii) $z = 1$, it is convergent if $R\ell(c-a-b) > 0$

and divergent if $R\ell(c-a-b) < 0$. (19c)

If either of the numerator or denominator parameter is zero or a negative integer, $-n$, say, then the series (16) has only $n+1$ terms and it becomes a polynomial of degree n in z. If the denominator parameter c is a negative integer; or, if the zero due to the denominator parameter occurs before the zero due to the numerator parameter; then, the hypergeometric series is not defined. This is the accepted convention.

Gauss established 15 relations between six contiguous functions:

$$ {}_2F(a\pm 1,\ b;\ c;\ z) \equiv F(a\pm 1), \quad F(b\pm 1), \quad F(c\pm 1) \tag{20}$$

and showed that there exists between any two of these contiguous functions linear relations with coefficients, at most linear in z.

Gauss proved the famous (Gauss) summation theorem (17) and gave many recurrence relations between two or more of these series. The summation theorem is easily proved from the Euler integral representation for the Gauss function:

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt,\ c > b > 0, \tag{21}$$

for $c \neq 0, -1, -2, \cdots$ and $R\ell(c-a-b) > 0$.

When $z = 1$ in the Euler integral representation becomes:

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}\,(1-t)^{c-a-b-1}\, dt \tag{22}$$

where the integral can be recognized as the beta function which has the property:

$$B(m,n) = \int_0^1 t^{m-1}\,(1-t)^{n-1}\, dt = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},\ R\ell(m) > 0,\ R\ell(n) > 0, \tag{23}$$

so that

$$ {}_2F_1(a,b;c;1) \;=\; \frac{B(b,c-a-b)}{B(b,c-b)} \;=\; \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{24} $$

for $c-a-b>0$ for all a,b,c complex (even without $R\ell(c-a-b)>0$).

Ramanujan came across the Gauss summation theorem (4) in Carr's *Synopsis* and he was able to provide an alternate proof of this theorem without recourse to the Euler integral representation for the beta function. Entry 8, in Chapter X of Ramanujan's Notebook 2 contains this remarkable theorem and its proof:

Theorem: If $R\ell(x+y+n+1)>0$, then

$$ {}_2F_1(-x,-y;\; n+1;\; 1) \;=\; \frac{\Gamma(n+1)\Gamma(x+y+n+1)}{\Gamma(x+n+1)\Gamma(y+n+1)} \tag{25} $$

Proof: Assume n,x are integers, $n\geq 0$, $x\geq 0$, and let

$$ (1+u)^{y+n} \;=\; \sum_k {}^{y+n}C_k \qquad \text{and} \qquad \left(1+\frac{1}{u}\right)^x \;=\; \sum_\ell {}^xC_\ell \left(\frac{1}{u}\right)^\ell, \tag{26} $$

then

(i) $(1+u)^{y+n}\ \left(1+\frac{1}{u}\right)^x \;=\; \sum_{k,\ell}\ {}^{y+n}C_k\ {}^xC_\ell \;=\; \sum_n\ a_n\ u^n$ (say), where

$$ a_n \;=\; \sum_\ell {}^{y+n}C_{\ell+n}\ {}^xC_\ell \;=\; \frac{\Gamma(y+n+1)}{\Gamma(n+1)\Gamma(y+1)}\ {}_2F_1(-x,-y;\; n+1;\; 1). \tag{27} $$

(ii) $$ \frac{(1+u)^{x+y+n}}{u^x} \;=\; \sum_\ell\ {}^{x+y+n}C_\ell\ u^{\ell-x} \;=\; \sum_n\ a_n\ u^n \text{ (say)}, \tag{28} $$

then

$$ a_n \;=\; {}^{x+y+n}C_{x+n} \;=\; \frac{\Gamma(x+y+n+1)}{\Gamma(x+n+1)\Gamma(y+1)} \tag{29} $$

and hence,

$$ {}_2F_1(-x,-y;\; n+1;\; 1) \;=\; \frac{\Gamma(n+1)\Gamma(x+y+n+1)}{\Gamma(x+n+1)\Gamma(y+n+1)}, \tag{30} $$

which for

$$ x\to -a, \quad y\to -b, \quad n\to c-1, \tag{31} $$

yields the Gauss summation theorem for ${}_2F_1(a,b;\; c;\; 1)$ given in (17) above.

Barnes [1], in 1907, developed the modern day notation and introduced the integral representations for the hypergeometric functions, first introduced by Euler, in his '*Introduction in Analysis Infinitorum*', Lusanne (1748), vol. I. Conventionally, in the notation due to Barnes, the Gauss hypergeometric function is written as:

$$ {}_2F_1(a, b; c; z) \; = \; \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \, \frac{z^n}{n!}, \tag{32}$$

where the symbol

$$ (a)_n \; = \; \frac{\Gamma(a+n)}{\Gamma(a)} \; = \; a(a+1)(a+2)\cdots(a+n-1), \quad (a)_0 = 1, \tag{33}$$

is called the Pochammer symbol [10], or the shifted / raising factorial, or a generalized power.

Kummer, in 1836, showed that there exists in all 24 solutions for the Gauss hypergeometric 2nd Order Ordinary Differential Equation:

$$ \frac{d^2y}{dz^2} + \left[\frac{c}{z(1-z)} - \frac{1+a+b}{1-z} \right] \frac{dy}{dz} - \frac{ab}{z(1-z)} y \; = \; 0, \tag{34}$$

from which 0 and 1 are seen to be regular singular points of the equation. If $1/z$ replaces z, then ∞ is also a regular singular point of the Gauss equation [15]. Kummer's 24 solutions to the Gauss equation, cover the entire complex z-plane, provided that a, b and c are not integers nor zero [13]. In a self-contained, short communication [8] Lievens, Srinivasa Rao and Van der Jeugt showed that the set of 24 Kummer solutions of the classical second order ordinary differential equation of Gauss has an *elegant, simple group theoretic structure associated with the symmetries of a cube*, or, in other words, that the underlying symmetry group is the Symmetric group S_4.

The hypergeometric function has retained its significance in modern mathematics because of its powerful unifying influence, since an intimate relationship exists between the ${}_2F_1(z)$ and the special functions of mathematics Physics – ref. Theorem 9.1 in W.W. Bell [2].

Clausen (1828) introduced a series which is a generalization of (3), by increasing the number of numerator and denominator parameters to 3 and 2, respectively, ${}_3F_2(a, b, c; d, e; z)$. This led to further generalizations in this direction and summation theorems, like the Gauss summation theorem for the ${}_2F_1(1)$, have been obtained by Saalschütz, Dixon (1903) and Dougall (1907) for generalized hypergeometric series. The first monograph of W.N. Bailey on 'Generalized Hypergeometric Series', provides an excellent summary of the whole theory, from its origins, as it existed then. Some of the subsequent books on generalized hypergeometric functions of interest are due to Lucy Slater [13]; G. Gasper and M. Rahman [4]. See also, William J. Thompson [14].

What is remarkable about Ramanujan's 'natural' genius is that with only the hint of the Gauss summation theorem, which he found in Carr's *Synopsis*, Ramanujan starts the Chapter XII of Notebook 1, and Chapter X of his Notebook 2, with the most general summation theorem, as Entry 1:

$$\begin{aligned}
&\frac{n\,\Gamma(x+n+a, y+n+1, z+n+1, u+n+1)}{n!\,\Gamma(x+y+z+u+n+1, x+y+n+1, x+z+n+1, x+u+n+1)} \\
&\quad\times\frac{\Gamma(x+y+z+n, y+z+u+n, z+u+x+n, x+y+u+n)}{\Gamma(y+z+n+1, y+u+n+1, z+u+n+1)} \\
&= n \;-\; (n+2)\frac{x}{1!}\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)} \\
&\quad\times\frac{z(z-1)}{(z+n+1)(z+n+2)}\cdot\frac{u(u-1)}{(u+n+1)(u+n+2)} \\
&\quad\times\frac{(x+y+z+u+2n+1)(x+y+z+u+2n+2)}{(x+y+z+u+n)(x+y+z+u+n+1)} \\
&\quad- \&c.
\end{aligned} \tag{35}$$

This Entry is followed by Ramanujan's note:

"NB. The above result is not true for all values of x, y, z, u and n. For example it is not true when $z+y+z+u+n = x, y, z$ *or* u owing to the extraneous factor containing all the quantities x, y, z & u in each term, unless we get rid of this factor identities deduced from the above won't be true for all values. The only way to get rid of this is to make u infinitely great. The solution for the theorem is evident from the result."

The historic letter from an 'unknown clerk' to G.H. Hardy, at Trinity College, Cambridge, dated 16 January 1913, was a bolt from the blue, for which Hardy, with his penchant for the unorthodox and the unexpected, opened his mind and heart to groom, nurture and mentor an un-schooled mathematical 'natural' genius, from then on till his (Hardy's) death, on Dec. 1, 1947, more than 27 years after the untimely death of his protege Ramanujan, who died on 26 April 1920. Incidentally the year 2020, marks the Ramanujan Remembrance Centenary, and this article is our tribute to the 'natural' mathematical genius of India.

Hardy was kind and considerate and concerned to the happy mental state of Ramanujan, due to his loneliness, in an alien country, compounded by his orthodoxy and vegetarianism. Hardy's efforts to submit the longest paper of Ramanujan on 'Highly Composite Numbers', along with a few other papers, as a Thesis to obtain for Ramanujan, (a failed F.A. of the University of Madras), the B.A. degree in Matheamtics, by Research; and in 1918, Hardy got him elected to the Fellowship of the Royal Society of London, despite an attempt by Arthur Eddington to postpone the same to the next year. It is to be noted that Ramanujan is the first F.R.S. in Mathematics of India.

Finally, an appeal: The original Notebooks of Ramanujan, are unfortunately in an extremely brittle dilapidated state, despite several appeals by the author. The author fortunately scanned the Notebooks in 2002-2003, and included them in the website:

www.imsc.res.in/ ~ rao/ramanujan

created along with the CD ROMs produced under the DST Project, mentioned earlier. This is considered by George Andrews[2] as a contribution to Mathematics and mathematicians. It may be worthwhile to do the same for the 'Lost' Notebook of Ramanujan, with the help of Professors George Andrews and Bruce Berndt, who are continuing to work on the same even now.

The original Notebooks of Ramanujan are kept 'SAFE' in a closed Godrej Bureau, as Reference volumes, with the key in the custody of the Librarian. In 2012, on the occasion of the 125th Birth Anniversary of Ramanujan, an excellent edition of the original Notebooks, digitally restored has been brought out by the National Board of Higher Mathematics. However, the original Notebooks are not accessible to any but the most persistent, since the authorsaw them in 1987 and the author accompanied many including the Ramanujan followers, mathematicians of repute: Richard Askey, George Andrews, Bruce Berndt, Ratan P Agarwal, Arun Verma, Ken Ono, A. Raghuram, to see the original Notebooks. Suggestions [by the author] for preserving them in an air-conditioned box, at a proper temperature with feasibility to turn the pages using an electro-mechanical device[3], have fallen on unresponsive ears of the unconcerned authorities till date. As a tribute to Ramanujan, who has made Carr's 'Synopsis' famous, this book was reprinted by the American Mathematical Society - London Mathematical Society, at the beginning

[2]In a e-mail communication to the author.

[3]This can be done by my dear friend Dr. T.S. Radhakrishnan, Former Head of the Material Sciences Division of IGCAR, Kalpakkam, who is aware of this problem as we had discussions about the same, on a few occasions.

of the new Millennium (in 1999) and digitized the same, in May 2006.

Hopefully, at least now, in this Ramanujan Remembrance Centenary Year (April 26, 2020 – April 26, 2021), let us hope that efforts will be made to preserve these National Treasures, along with a few remaining artifacts – the invaluable Original Notebooks of Ramanujan, the papers of Ramanujan in the Wren Library, Trinity College archives; the Ramanujan Personal files in the National and Tamilnadu Archives (listed in the Appendix 2 and Appendix 3, of 'Srinivasa Ramanujan: a Mathematical Genius', by the author (K Srinivasa Rao), East West Books (Madras) Pvt. Ltd, 2004); the slate used by Mr. S. Narayana Iyer, the brass vessel used to boil water to give him fomentations; and the preserved works of other great Indians, including Sir C.V. Raman – in a National Ramanujan Museum, for posterity, as the Notebooks must be considered as our National Heritage, like the works of Newton and Einstein preserved in Developed Nations.

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References

[1] Barnes, E. W., Proc. London Math. Soc., (2) 5, (1907) 59–116, 6 (1907), 141–177.

[2] Bell, W. W., Special Functions for Scientists and Engineers, von Nostrand, New York (1968).

[3] Collected Papers of Srinivasa Ramanujan, Ed. by G. H. Hardy, P. V. Seshu Aiyar and B. M. Wilson, AMS Chelsea Publishing, AMS, Providence, Rhode Island (2000) p.xxxiv.

[4] Gasper G. and Rahman M., Encyclopedia of Mathematics and its Applications 35, Cambridge University Press, Cambridge (1990).

[5] Kanigel, Robert, The Man Who Knew Infinity: the life of the genius Ramanujan, Charles Scribner's Sons, New York (1991); Rupa & Co.'s Indian edition, with a Foreword by KSR (1991), has been reprinted on the occasion of the 125th Birth Anniversary of Ramanujan, in 2012.

[6] Kanigel, Robert, The Man who knew Infinity: a life of the Genius Ramanujan, Charles Scribner's Sons, New York (1991); Indian Edition published by Rupa & Co. (1994).

[7] Kummer E. E., $F(\alpha, \beta, x)$, J. für Math., 15 (1836), 39 – 83 and 127 – 172.

[8] Lievens S., Rao K. Srinivasa and Van der Jeugt J., Int. Trans. Spl. Func., 16 (2005), 153-158.

[9] Nandy, Ashish, Alternative Sciences, Allied Publishers, New Delhi, 1980.

[10] Pochammer, L., Über die hypergeometrische Funktionen n-ter Ordnung, J. für Math. (Crelle), 71 (1870) 316 – 340.

[11] Ramanujan S., Letters and Commentary, B. C. Berndt and R. A. Rankin, AMS-LMS, (1995) p. 137.

[12] Ramanujan, S., Letters and Commentary, Ed. P. K. Srinivasan, Muthialpet High School, Madras-1, (1968) p. 115.

[13] Slater Lucy J., Generalized Hypergeometric Functions, Cambridge University Press, Cambridge (1966).

[14] Thompson William J., Angular Momentum - An Illustrated Guide to Rotational Symmetries for Physical Systems, John Wiley & Sons, Inc. (1994) and Wiley-VCH Verlag GmbH & Co. KGaA, Weinheim (2004).

[15] Whittaker E. T. and Watson G. N., A course of Modern Analysis, Cambridge University Press, Cambridge (1902, Fourth Edn. 1927), x10.3