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ROGERS–RAMANUJAN TYPE IDENTITIES FOR $(n+t)$–COLOR OVERPARTITIONS

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Abstract: In this paper, we provide combinatorial interpretations of some Rogers–Ramanujan type identities listed in Chu-Zhang and Slater's Compendium using n–color overpartitions.

Keywords and Phrases: Rogers–Ramanujan type identities, n–color overpartitions and combinatorial interpretations.

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1. Introduction

The history of partitions goes by fits and starts. Euler's deep analysis on partitions stood for over a century before other mathematicians made considerable progress. According to Euler's insight, the different kinds of two sets of partitions is seen to have the same count of elements. The most eminent of these outcomes are called the Rogers–Ramanujan identities (RRI). These identities were first discovered by Rogers in 1894 but were appreciated only after Ramanujan rediscovered these in 1913.

$$\sum_{\alpha=0}^{\infty}\frac{q^{\alpha^2}}{(q;q)_\alpha} = \frac{(q^2,q^3;q^5)_\infty}{(q;q)_\infty},$$

$$\sum_{\alpha=0}^{\infty}\frac{q^{\alpha(\alpha+1)}}{(q;q)_\alpha} = \frac{(q,q^4;q^5)_\infty}{(q;q)_\infty},$$

where

$$(x_1, x_2, \cdots, x_r; q)_\infty = \prod_{j=0}^{\infty}(1 - x_1 q^j)\cdots(1 - x_r q^j).$$

It was MacMahon, who first analysed these combinatorially in [8] as follows.
First Rogers-Ramanujan Identity. The number of partitions of α into parts with minimal difference 2 equals the number of partitions of α into parts which are congruent to $\pm 1 (mod\ 5)$.
Second Rogers-Ramanujan Identity. The number of partitions of α with minimal part 2 and minimal difference 2 equals the number of partitions of α into parts which are congruent to $\pm 2 (mod\ 5)$.
There are several mathematicians who gave interesting proofs for RRI. Among them, Schur was apparently the one who gave the first bijective proof of these identities in [10] and these identities are used in the solutions of hard hexagon model [5]. Recently, these identities combinatorial interpretations in terms of n-color partitions have been studied by Sharma and second author of this paper in [11]. The n-color partitions are introduced by Agarwal and Andrews in [2] and this partition object is linked with several other combinatorial objects such as lattice paths, generalized Frobenius partitions, and plane partitions [1, 3, 5].

Definition 1.1. *An n–color partition is a partition where a part n can appear in n colors denoted by subscripts: $n_1, n_2, \cdots, n_n$. The parts are ordered first by size and then by color. For any integer $t \geq 0$, an $(n+t)$–color partition, is a partition in which a part 'n' can appear in $(n+t)$–colors as $n_1, n_2, \cdots, n_{n+t}$. Note that if $t > 0$ the partition can contain a part of size 0 but only one copy of zero '0_t' is allowed. The weighted difference of two parts m_x, n_y, $m \geq n$ in an $(n+t)$–color partition $(m_r)_{x_r} + (m_{r-1})_{x_{r-1}} + \cdots + (m_1)_{x_1}$ such that $(m_r)_{x_r} \geq (m_{r-1})_{x_{r-1}} \geq \cdots \geq (m_1)_{x_1}$, is $m - n - x - y$ and denoted by $\big((m_x - n_y)\big)$.*
Analogues to n–color partitions, Lovejoy and Mallet in [7] introduced n–color overpartitions, and Mallet in [9] extended it to $(n+t)$–color overpartitions.

Definition 1.2. *The n–color overpartition is an n–color partition in which the final occurrence of a part n_j may be overlined.*

Example 1.1. For $\alpha = 3$, there are 16 n–color overpartitions.

$$\begin{array}{lllll} 3_3 & \overline{3}_3 & 2_2 1_1 & 2_1 1_1 & 1_1 1_1 1_1 \\ 3_2 & \overline{3}_2 & \overline{2}_2 1_1 & \overline{2}_1 1_1 & 1_1 1_1 \overline{1}_1 \\ 3_1 & \overline{3}_1 & 2_2 \overline{1}_1 & 2_1 \overline{1}_1 & \\ & & \overline{2}_2 \overline{1}_1 & \overline{2}_1 \overline{1}_1 & \end{array}$$

In this paper, we provide the combinatorial interpretations of fourteen Rogers–Ramanujan type identities (RRTI) in terms of $(n+t)$–color overpartitions. These identities are listed in Table 1. The first identity appeared in [12] as Identity No. 29 and remaining identities are from [6] as Identity No. 104, 102, 29, 27, 25, 195, 45, 46, 11, 12, 37, 106, 40 respectively. The sum side of RRTI is the generator for the partitions enumerated by $M_l(\alpha)$ in terms of $(n+t)$–color overpartitions and the product side is enumerated by $N_l(\alpha)$ with ordinary partitions which lead to the two-way combinatorial interpretations and satisfies

$$h_l(q) = \sum_{\alpha=0}^{\infty} M_l(\alpha)q^{\alpha} = \sum_{\alpha=0}^{\infty} N_l(\alpha)q^{\alpha}, \quad 1 \leq l \leq 14 \tag{1.1}$$

for all non-negative integral values of α.

Table 1

Sr.no.	$\sum_{\alpha=0}^{\infty} M_l(\alpha)q^{\alpha}$	$\sum_{\alpha=0}^{\infty} N_l(\alpha)q^{\alpha}$
1.	$\sum_{\alpha=0}^{\infty} \frac{(-q;q^2)_{\alpha}q^{\alpha^2}}{(q;q)_{2\alpha}}$	$\frac{(-q;q^2)_{\infty}}{(q^2;q^2)_{\infty}}[-q^2,-q^4,q^6;q^6]_{\infty}$
2.	$\sum_{\alpha=0}^{\infty} \frac{(-q;q^2)_{\alpha}q^{\alpha(\alpha+1)}}{(q;q^2)_{\alpha+1}(q^2;q^2)_{\alpha}}$	$\frac{1}{(q;q)_{\infty}}[q^4,q^8,q^{12};q^{12}]_{\infty}$
3.	$\sum_{\alpha=0}^{\infty} \frac{(-q;q^2)_{\alpha}q^{\alpha(\alpha+2)}}{(q;q^2)_{\alpha+1}(q^2;q^2)_{\alpha}}$	$\frac{1}{(q;q)_{\infty}}[q^2,q^{10},q^{12};q^{12}]_{\infty}$
4.	$\sum_{\alpha=0}^{\infty} \frac{(-1)^{\alpha}(q;q^2)_{\alpha}q^{\alpha^2}}{(-q;q^2)_{\alpha}(q^4;q^4)_{\alpha}}$	$\frac{(-q;q^2)_{\infty}}{(q^2;q^2)_{\infty}}[q^5,-q^2,-q^3;q^5]_{\infty}$
5.	$\sum_{\alpha=0}^{\infty} \frac{(-1)^{\alpha}(q;q^2)_{\alpha}q^{\alpha(\alpha+2)}}{(-q;q^2)_{\alpha}(q^4;q^4)_{\alpha}}$	$\frac{(-q;q^2)_{\infty}}{(q^2;q^2)_{\infty}}[q^5,-q,-q^4;q^5]_{\infty}$
6.	$\sum_{\alpha=0}^{\infty} \frac{(-1)^{\alpha}(q;q^2)_{\alpha}q^{\alpha(\alpha+2)}}{(-q;q^2)_{\alpha+1}(q^4;q^4)_{\alpha}}$	$\frac{(-q;q^2)_{\infty}}{(q^2;q^2)_{\infty}}[q^5,-q^5,-q^5;q^5]_{\infty}$
7.	$\sum_{\alpha=1}^{\infty} \frac{(-q^2;q^2)_{\alpha-1}q^{\alpha^2}}{(q;q)_{2\alpha}}$	$\frac{[q^{16},q^2,q^{14};q^{16}]_{\infty}[q^{20},q^{12};q^{32}]_{\infty}}{(q;q)_{\infty}}$
8.	$\sum_{\alpha=0}^{\infty} \frac{(-1;q^2)_{\alpha}q^{\alpha(\alpha+1)}}{(q;q)_{2\alpha}}$	$\frac{(-q^2;q^2)_{\infty}}{(q^2;q^2)_{\infty}}[q^6,-q^3,-q^3;q^6]_{\infty}$
9.	$\sum_{\alpha=0}^{\infty} \frac{(-q^2;q^2)_{\alpha}q^{\alpha(\alpha+1)}}{(q;q)_{2\alpha+1}}$	$\frac{(-q^2,q^2)_{\infty}}{(q^2;q^2)_{\infty}}[q^6,-q,-q^5;q^6]_{\infty}$
10.	$\sum_{\alpha=0}^{\infty} \frac{(-1;q^4)_{\alpha}q^{\alpha^2}}{(q;q^2)_{\alpha}(q^4;q^4)_{\alpha}}$	$\frac{(-q;q^2)_{\infty}}{(q^2;q^2)_{\infty}}[-q^4,-q,q^3;-q^4]_{\infty}$
11.	$\sum_{\alpha=0}^{\infty} \frac{(-1;q^4)_{\alpha}q^{\alpha(\alpha+2)}}{(q;q^2)_{\alpha}(q^4;q^4)_{\alpha}}$	$\frac{(-q;q^2)_{\infty}}{(q^2;q^2)_{\infty}}[-q^4,q,-q^3;-q^4]_{\infty}$

12.	$\sum_{\alpha=0}^{\infty} \frac{(-1;q)_\alpha q^{\alpha^2}}{(q;q^2)_\alpha (q;q)_\alpha}$	$\frac{(-q;q)_\infty}{(q;q)_\infty}[q^6, q^3, q^3; q^6]_\infty$
13.	$\sum_{\alpha=1}^{\infty} \frac{(-q;q)_{\alpha-1} q^{\alpha^2}}{(q;q^2)_\alpha (q;q)_\alpha}$	$\frac{[q^{12}, -q^5, -q^7; q^{12}]_\infty}{(q;q)_\infty}$
14.	$\sum_{\alpha=0}^{\infty} \frac{(-q;q)_\alpha q^{\alpha(\alpha+1)}}{(q;q^2)_{\alpha+1} (q;q)_\alpha}$	$\frac{(-q;q)_\infty}{(q;q)_\infty}[q^6, q, q^5; q^6]_\infty$

2. Combinatorial interpretations using $(n+t)$–color overpartitions

Theorem 2.1. *For $\alpha \geq 0$, let $M_1(\alpha)$ represent the number of n–color overpartitions of α satisfying;*

(2.1.a) $m_k \equiv x_k \ (mod\ 2), \ \forall\ k$

(2.1.b) *In the overlined part the subscripts are always greater than* 1.

(2.1.c) $\big(((m_k)_{x_k} - (m_{k-1})_{x_{k-1}})\big) \geq 0$ *and* $\equiv 0 \ (mod\ 2)\ \forall\ k > 1.$

Let $N_1(\alpha)$ represent the count of partitions of α such that the odd parts are distinct, even parts are $\equiv \pm 2, \pm 4 (\text{mod } 12)$ and two copies of the parts which are $\equiv \pm 2 (\text{mod } 12)$ are used.

Example 2.1. For $\alpha = 8$, $M_1(8) = N_1(8) = 18$ the relevant n–color overpartitions of $M_1(8)$ are

$$\begin{array}{llllll} 8_8 & 8_2 & \overline{8}_4 & \overline{6}_2 2_2 & 7_5 1_1 & \overline{7}_3 1_1 \\ 8_6 & \overline{8}_8 & \overline{8}_2 & 6_2 \overline{2}_2 & \overline{7}_5 1_1 & 7_1 1_1 \\ 8_4 & \overline{8}_6 & 6_2 2_2 & \overline{6}_2 \overline{2}_2 & 7_3 1_1 & 5_1 3_1. \end{array}$$

And the partitions corresponding to $N_1(8)$ are

$$\begin{array}{llllll} 8 & 44 & 431 & 42_2 2_2 & 32_1 2_2 1 & 2_1 2_1 2_2 2_2 \\ 71 & 52_1 1 & 42_1 2_1 & 32_1 2_1 1 & 2_1 2_1 2_1 2_1 & 2_1 2_2 2_2 2_2 \\ 53 & 52_2 1 & 42_1 2_2 & 32_2 2_2 1 & 2_1 2_1 2_1 2_2 & 2_2 2_2 2_2 2_2. \end{array}$$

Theorem 2.2. *For $\alpha \geq 0$, let $M_2(\alpha)$ represent the number of $(n+1)$–color overpartitions satisfying* (2.1.b), (2.1.c) *and $x_1 = m_1 + 1$ with $(m_1)_{x_1}$ is not overlined.* Let $N_2(\alpha)$ represent the count of partitions of α in which the parts are $\equiv \pm 1$, ± 2, ± 3, ± 5, $6 (\text{mod } 12)$.

Theorem 2.3. *For $\alpha \geq 0$, let $M_3(\alpha)$ represent the number of $(n+2)$–color overpartitions satisfying* (2.1.a)–(2.1.c) *and $x_1 = m_1 + 2$ with $(m_1)_{x_1}$ is not overlined.* Let $N_3(\alpha)$ represent the count of partitions of α in which the parts are $\equiv \pm 1$, ± 3,

± 4, ± 5, 6(mod 12).

Theorem 2.4. *For $\alpha \geq 0$, let $M_4(\alpha)$ represent the number of n–color overpartitions satisfying the following conditions along with* (2.1.a) *and* (2.1.b)

(2.4.a) $m_1 - x_1 \equiv 0 \ (mod\ 4)$,

(2.4.b) $\big(((m_k)_{x_k} - (m_{k-1})_{x_{k-1}})\big) \geq 0$ *and* $\equiv 0 \ (mod\ 4)$. $\forall\ k > 1$

Let $N_4(\alpha) = \sum_{i=0}^{\alpha} X_4(\alpha-i)Y_4(i)$, where $X_4(\alpha)$ represent the count of partitions of α into distinct parts such that two copies of $5(mod\ 10)$ and one copies of $\pm 1(mod\ 10)$ are allowed and $Y_4(\alpha)$ represent the count of partitions of α into two copies of $\pm 2(mod\ 10)$ are allowed.

Theorem 2.5. *For $\alpha \geq 0$, let $M_5(\alpha)$ represent the number of n–color overpartitions satisfying* (2.1.a), (2.1.b), (2.4.b) *and $m_1 \geq 3$ with $m_1 - x_1 \equiv 2 \ (mod\ 4)$.*
Let $N_5(\alpha) = \sum_{i=0}^{\alpha} X_5(\alpha-i)Y_5(i)$, where $X_5(\alpha)$ represent the count of partitions of α in which the two copies of distinct parts $5(mod\ 10)$ and one copies of $\pm 3(mod\ 10)$ are allowed and $Y_5(\alpha)$ represent the count of partitions of α into the two copies of $\pm 4(mod\ 10)$ are allowed.

Theorem 2.6. *For $\alpha \geq 0$, let $M_6(\alpha)$ represent the number of $(n+1)$–color overpartitions satisfying* (2.1.b), (2.2.c) *and* (2.4.d).
Let $N_6(\alpha) = \sum_{i=0}^{\alpha} X_6(\alpha-i)Y_6(i)$, where $X_6(\alpha)$ represent the count of partitions of α in which the parts are $\equiv \pm 1, \pm 3(mod\ 10)$ and $Y_6(\alpha)$ represent the count of partitions of α in which the parts are $\equiv \pm 2, \pm 4(mod\ 10)$.

Theorem 2.7. *For $\alpha \geq 0$, let $M_7(\alpha)$ represent the number of n–color overpartitions satisfying the following conditions along with* (2.1.a)

(2.7.a) *the occurrence of m_1 is not overlined,*

(2.7.b) $\big(((m_k)_{x_k} - (m_{k-1})_{x_{k-1}})\big) \geq 0$ *and* $\equiv 0 \ (mod\ 2)$ $\forall\ k > 1$. *For* $\big(((m_k)_{x_k} - (m_{k-1})_{x_{k-1}})\big) = 0$, *$m_k$ is not overlined.*

Let $N_7(\nu)$ represent the count of partitions of α in which the parts are $\equiv \pm 1$, ± 3, ± 5, ± 6(mod 32).

Theorem 2.8. *For $\alpha \geq 0$, let $M_8(\alpha)$ represent the number of n–color overpartitions satisfying* (2.1.a) *and*

(2.8.a) *for $m_1 = x_1$, the occurrence of m_1 is not overlined,*

(2.8.b) $\big(((m_k)_{x_k} - (m_{k-1})_{x_{k-1}})\big) \geq -2$ *and* $\equiv 0 \ (mod\ 2)$ $\forall\ k > 1$. *For* $\big(((m_k)_{x_k} - (m_{k-1})_{x_{k-1}})\big) = -2$, *$m_k$ is not overlined.*

Let $N_8(\alpha) = \sum_{i=0}^{\alpha} X_8(\alpha - i)Y_8(i)$, where $X_8(\alpha)$ represent the count of partitions of α in which the parts are $\equiv \pm 2, \pm 3, \pm 4(mod\ 12)$ and $Y_8(\alpha)$ represent the count of partitions of α into distinct parts $\equiv \pm 2, \pm 3, \pm 4(mod\ 12)$.

Theorem 2.9. *For $\alpha \geq 0$, let $M_9(\alpha)$ represent the number of $(n+1)$–color overpartitions satisfying* (2.7.b) *and $x_1 = m_1 + 1$ with $(m_1)_{x_1}$ is not overlined.*
Let $N_9(\alpha) = \sum_{i=0}^{\alpha} X_9(\alpha - i)Y_9(i)$, where $X_9(\alpha)$ represent the count of partitions of α in which the parts are $\equiv 2, 4(mod\ 6)$ and $Y_9(\alpha)$ represent the count of partitions of α into distinct parts are $\equiv 0, \pm 1, \pm 2(mod\ 6)$.

Theorem 2.10. *For $\alpha \geq 0$, let $M_{10}(\alpha)$ represent the number of n–color overpartitions satisfying*

(2.10.a) $m_1 \equiv x_1\ (mod\ 4)$,

(2.10.b) *the occurrence of m_1 is not overlined,*

(2.10.c) $\big(((m_k)_{x_k} - (m_{k-1})_{x_{k-1}})\big) \geq 0$ *and* $\equiv 0\ (mod\ 4)\ \forall\ k > 1$. *For* $\big(((m_k)_{x_k} - (m_{k-1})_{x_{k-1}})\big) = 0$ *then m_k is not overlined.*

Let $N_{10}(\alpha) = \sum_{i=0}^{\alpha} X_{10}(\alpha - i)Y_{10}(i)$, where $X_{10}(\alpha)$ represent the count of partitions of α in which the parts are $\equiv \pm 1, 4(mod\ 8)$ and $Y_{10}(\alpha)$ represent the count of partitions of α in which the parts are $\equiv 0, \pm 1, 4(mod\ 8)$.

Theorem 2.11. *For $\alpha \geq 0$, let $M_{11}(\alpha)$ represent the number of n–color overpartitions satisfying* (2.10.a)–(2.10.c) *and $m_1 > 2$ with $m_1 - x_1 \equiv 2\ (mod\ 4)$.*
Let $N_{11}(\alpha) = \sum_{i=0}^{\alpha} X_{11}(\alpha - i)Y_{11}(i)$, where $X_{11}(\alpha)$ represent the count of partitions of α in which the parts are $\equiv \pm 3, 4(mod\ 8)$ and $Y_{11}(\alpha)$ represent the count of partitions of α into distinct parts are $\equiv \pm 3, 4(mod\ 8)$.

Theorem 2.12. *For $\alpha \geq 0$, let $M_{12}(\alpha)$ represent the number of n–color overpartitions satisfying*

(2.12.a) *the occurrence of m_1 is not overlined,*

(2.12.b) $\big(((m_k)_{x_k} - (m_{k-1})_{x_{k-1}})\big) \geq 0\ \forall\ k > 1$ *if* $\big(((m_k)_{x_k} - (m_{k-1})_{x_{k-1}})\big) = 0$ *then m_k is not overlined.*

Let $N_{12}(\alpha) = \sum_{i=0}^{\alpha} X_{12}(\alpha - i)Y_{12}(i)$, where $X_{12}(\alpha)$ represent the count of partitions of α in which the parts are $\equiv \pm 1, \pm 2(mod\ 6)$ and $Y_{12}(\alpha)$ represent the count of partitions of α into distinct parts are $\equiv \pm 1, \pm 2(mod\ 6)$.

Theorem 2.13. *For $\alpha \geq 0$, let $M_{13}(\alpha)$ represent the number of n–color overpartitions satisfying* (2.12.a) *and* (2.12.b)

Let $N_{13}(\alpha) = \sum_{i=0}^{\alpha} X_{13}(\alpha-i)Y_{13}(i)$, where $X_{13}(\alpha)$ represent the count of partitions of α in which the parts are $\equiv \pm1, \pm2, \pm3, \pm4, \pm5, 6(mod\ 12)$ and $Y_{13}(\alpha)$ represent the count of partitions of α in which the parts are $\equiv \pm5(mod\ 12)$.

Theorem 2.14. *For $\alpha \geq 0$, let $M_{14}(\alpha)$ represent the number of $(n+1)$–color overpartitions satisfying* (2.12.b) *and $x_1 = m_1 + 1$ with $(m_1)_{x_1}$ is not overlined.* Let $N_{14}(\alpha) = \sum_{i=0}^{\alpha} X_{14}(\alpha - i)Y_{14}(i)$, where $X_{14}(\alpha)$ represent the count of partitions of α in which the parts are $\equiv \pm2, 3(mod\ 6)$ and $Y_{14}(\alpha)$ represent the count of partitions of α into distinct parts.

3. Proofs

Before constructing the proofs of Theorem 2.1-2.14, let us consider

$$h_l(w,q) = \sum_{\alpha=0}^{\infty}\sum_{r=0}^{\infty} M_l(r,\alpha)w^r q^\alpha, \tag{3.1}$$

for $l = 1$–14 and $M_l(\alpha)$ denote the partitions with some conditions in any number of parts and $M_l(r,\alpha)$ will denote the partitions with the same conditions into r parts.

Proof of Theorem 2.1. We split the partitions enumerated by $M_1(r,\alpha)$ into four classes, viz.,

(i) those that have 1_1 as a part,

(ii) those that have a part like 2_2 or $\overline{2}_2$,

(iii) those that don't have a part like $(m_k)_{m_k}$ or $(\overline{m}_k)$, and

(iv) those that have a part of the form $(m_k)_{m_k}$ or $(\overline{m}_k)$ for $k \geq 3$.

In the first class, partitions contain 1_1 as the least part. Deleting 1_1 and subtracting 2 from all the remaining parts without any change in the subscripts, it will not disturb the inequalities between the parts and transformed partitions will be of the type enumerated by $M_1(r-1, \alpha - 2r + 1)$. We first delete 2_2 or $\overline{2}_2$, as the case may be, and then subtract 4 from each of the remaining parts. The transformed partition will be of the type enumerated by $M_1(r-1, \alpha-4r+2)$. Third class contains the partitions that do not have $(x_k)_{x_k}$ or $(\overline{x_k})_{x_k}$ as the least part. Transform the partition by subtracting 2 from subsequent parts without disturbing subscripts. The transformed partitions are enumerated by $M_1(r, \alpha - 2r)$. In the last class, the partition involves $(x_k)_{x_k}$ $(x_k \geq 2)$ or $(\overline{x_k})_{x_k}$ $(x_k \geq 3)$ as a part. Transformed by replacing $(x_k)_{x_k}$ by $(x_k - 1)_{x_k-1}$ or $(\overline{x_k})_{x_k}$ by $(\overline{x_k} - 1)_{x_k-1}$ and subtract 2 from the remaining parts without disturbing the subscripts. This will result in partitions enumerated by $M_1(r, \alpha-2r+1)$. It should be noted here that we are obtaining those partitions of $\alpha-2r+1$ which have a part of the type $(x_k)_{x_k}$ and $(\overline{x_k})_{x_k}$ so the number of partitions in this class enumerated by $M_1(r, \alpha-2r+1) - M_1(r, \alpha-4r+1)$. Thus, the transformed partition are enumerated by $M_1(r, \alpha-2r+1) - M_1(r, \alpha-4r+1)$. Hence we get the following recurrence formula for $M_1(r, \alpha)$

$$\begin{aligned} M_1(r,\alpha) =& M_1(r-1,\alpha-2r+1) + M_1(r-1,\alpha-4r+2) + M_1(r,\alpha-2r) \\ &+ M_1(r,\alpha-2r+1) - M_1(r,\alpha-4r+1), \end{aligned} \tag{3.2}$$

where $M_1(0,0) = 1$ and $M_1(r,\alpha) = 0$ for $\alpha < 0$.
For $|q| < 1$ and $|w| < |q|^{-1}$, let $h_1(w,q)$ be defined by

$$h_1(w,q) = \sum_{\alpha=0}^{\infty}\sum_{r=0}^{\infty} M_1(r,\alpha) w^r q^\alpha. \tag{3.3}$$

Substitute $M_1(r,\alpha)$ from (3.2) in (3.3), we get q–functional equation

$$\begin{aligned} h_1(w,q) =& wqh_1(wq^2,q) + wq^2h_1(wq^4,q) + h_1(wq^2,q) \\ &+ q^{-1}h_1(wq^2,q) - q^{-1}h_1(wq^4,q). \end{aligned} \tag{3.4}$$

Setting

$$h_1(w,q) = \sum_{\alpha=0}^{\infty} z_1(\alpha,q) w^\alpha. \tag{3.5}$$

Using (3.4) in (3.5) and then examining the coefficients of w^α, we get

$$z_1(\alpha,q) = \frac{q^{2\alpha-1}(1+q^{2\alpha-1})}{(1-q^{2\alpha})(1-q^{2\alpha-1})} z_1(\alpha-1,q). \tag{3.6}$$

Iterating (3.6) α times and note that $z_1(0,q) = 1$, we find that

$$z_1(\alpha,q) = \frac{(-q;q^2)_\alpha q^{\alpha^2}}{(q^2;q^2)_\alpha (q;q^2)_\alpha}. \tag{3.7}$$

Therefore,

$$\begin{aligned} h_1(w,q) =& \sum_{\alpha=0}^{\infty} \frac{(-q;q^2)_\alpha q^{\alpha^2}}{(q^2;q^2)_\alpha (q;q^2)_\alpha} w^\alpha \\ =& M_1(w,q), \end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\sum_{\alpha=0}^{\infty} M_1(\alpha)q^{\alpha} &= \sum_{\alpha=0}^{\infty}\left(\sum_{r=0}^{\infty} M_1(r,\alpha)\right)q^{\alpha}\\ &= h_1(1,q)\\ &= h_1(q).\end{aligned}$$

Proof of Theorem 2.2. $M_2(r+1,\alpha)$ represent the color overpartitions enumerated by $M_2(\alpha)$ of α into $r+1$ columns. The partitions enumerated by $M_2(r+1,\alpha)$ split into two classes, first containing the partitions with 0_1 and second containing the partitions with $(x_k)_{x_k}, x_k > 0$. The partitions in the first class are enumerated by $M_1(r,\alpha-r)$ and in the second class are enumerated by $M_2(r+1,\alpha-2r-1)$. Thus the recurrence relation becomes

$$\begin{aligned}M_2(r+1,\alpha) &= M_1(r,\alpha-r) + M_2(r+1,\alpha-2r-1)\\ \sum_{r+1=0}^{\infty} M_2(r+1,\alpha)w^{r+1}q^{\alpha} &= \sum_{r+1=0}^{\infty} M_1(r,\alpha-r)w^{r+1}q^{\alpha}\\ &\quad+\sum_{r+1=0}^{\infty} M_2(r+1,\alpha-2r-1)w^{r+1}q^{\alpha}\\ \sum_{r=0}^{\infty} M_2(r,\alpha)w^{r}q^{\alpha} &= w\sum_{r=0}^{\infty} M_1(r-1,\alpha-r+1)(wq)^{r-1}q^{(\alpha-r+1)}\\ &\quad+q^{-1}\sum_{r=0}^{\infty} M_2(r,\alpha-2r+1)(wq^2)^{r}q^{\alpha-2r+1}.\end{aligned}$$

We follow the same technique as interpreting $M_1(\alpha)$ and the corresponding q–functional equation becomes

$$h_2(w,q) = wh_2(wq,q) + q^{-1}h_1(wq^2,q). \tag{3.9}$$

Sketch proofs of Theorem 2.3–2.6

- The proof of Theorem 2.3 can be supplied by the reader on lines of Theorem 2.2
- Splitting the partitions enumerated by $M_4(r,\alpha)$ into four classes:
  Class (i): that contains 1_1 as the least part,
  Class (ii): that contains $\overline{2}_2$ as the least part,

Class (iii): that does not contains $(x_k)_{x_k}$ or $(\overline{x}_k)_{x_k}$ as the least part,
Class (iv): that contains $(x_k)_{x_k}$ $(x \geq 2)$ and $(\overline{x}_k)_{x_k}$ $(x \geq 3)$ as a least part.
Proceeding as Theorem 2.1, one can easily obtain the following q–functional equation,

$$\begin{aligned} h_4(w,q) &= wqh_4(wq^2,q) + wq^2h_4(wq^4,q) + h_4(wq^4,q) \\ &\quad +q^{-1}h_4(wq^2,q) - q^{-1}h_4(wq^6,q). \end{aligned}$$

- For the proof of Theorem 2.5 one can proceed as Theorem 2.2 and get following q–functional equation,

$$h_5(w,q) = wh_5(wq,q) + q^{-1}h_4(wq^2,q).$$

- Proceeding as Theorem 2.3 one can easily obtain the proof of Theorem 2.6 and corresponding q–functional equation as follows,

$$h_6(w,q) = wh_6(wq,q) + q^{-1}h_4(wq^2,q).$$

Proofs of Theorem 2.7–Theorem 2.14

To interpret $M_l(\alpha)$, $k = 7$–14 in terms of n–color overpartition we have to consider some more q–series listed in Table 2.

Table 2

Sr no.	$\sum_{\alpha=0}^{\infty} W_j(\alpha)q^{\alpha}$
1.	$\sum_{\alpha=0}^{\infty} \frac{(-q^2;q^2)_{\alpha}q^{\alpha^2}}{(q;q)_{2\alpha}}$
2.	$\sum_{\alpha=0}^{\infty} \frac{(-q^2;q^2)_{\alpha}q^{\alpha(\alpha+1)}}{(q;q)_{2\alpha}}$
3.	$\sum_{\alpha=0}^{\infty} \frac{(-q^4;q^4)_{\alpha}q^{\alpha^2}}{(q^4;q^4)_{\alpha}(q;q^2)_{\alpha}}$
4.	$\sum_{\alpha=0}^{\infty} \frac{(-q^4;q^4)_{\alpha}q^{\alpha(\alpha+2)}}{(q;q^2)_{\alpha}(q^4;q^4)_{\alpha}}$
5.	$\sum_{\alpha=0}^{\infty} \frac{(-q;q)_{\alpha}q^{\alpha^2}}{(q;q^2)_{\alpha}(q;q)_{\alpha}}$

Firstly, we enumerate $W_j(\alpha)$, $1 \leq j \leq 5$ in terms of n–color overpartitions. Let $W_j(r,\alpha,)$ denote the partitions enumerated by $W_j(\alpha)$ into r parts, and so we let

$$g_j(w,q) = \sum_{\alpha=0}^{\infty}\sum_{r=0}^{\infty} W_j(r,\alpha)w^r q^{\alpha}. \tag{3.10}$$

And noting that $h_l(q)$, $l = 8, 10$–12 can be rewritten in terms of $\hat{h}_l(q)$ where,

$$\begin{aligned} h_8(q) =& \sum_{\alpha=0}^{\infty} M_8(\alpha) q^{\alpha} \\ =& \sum_{\alpha=0}^{\infty} \frac{(-1;q^2)_{\alpha} q^{\alpha(\alpha+1)}}{(q;q)_{2\alpha}} \\ =& 1 + 2\sum_{\alpha=1}^{\infty} \frac{(-q^2;q^2)_{\alpha-1} q^{\alpha(\alpha+1)}}{(q;q)_{2\alpha}} \\ =& 1 + 2\sum_{\alpha=1}^{\infty} \hat{M}_8(\alpha) q^{\alpha} \\ =& 1 + 2\hat{h}_8(q), \qquad \text{and} \qquad \hat{h}_l(q) = \sum_{\alpha=1}^{\infty} \hat{M}_l(\alpha) q^{\alpha}. \end{aligned} \tag{3.11}$$

For the interpretations of $\hat{M}_l(\alpha)$, $l = 8, 10$–12 suppose

$$\hat{h}_l(w, q) = \sum_{\alpha=0}^{\infty} \sum_{r=0}^{\infty} \hat{M}_l(r, \alpha) w^r q^{\alpha}. \tag{3.12}$$

Lemma 3.1. *For $\alpha \geq 0$, let $W_1(\alpha)$ represent the number of n–color overpartitions of α such that*

(2.3.1.a) $m_k \equiv x_k \ (mod\ 2)$, $\forall\ k$

(2.3.1.b) *if $m_1 = x_1$ then the occurrence of m_1 is not overlined,*

(2.3.1.c) $\big(((m_k)_{x_k} - (m_{k-1})_{x_{k-1}})\big) \geq 0$ *and* $\equiv 0\ (mod\ 2)\ \forall\ k$ *if* $\big(((m_k)_{x_k} - (m_{k-1})_{x_{k-1}})\big)$ $=$ *then m_k is not overlined.*

Proof. Let $W_1(\alpha)$ denote the number of n–color overpartition of α enumerated by $W_1(\alpha)$ into r parts. Divide the partitions into four classes. First class has partitions that do not involve $(x_k)_{x_k}$ or $(\overline{x_k})_{x_k-2}$ as a part. Subtracting 2 from all the parts, we get transformed partition enumerated by $W_1(r, \alpha - 2r)$. The second class has the partitions that involve 1_1 as a part. Deleting 1_1 part and then subtracting 2 from all the remaining parts, corresponding transformed partition enumerated by $W_1(r-1, \alpha - 2r + 1)$. The next class has the partitions which involve $\overline{3}_1$ as a part. Deleting $\overline{3}_1$ and subtracting 4 from the remaining parts. The transformed partition enumerated by $W_1(r-1, \alpha - 4r + 1)$. The last class has the partitions which involve

$(x_k)_{x_k}(x_k \geq 2)$ and $(\overline{x_k})_{x_k-2}(x_k \geq 4)$ as a part. Replace $(x_k)_{x_k}$ by $(x_k-1)_{x_k-1}$ and $(\overline{x_k})_{x_k-2}$ by $(\overline{x_k-1})_{x_k-3}$ and subtract 2 from the remaining parts. This will result in partitions enumerated by $W_1(r, \alpha-2r+1)$. It should be noted here that we are obtaining only those partitions of $\alpha-2r+1$ which involve a part of the type $(x_k)_{x_k}$ and $(\overline{x_k})_{x_k-2}$ and any other repeated part is added correspondingly, so the number of partitions in last class is enumerated by $W_1(r, \alpha-2r+1)-W_1(r, \alpha-4r+1)$. The transformed classes are reversible. There is one to one correspondence between the classes enumerated by $W_1(r, \alpha)$ and those by

$$\begin{aligned} W_1(r,\alpha) =& W_1(r,\alpha-2r)+W_1(r-1,\alpha-2r+1)+W_1(r-1,\alpha-4r+1) \\ &+W_1(r,\alpha-2r+1)-W_1(r,\alpha-4r+1), \end{aligned} \tag{3.13}$$

where $W_1(0,0)=1$ and $W_1(r,\alpha)=0$ for $\alpha<0$. For $|q|<1$ and $|w|<|q|^{-1}$ and let $g_1(w,q)$ be defined by

$$g_1(w,q)=\sum_{\alpha=0}^{\infty}\sum_{r=0}^{\infty} W_1(r,\alpha)w^r q^{\alpha} \tag{3.14}$$

Now substituting (3.14) in (3.13) and we get the q-functional equation as,

$$\begin{aligned} g_1(w,q) &= g_1(wq^2,q)+wqg_1(wq^2,q)+wq^3g_1(wq^4,q) \\ &\quad +q^{-1}g_1(wq^2,q)-q^{-1}g_1(wq^4,q). \end{aligned}$$

By proceeding in the same manner as in the proof of Theorem 2.1, we can get the desired result.

Sketch proof of Theorem 2.7

Now, we shall prove

$$\sum_{\alpha=0}^{\infty} M_7(\alpha)q^{\alpha}=\sum_{\alpha=0}^{\infty}\frac{(-q^2;q^2)_{\alpha-1}q^{\alpha^2}}{(q;q)_{2\alpha}}.$$

Split the partitions enumerated by $M_7(r,\alpha)$ into four classes. In the first class, partitions contain 1_1 as their least part. Remove 1_1 and subtract 2 from all the remaining parts. Transformed partitions are enumerated by $W_1(r-1,\alpha-2r+1)$. Second class contains the partitions which have 3_1. By removing 3_1 and subtracting 4 from all the remaining parts, we get the partition enumerated by $W_1(r-1,\alpha-4r+1)$. The partition in the third class contains the partitions which do not have $(x_k)_{x_k}$ and $(x_k)_{x_k-2}$ as the least part. Subtracting 4 from all the parts and we get the partitions enumerated by $M_7(r,\alpha-4r)$. The last class contains the partitions of the form $(x_k)_{x_k}$ and $(x_k)_{x_k-2}$. Replace $(x_k)_{x_k}$ into $(x_k-1)_{x_k-1}$ or $(x_k)_{x_k-2}$ into $(x_k-1)_{x_k-3}$ and then subtract 2 from remaining parts and can get enumerated by $M_7(r,\alpha-2r+1)-M_1(r,\alpha-6r+1)$. Thus, the recurrence relation for $M_7(r,\alpha)$ is given by

$$\begin{aligned} M_7(r,\alpha) &= W_1(r-1,\alpha-2r+1)+W_1(r-1,\alpha-4r+1) \\ &\quad +M_7(r,\alpha-4r)+M_7(r,\alpha-2r+1)-M_7(r,\alpha-6r+1), \end{aligned}$$

and the corresponding q-functional equation is

$$\begin{aligned} h_7(w,q) &= wqg_1(wq^2,q)+wq^3g_1(wq^4,q)+h_7(wq^4,q) \\ &\quad +q^{-1}h_7(wq^2,q)-q^{-1}h_7(wq^6,q). \end{aligned}$$

Proceeding in the same manner as in the proof of Lemma 3.1, we get the desired result.

Now, we give the enumeration of $W_j(\alpha)$ $2\le j\le 5$ in following Lemma 3.2–3.5 respectively. Then, we give only necessary outlines of the proofs. Using these Lemmas, one can elaborates the proofs of Theorem 2.8–2.14.

Lemma 3.2. *For $\alpha\ge 0$, let $W_2(\alpha)$ represent the number of n–color overpartitions satisfying* (2.1.a), (2.8.b)*, $m_1,x_1\ge 2$, and if $m_1=x_1$ then the occurrence of m_1 is not overlined.*

Lemma 3.3. *For $\alpha\ge 0$, let $W_3(\alpha)$ represent the number of n–color overpartitions satisfying* (2.10.a), (2.10.c) *and if $m_1=x_1$ then the occurrence of m_1 is not overlined.*

Lemma 3.4. *For $\alpha\ge 0$, let $W_4(\alpha)$ represent the number of n–color overpartitions satisfying* (2.10.a), (2.10.c), *$m_1\ge 3$, $m_1-x_1\equiv 2 \pmod 4$ and if $x_1=m_1-2$ then the occurrence of m_1 is not overlined.*

Lemma 3.5. *For $\alpha\ge 0$, let $W_5(\alpha)$ represent the number of n–color overpartitions satisfying* (2.12.b) *and if $m_1=x_1$ then the occurrence of m_1 is not overlined.*

Sketch proofs of Lemma 3.2–3.5 Since the proofs of Lemma 3.2–3.5 are similar to that of Lemma 3.1, we omit the details and give only the q–functional equations which is used in each case.

$$
\begin{aligned}
g_2(w,q) &= wq^2g_2(wq^2,q)+wq^4g_2(wq^4,q)+g_2(wq^2,q)\\
&\quad +q^{-1}g_2(wq^2,q)-q^{-1}g_2(wq^4,q).\\
g_3(w,q) &= wqg_3(wq^2,q)+wq^5g_3(wq^6,q)+g_3(wq^4,q)\\
&\quad +q^{-1}g_3(wq^2,q)-q^{-1}g_3(wq^6,q).\\
g_4(w,q) &= wq^3g_4(wq^2,q)+wq^7g_4(wq^6,q)+g_4(wq^4,q)\\
&\quad +q^{-1}g_4(wq^2,q)-q^{-1}g_4(wq^6,q).\\
g_5(w,q) &= wqg_5(wq^2,q)+wq^2g_5(wq^3,q)+g_5(wq,q)\\
&\quad +q^{-1}g_5(wq^2,q)-q^{-1}g_5(wq^3,q).
\end{aligned}
$$

The proofs of the Theorem 2.7–2.14 are also established by splitting partitions enumerated by $M_l(\alpha)$ into classes. Also, it is evident from the proof of Theorem 2.7 that classes depend on the least part of the partitions. One can easily establish proofs so only q–functional equations are provided.

Table 3

Enumerator	class 1	class 2	class 3 do not contains	class 4 contains
$M_8(\alpha)$	2_2	4_2	$(x_k)_{x_k}$ or $(x_k)_{x_k-2}$	$(x_k)_{x_k}$ $(x_k \geq 3)$ or $(x_k)_{x_k-2}$ $(x_k \geq 5)$
$M_9(\alpha)$	1_1	$(x_k)_{x_k}$		
$M_{10}(\alpha)$	1_1	5_1	$(x_k)_{x_k}$ or $(x_k)_{x_k-4}$	$(x_k)_{x_k}$ $(x_k \geq 2)$ or $(x_k)_{x_k-4}$ $(x_k \geq 6)$
$M_{11}(\alpha)$	3_3	7_3	$(x_k)_{x_k}$ or $(x_k)_{x_k-4}$	$(x_k)_{x_k}$ $(x_k \geq 4)$ or $(x_k)_{x_k-4}$ $(x_k \geq 8)$
$M_{12}(\alpha)$	1_1	2_1	$(x_k)_{x_k}$ or $(x_k)_{x_k-1}$	$(x_k)_{x_k}$ $(x_k \geq 2)$ or $(x_k)_{x_k-1}$ $(x_k \geq 3)$
$M_{14}(\alpha)$	1_1	$(x_k)_{x_k}$		

Remark 3.1. *The classes for $M_{13}(\alpha)$ is same as $M_{12}(\alpha)$.*

$$\begin{aligned}
\hat{h_8}(w,q) &= \hat{h_8}(wq^4,q) + wq^2g_2(wq^2,q) + wq^4g_2(wq^4,q) \\
&\quad + q^{-1}\hat{h_8}(wq^2,q) - q^{-1}\hat{h_8}(wq^6,q). \\
h_9(w,q) &= wqg_1(wq^2,q) + q^{-1}h_9(wq^2,q). \\
\hat{h_{10}}(w,q) &= \hat{h_{10}}(wq^8,q) + wqg_3(wq^2,q) + wq^5g_3(wq^6,q) \\
&\quad + q^{-1}\hat{h_{10}}(wq^2,q) - q^{-1}\hat{h_{10}}(wq^{10},q). \\
\hat{h_{11}}(w,q) &= \hat{h_{11}}(wq^8,q) + wq^3g_4(wq^2,q) + wq^7g_4(wq^6,q) \\
&\quad + q^{-1}\hat{h_{11}}(wq^2,q) - q^{-1}\hat{h_{11}}(wq^{10},q). \\
\hat{h_{12}}(w,q) &= \hat{h_{12}}(wq^2,q) + wqg_5(wq^2,q) + wq^2g_5(wq^3,q) \\
&\quad + q^{-1}\hat{h_{12}}(wq^2,q) - q^{-1}\hat{h_{12}}(wq^4,q). \\
h_{13}(w,q) &= \hat{h_{12}}(w,q). \\
h_{14}(w,q) &= wqg_5(wq^2,q) + q^{-1}h_{14}(wq^2,q).
\end{aligned}$$

Remark 3.2. *To calculate the q-functional equations of $h_9(q)$ and $h_{14}(q)$, follow a displacement of $\alpha \mapsto \alpha+1$ due to the factor $(1+q^{\alpha+1})$.*

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