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RELIABILITY INVOLVING GENERALIZED GAMMA, GENERALIZED FOLDED LOGISTIC DISTRIBUTIONS AND FGM COPULA

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Abstract: This paper deals with reliability measure $P(Y < X)$ using FGM copula, when X and Y follow (a) Weibull and generalized gamma distributions, and (b) Rathie-Swamee generalized folded logistic distributions. The use of copula is better and widely employed than doing the classical joint distribution dependence. A few particular cases are also indicated.

Keywords and Phrases: Stress-strength reliability, $P(Y < X)$, FGM copula, Folded Rathie-Swamee, Gamma and Weibull distributions.

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1. Introduction

In literature, $R = P(Y < X)$ is a measure of component reliability when the component is subjected to a random stress Y and a random strength X. The component fails when the applied stress exceeds the strength. In the book by [6],

X and Y are assumed to be independent or having a joint distribution function. The evaluation of R is more realistic when X and Y are assumed to be dependent. The copula-based approach has been employed by [1, 2], and [10].

For positive X and Y, the measure R is given by

$$R = P(Y < X) = \int_0^\infty \int_0^x h(x, y)dxdy, \tag{1}$$

where

$$h(x, y) = c(F(x), G(y))f(x)g(y). \tag{2}$$

In this article, we will use the Farlie-Gumbel-Morgenstern (FGM) copula density defined by

$$c(F(x), G(y)) = 1 + \theta(1 - 2F(x))(1 - 2G(y)). \tag{3}$$

Using (2) and (3) in (1), we arrive at the following expression for R:

$$R = R_I + \theta D, \tag{4}$$

where θ is the dependency parameter,

$$R_I = \int_0^\infty G(x)f(x)dx \tag{5}$$

and

$$D = \int_0^\infty G(x)(1 - G(x))(1 - 2F(x))f(x)dx. \tag{6}$$

For reliability analysis $P(Y < X)$, bivariate distributions for strength and stress considered were: bivariate normal [3], bivariate exponential [9], bivariate gamma [8], bivariate Pareto [5] and bivariate log-normal [4]. The advantage of copula lies in separating the dependence from joint distribution in a more general setting.

The reliability $R = P(Y < X)$ has applications in physics, engineering, quality control, economics, medicine, etc. See, for example, [12], [11] and [13].

This paper is written as follows: Section 2 presents some known results, definitions of generalized gamma, Weibull and Rathie-Swamee generalized folded logistic distributions. In Section 3, reliability $P(Y < X)$ using FGM copula is obtained for X and Y following (a) Weibull and generalized gamma distributions, and (b) folded Rathie-Swamee distributions, respectively. A few particular cases of the main theorems are mentioned which involve Weibull and folded logistic distributions. In the last section, we conclude the paper by providing a few conclusions and outline of possible future work.

2. Known Results and Statistical Distributions

The H-function (see [7]), is defined by

$$H_{p,q}^{m,n}\left[x\middle|\begin{matrix}(a_1,A_1),\dots,(a_n,A_n),(a_{n+1},A_{n+1}),\dots,(a_p,A_p)\\(b_1,B_1),\dots,(b_m,B_m),(b_{m+1},B_{m+1}),\dots,(b_q,B_q)\end{matrix}\right]$$
$$=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-B_js)\prod_{j=1}^{n}\Gamma(1-a_j+A_js)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+B_js)\prod_{j=n+1}^{p}\Gamma(a_j-A_js)}x^s ds, \tag{7}$$

where A_j and B_j are assumed to be positive quantities and all the a_j and b_j may be complex. The contour L runs from $c-i\infty$ to $c+i\infty$ such that poles of $\Gamma(b_j-B_js)$, $j=1,\dots,m$, lie to the right of L and the poles of $\Gamma(1-a_j+A_js)$, $j=1,\dots,n$, lie to the left of L.

The H-function can be expressed in computable form as (see [7]):

When the poles of $\prod_{j=1}^{m}\Gamma(b_j-B_js)$ are simple, we have

$$H_{p,q}^{m,n}(x)=\sum_{h=1}^{m}\sum_{\nu=0}^{\infty}\frac{\prod\limits_{j=1\neq h}^{m}\Gamma\left(b_j-B_j\dfrac{b_h+\nu}{B_h}\right)}{\prod\limits_{j=m+1}^{q}\Gamma\left(1-b_j+B_j\dfrac{b_h+\nu}{B_h}\right)}$$
$$\times\frac{\prod\limits_{j=1}^{n}\Gamma\left(1-a_j+A_j\dfrac{b_h+\nu}{B_h}\right)}{\prod\limits_{j=n+1}^{p}\Gamma\left(a_j-A_j\dfrac{b_h+\nu}{B_h}\right)}\frac{(-1)^\nu x^{(b_h+\nu)/B_h}}{\nu!B_h}, \tag{8}$$

for $x\neq 0$ if $\delta>0$ and for $0<|x|<w$ if $\delta=0$, where $\delta=\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j$ and $w=\prod_{j=1}^{p}A_j^{A_j}/\prod_{j=1}^{q}B_j^{B_j}$.

When the poles of $\prod_{j=1}^{n}\Gamma(1-a_j+A_js)$ are simple, we have

$$H_{p,q}^{m,n}(x)=\sum_{h=1}^{n}\sum_{\nu=0}^{\infty}\frac{\prod_{j=1\neq h}^{n}\Gamma\left(1-a_j-A_j\frac{1-a_h+\nu}{A_h}\right)}{\prod_{j=n+1}^{p}\Gamma\left(a_j+A_j\frac{1-a_h+\nu}{A_h}\right)}$$
$$\times\frac{\prod_{j=1}^{m}\Gamma\left(b_j+B_j\frac{1-a_h+\nu}{A_h}\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j-B_j\frac{1-a_h+\nu}{A_h}\right)}\frac{(-1)^{\nu}(1/x)^{(1-a_h+\nu)/A_h}}{\nu!A_h},\tag{9}$$

for $x\neq 0$ if $\delta<0$ and for $|x|>w$ if $\delta=0$.

$$I_{s,a,b}^{\lambda,\lambda_1}=\int_0^{\infty}y^{s-1}\exp(-ay^{\lambda}-by^{\lambda_1})dy\tag{10}$$

(using [7, p.60])

$$=\lambda^{-1}a^{-\frac{s}{\lambda}}H_{1,1}^{1,1}\left[ba^{\frac{-\lambda_1}{\lambda}}\middle|\begin{matrix}(1-\frac{s}{\lambda},\frac{-\lambda_1}{\lambda})\\(0,1)\end{matrix}\right]\tag{11}$$

(using(8))

$$=\lambda^{-1}a^{-\frac{s}{\lambda}}\sum_{\nu=0}^{\infty}\Gamma\left(\frac{s+\lambda_1\nu}{\lambda}\right)\frac{(-ba^{-\frac{\lambda_1}{\lambda}})^{\nu}}{\nu!},\tag{12}$$

valid for $\mathrm{Re}(a,b,\lambda,\lambda_1,s)>0$.

Generalized-Gamma distribution has density and distribution functions given by

$$f(x)=\frac{\gamma_1\beta_1^{\frac{\alpha_1}{\gamma_1}}}{\Gamma\left(\frac{\alpha_1}{\gamma_1}\right)}x^{\alpha_1-1}\exp(-\beta_1x^{\gamma_1})\tag{13}$$

and

$$F(x)=\frac{(\beta_1x^{\gamma_1})^{\frac{\alpha_1}{\gamma_1}}}{\Gamma\left(1+\frac{\alpha_1}{\gamma_1}\right)}{}_1F_1\left(\frac{\alpha_1}{\gamma_1};\frac{\alpha_1}{\gamma_1}+1;-\beta_1x^{\gamma_1}\right),\tag{14}$$

respectively, where α_1, β_1, γ_1, $x>0$.

Alternately, (14) may be written as

$$F(x)=a\sum_{r=0}^{\infty}a_rx^{\gamma_1r+\alpha_1},\tag{15}$$

where

$$a = \frac{\beta_1^{\frac{\alpha_1}{\gamma_1}}}{\Gamma\left(\frac{\alpha_1}{\gamma_1}\right)} \tag{16}$$

and

$$a_r = \frac{(-\beta_1)^r}{r!(\frac{\alpha_1}{\gamma_1} + r)}. \tag{17}$$

Weibull distribution has density and distribution functions as

$$g(x) = \beta_2\gamma_2 x^{\gamma_2 - 1}\exp(-\beta_2 x^{\gamma_2}) \tag{18}$$

and

$$G(x) = 1 - \exp(-\beta_2 x^{\gamma_2}), \tag{19}$$

respectively, where β_2, γ_2, $x > 0$.

For $X \sim RS(a, b, p)$, $a, b \geq 0$ (both a and b are not zeros simultaneously), $p \geq -1$, $x \in (0, \infty)$, the Rathie-Swamee [14, 15] density and distribution functions are defined as

$$f(x) = \frac{2[a + b(p+1)x^p]\exp[-x(a + bx^p)]}{\{1 + \exp[-x(a + bx^p)]\}^2} \tag{20}$$

and

$$F(x) = \frac{1 - \exp[-x(a + bx^p)]}{1 + \exp[-x(a + bx^p)]} = \frac{2}{1 + \exp[-x(a + bx^p)]} - 1. \tag{21}$$

3. Reliability using FGM Copula

In this section, we derive two main theorems for the reliability R using FGM copula given in (1) when (a) X has generalized gamma distribution and Y has Weibull distribution and (b) X and Y have folded Rathie-Swamee distributions with different parameters.

3.1. Reliability using Weibull and Generalized Gamma Distributions

The following theorem for reliability $R = P(Y < X)$ is established in this section:

Theorem 1. *Let* $X \sim GG(\alpha_1, \beta_1, \gamma_1)$ *and* $Y \sim GG(\gamma_2, \beta_2, \gamma_2)$. *Then*

$$R = P(Y < X) = R_I + \theta D, \tag{22}$$

where

$$R_I = 1 - \frac{\gamma_1 \beta_1^{\frac{\alpha_1}{\gamma_1}}}{\Gamma\left(\frac{\alpha_1}{\gamma_1}\right)} I^{\gamma_1,\gamma_2}_{\alpha_1,\beta_1,\beta_2} \tag{23}$$

and

$$D = \gamma_1 a \left[I^{\gamma_1,\gamma_2}_{\alpha_1,\beta_1,\beta_2} - I^{\gamma_1,\gamma_2}_{\alpha_1,\beta_1,2\beta_2} - 2a\sum_{r=0}^{\infty} a_r I^{\gamma_1,\gamma_2}_{2\alpha_1+\gamma_1 r,\beta_1,\beta_2} + 2a\sum_{r=0}^{\infty} a_r I^{\gamma_1,\gamma_2}_{2\alpha_1+\gamma_1 r,\beta_1,2\beta_2} \right], \tag{24}$$

with $I^{\lambda,\lambda_1}_{s,a,b}$, a, a_r *given respectively in (10), (16) and (17).*

Proof. From (5), (13) and (19), we have

$$\begin{aligned} R_I &= \int_0^\infty [1-\exp(-\beta_2 x^{\gamma_2})] \frac{\gamma_1 \beta_1^{\frac{\alpha_1}{\gamma_1}}}{\Gamma\left(\frac{\alpha_1}{\gamma_1}\right)} x^{\alpha_1-1}\exp(-\beta_1 x^{\gamma_1})dx \\ &= 1 - \frac{\gamma_1 \beta_1^{\frac{\alpha_1}{\gamma_1}}}{\Gamma\left(\frac{\alpha_1}{\gamma_1}\right)} \int_0^\infty x^{\alpha_1-1}\exp(-\beta_1 x^{\gamma_1} - \beta_2 x^{\gamma_2})dx \\ &= 1 - \frac{\gamma_1 \beta_1^{\frac{\alpha_1}{\gamma_1}}}{\Gamma\left(\frac{\alpha_1}{\gamma_1}\right)} I^{\gamma_1,\gamma_2}_{\alpha_1,\beta_1,\beta_2}. \end{aligned}$$

Also, from (6), (13), (14) and (19) we get

$$\begin{aligned} D &= \int_0^\infty G(x)[1-G(x)][1-2F(x)]f(x)dx \\ &= \int_0^\infty [1-\exp(-\beta_2 x^{\gamma_2})]\exp(-\beta_2 x^{\gamma_2}) \left(1 - 2a\sum_{r=0}^{\infty} a_r x^{\gamma_1 r+\alpha_1}\right) \\ &\quad \gamma_1 a x^{\alpha_1-1}\exp(-\beta_1 x^{\gamma_1})dx \\ &= \gamma_1 a \int_0^\infty x^{\alpha_1-1}\exp(-\beta_1 x^{\gamma_1} - \beta_2 x^{\gamma_2}) \Big[1-\exp(-\beta_2 x^{\gamma_2}) \\ &\quad -2a\sum_{r=0}^{\infty} a_r x^{\gamma_1 r+\alpha_1} + 2a\sum_{r=0}^{\infty} a_r x^{\gamma_1 r+\alpha_1}\exp(-\beta_2 x^{\gamma_2})\Big] dx \\ &= \gamma_1 a \left[I^{\gamma_1,\gamma_2}_{\alpha_1,\beta_1,\beta_2} - I^{\gamma_1,\gamma_2}_{\alpha_1,\beta_1,2\beta_2} - 2a\sum_{r=0}^{\infty} a_r I^{\gamma_1,\gamma_2}_{2\alpha_1+\gamma_1 r,\beta_1,\beta_2} + 2a\sum_{r=0}^{\infty} a_r I^{\gamma_1,\gamma_2}_{2\alpha_1+\gamma_1 r,\beta_1,2\beta_2} \right] \end{aligned}$$

The following particular case of Theorem 1 for $X \sim GG(\phi_1, \theta_1, \phi_1)$ and $Y \sim GG(\phi_2, \theta_2, \phi_2)$ can be derived or proved independently:

Corollary 1. *For the Weibull distributions, the reliability R is given by*

$$R = R_I + \theta D \tag{25}$$

where

$$R_I = 1 - \theta_1 \phi_1 I^{\phi_1,\phi_2}_{\phi_1,\theta_1,\theta_2} \tag{26}$$

and

$$D = \theta_1 \phi_1 \left[2I^{\phi_1,\phi_2}_{\phi_1,2\theta_1,\theta_2} + I^{\phi_1,\phi_2}_{\phi_1,\theta_1,2\theta_2} - 2I^{\phi_1,\phi_2}_{\phi_1,2\theta_1,2\theta_2} - I^{\phi_1,\phi_2}_{\phi_1,\theta_1,\theta_2} \right]. \tag{27}$$

3.2. Reliability using Folded Rathie-Swamee Distribution

In this subsection, we obtain the reliability R when X and Y follow generalized folded logistic distribution.

Theorem 2. *The probability $P(Y < X)$, $X \sim RS(a_1, b_1, p)$ and $Y \sim RS(a_2, b_2, p)$, is given by*

$$R = P(Y < X) = R_I + \theta D, \tag{28}$$

where

$$R_I = 4I_{1,2} - 1 \tag{29}$$

and

$$D = 4(9I_{1,2} - 6I_{2,2} - 3/2 - 12I_{1,3} + 8I_{2,3} + 4I_{0,3}), \tag{30}$$

with $I_{\alpha,\beta}$ given in (33) with $I^{\lambda,\lambda_1}_{s,a,b}$ in (10).

Proof. For $X \sim RS(a_1, b_1, p)$ and $Y \sim RS(a_2, b_2, p)$ with distribution functions $F(x)$ and $G(y)$, respectively, D is given by

$$D = \int_0^\infty \left\{ \frac{2}{1 + \exp[-x(a_2 + b_2 x^p)]} - 1 \right\} 2 \left\{ 1 - \frac{1}{1 + \exp[-x(a_2 + b_2 x^p)]} \right\} \left\{ 3 - \frac{4}{1 + \exp[-x(a_1 + b_1 x^p)]} \right\} \frac{2[a_1 + b_1(p+1)x^p] \exp[-x(a_1 + b_1 x^p)]}{\{1 + \exp[-x(a_1 + b_1 x^p)]\}^2} dx. \tag{31}$$

Let $D_1 = 1 + \exp[-x(a_1 + b_1 x^p)]$ and $D_2 = 1 + \exp[-x(a_2 + b_2 x^p)]$, then

$$\begin{aligned}
D =& 4\int_0^\infty \left[\frac{9}{D_2} - \frac{6}{D_2^2} - 3 - \frac{12}{D_1 D_2} + \frac{8}{D_2^2 D_1} + \frac{4}{D_1}\right] \\
& \frac{[a_1 + b_1(p+1)x^p]\exp[-x(a_1 + b_1 x^p)]}{D_1^2} dx \\
=& 4\int_0^\infty [a_1 + b_1(p+1)x^p]\exp[-x(a_1 + b_1 x^p)] \\
& \left[\frac{9}{D_2 D_1^2} - \frac{6}{D_2^2 D_1^2} - \frac{3}{D_1^2} - \frac{12}{D_2 D_1^3} + \frac{8}{D_2^2 D_1^3} + \frac{4}{D_1^3}\right] dx \\
=& 4[9I_{1,2} - 6I_{2,2} - 3I_{0,2} - 12I_{1,3} + 8I_{2,3} + 4I_{0,3}],
\end{aligned} \tag{32}$$

where

$$\begin{aligned}
I_{\alpha,\beta} =& \int_0^\infty \{a_1 + b_1(p+1)x^p]\exp[-x(a_1 + b_1 x^p)]\}\, D_2^{-\alpha} D_1^{-\beta} dx \\
=& \sum_{r=0}^\infty \frac{(-1)^r(\alpha)_r}{r!} \sum_{s=0}^\infty \frac{(-1)^s(\beta)_s}{s!} \int_0^\infty [a_1 + b_1(p+1)x^p] \\
& \exp[-x(a_1 + b_1 x^p) - rx(a_2 + b_2 x^p) - sx(a_1 + b_1 x^p)]dx \\
=& \sum_{r=0}^\infty \sum_{s=0}^\infty \frac{(-1)^{r+s}(\alpha)_r(\beta)_s}{r!s!} \int_0^\infty [a_1 + b_1(p+1)x^p] \\
& \exp\{-x[a_1(1+s) + a_2 r] - x^{p+1}[b_1(1+s) + b_2 r]\}dx \\
=& \sum_{r=0}^\infty \sum_{s=0}^\infty \frac{(-1)^{r+s}(\alpha)_r(\beta)_s}{r!s!} \Big[a_1 I^{1,\ p+1}_{1,\ a_1(1+s)+a_2 r,\ b_1(1+s)+b_2 r} \\
& + b_1(p+1) I^{1,p+1}_{p+1,\ a_1(1+s)+a_2 r,\ b_1(1+s)+b_2 r}\Big].
\end{aligned} \tag{33}$$

$$(1+x)^{-\gamma} = \sum_{r=0}^\infty \frac{(\gamma)_r(-x)^r}{r!}, \tag{34}$$

$$D_2^{-1} = \sum_{r=0}^\infty (-1)^r \exp[-rx(a_2 + b_2 x^p)], \tag{35}$$

$$D_2^{-2} = \sum_{r=0}^\infty (1+r)(-1)^r \exp[-rx(a_2 + b_2 x^p)], \tag{36}$$

$$D_1^{-2} = \sum_{s=0}^\infty (1+s)(-1)^s \exp[-sx(a_1 + b_1 x^p)], \tag{37}$$

$$D_1^{-3} = \frac{1}{2}\sum_{s=0}^{\infty}(2+s)(1+s)(-1)^s \exp[-sx(a_1+b_1x^p)]. \tag{38}$$

Now

$$\begin{aligned} R_I &= \int_0^{\infty} G(x)f(x)dx \\ &= \int_0^{\infty}\left\{\frac{2}{1+\exp[-x(a_2+b_2x^p)]}-1\right\}\frac{2[a_1+b_1(p+1)x^p]\exp[-x(a_1+b_1x^p)]}{\{1+\exp[-x(a_1+b_1x^p)]\}^2}dx \\ &= J_1 - J_2, \end{aligned} \tag{39}$$

where

$$J_1 = 4\int_0^{\infty}\frac{[a_1+b_1(p+1)x^p]\exp[-x(a_1+b_1x^p)]}{\{1+\exp[-x(a_2+b_2x^p)]\}\{1+\exp[-x(a_1+b_1x^p)]\}^2}dx = 4I_{1,2} \tag{40}$$

and

$$J_2 = 2\int_0^{\infty}\frac{[a_1+b_1(p+1)x^p]\exp[-x(a_1+b_1x^p)]}{\{1+\exp[-x(a_1+b_1x^p)]\}^2}dx = 2I_{0,2} = 1. \tag{41}$$

Hence,

$$R_I = 4I_{1,2} - 1.$$

For $b_1 = 0 = b_2$, in (33), we have

$$\begin{aligned} I_{\alpha,\beta} &= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^{r+s}(\alpha)_r(\beta)_s}{r!s!}\left[a_1 I_{1,\ a_1(1+s)+a_2r,\ 0}^{1,\ p+1}\right] \\ &= a_1\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^{r+s}(\alpha)_r(\beta)_s}{r!s!}\left[a_1(1+s)+a_2r\right]^{-1}. \end{aligned} \tag{42}$$

For $b = 0$, we get

$$I_{t,a,0}^{\lambda,\lambda_1} = \lambda^{-1}a^{-t/\lambda}\Gamma(t/\lambda). \tag{43}$$

Corollary 2. *For $b_1 = b_2 = 0$, we deduce the following result for the reliability for folded logistic distribution:*

$$P(Y < X) = R_I + \theta D, \tag{44}$$

where

$$\begin{aligned}R_I &= 4I_{1,2} - 1 \\ &= 4\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^{r+s}(1)_r(2)_s}{r!s!}\left[a_1 I^{1,\ p+1}_{1,\ a_1(1+s)+a_2r,\ 0} + 0\right] - 1 \\ &(using (43)) \\ &= 4\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}(1+s)a_1[a_1(1+s)+a_2r]^{-1}\Gamma(1) - 1 \\ &= 4a_1\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}\frac{1+s}{a_1(1+s)+a_2r} - 1 \qquad (45)\end{aligned}$$

and

$$\begin{aligned}D &= 4[9I_{1,2} - 6I_{2,2} - 3/2 - 12I_{1,3} + 8I_{2,3} + 4I_{0,3}] \\ &= 4a_1\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^{r+s}}{r!s!}[a_1(1+s)+a_2r]^{-1}A_{r,s} - 6, \qquad (46)\end{aligned}$$

where

$$\begin{aligned}A_{r,s} &= [9(1)_r(2)_s - 6(2)_r(2)_s - 12(1)_r(3)_s + 8(2)_r(3)_s + 4(0)_r(3)_s] \\ &= 9r!(s+1)! - 6(1+r)!(1+s)! - 6r!(2+s)! + 4(1+r)!(2+s)! \\ &= 3r!s![3(s+1) - 2(1+r)(1+s) - 2(2+s)(1+s) + \frac{4}{3}(1+r)(2+s)(1+s)] \\ &= r!s!(s+1)(1-2r)(7+2s). \qquad (47)\end{aligned}$$

Hence

$$D = 4a_1\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}[a_1(1+s)+a_2r]^{-1}\left[(s+1)(7+2s)(1-2r)\right] - 6. \qquad (48)$$

Corollary 3. *For X and Y independent, we have*

$$P(Y < X) = R_I = 4I_{1,2} - 1, \qquad (49)$$

where

$$\begin{aligned}I_{1,2} = &\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^{r+s}(1)_r(2)_s}{r!s!}\Big[a_1 I^{1,\ p+1}_{1,\ a_1(1+s)+a_2r,\ b_1(1+s)+b_2r} \\ &+ b_1(p+1)I^{1,p+1}_{p+1,\ a_1(1+s)+a_2r,\ b_1(1+s)+b_2r}\Big], \qquad (50)\end{aligned}$$

with $I^{\lambda,\lambda_1}_{t,a,b}$ *given in (10).*

For $p = 0$ *in (50), we have*

$$I_{1,2} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}(1+s)\left[a_1 I^{1,1}_{1,\ a_1(1+s)+a_2r,\ b_1(1+s)+b_2r} + b_1 I^{1,1}_{1,\ a_1(1+s)+a_2r,\ b_1(1+s)+b_2r}\right]$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^{r+s}(1+s)(a_1+b_1)}{(a_1+b_1)(1+s)+(a_2+b_2)r}. \tag{51}$$

Thus with $p = 0$, X *and* Y *independent:*

$$P(Y < X) = 4(a_1+b_1)\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^{r+s}(1+s)}{(a_1+b_1)(1+s)+(a_2+b_2)r} - 1. \tag{52}$$

Corollary 4. *When* $b_1 = b_2 = 0$*, and* X *and* Y *independent folded logistic distributions, we have the following result on using (45):*

$$P(Y < X) = 4a_1\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^{r+s}(1+s)}{a_1(1+s)+a_2r} - 1. \tag{53}$$

4. Conclusions

Mathematical expressions for the reliability measure $P(Y < X)$ are obtained by using FGM copula when X and Y follow (a) generalized gamma and Weibull distributions, and (b) Rathie-Swamee generalized folded logistic distributions. A few particular cases of importance are mentioned.

In a future paper, the authors plan to apply FGM and other copulas to other statistical distributions to calculate the reliability $P(Y < X)$ and to analyze real data sets as possible applications.

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