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STUDY OF STABILITY CRITERION FOR SOME ITERATIVE METHODS FOR SOLVING NON LINEAR EQUATION

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Abstract: Research is concerned with the study of stability criterion for some iterative methods for solving non linear equations. We introduce the comparative study for stability of iterative methods for solving nonlinear equations along with several example. Some simple but powerful criterion for local stability of fixed points are established. It is also established that the iterative method New Two Step - 2 (NTS-2) and New Three Step Method - 2 (NTSM-2) comparably more hyperbolic asymptotically stable and provides better result.

Keywords and Phrases: Stability, Two - Step, Three - Step, nonlinear equations, iterative methods.

2020 Mathematics Subject Classification: 65H05.

1. Introduction

Numerical analysis is the area of mathematics and computer sciences that creates, analyzes and numerically the problems of continuous mathematics. Such problems originate generally from real - world applications of algebra, geometry and calculus and they involves variable which very continuously : these problems occur throughout the natural sciences, social sciences, engineering, medicine and business. The concepts of stability can be understood by analyzing the numerical solutions of the nonlinear equation $f(x) = 0$ are being developed using stability of fixed points, see [14, 11, 15, 3, 1, 2, 8] and the references there in.

It was rather surprising to learn that a complete stability theory for non - hyperbolic fixed points of one - dimensional maps is not known even to specialists in dynamical systems and difference equations. This is despite the fact that the study of the dynamics of one - dimensional maps is central to many fields including discrete dynamics of one dynamical systems [12, 14, 3, 13]. Our main objective here is to present a complete theory for the stability of non - hyperbolic fixed points of one - dimensional continuous maps f. [11, 15, 3, 6, 7, 8, 9]

2. Definition

Criteria for Stability:

In this section, we will established some simple but powerful criteria for local stability of fixed point. Fixed (equilibrium) points may be divided into two type : hyperbolic and non-hyperbolic. A fixed point α of a map f is said to be hyperbolic if $|f'(\alpha)| \neq 1$ otherwise it is non-hyperbolic. We will treat the stability of each-type-separately [5].

Hyperbolic Fixed Points:

Let α be a hyperbolic fixed point of a map f. Where f is continuously differentiable at α. The following statements then hold true.
1: If $|f'(\alpha)| < 1$, then α is asymptotically stable.
2: If $|f'(\alpha)| > 1$, then α is unstable.

Non-Hyperbolic Fixed Points:

The stability criteria for non-hyperbolic fixed points are more involved. They will be summarized in the next two results the first of which treats the case when $f'(\alpha) = 1$ and the second for $f'(\alpha) = -1$.

Theorem 2.1. *Let α be a fixed point of a map f such that $f'(\alpha) = 1$ if $f'''(\alpha) = 1$ and continuous. then the following statements hold:*
1: if $f''(\alpha) \neq 0$ then α is unstable.
2: $f''(\alpha) = 0$ and $f'''(\alpha) > 0$ then α is unstable.
3: if $f''(\alpha) = 0$ and $f'''(\alpha) < 0$ then α is asymptotically stable.

The Schwarzian derivative:

Sf of a function f is defined by [6, 7, 8]

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2.$$

Theorem 2.2. *Let α be a fixed point of a map f such that $f'(\alpha) = -1$ if $f'''(\alpha)$ is continuous, then the following statements hold:*
1: if $Sf(\alpha) < 0$, then α is asymptotically stable.
2: if $Sf(\alpha) > 0$, then α is unstable.

3. Criteria for stability method

In this section, we will established some simple but powerful criteria for local stability of fixed point.

Recently, Najmuddin Ahmad and Vimal Pratap Singh [2] have obtained some new two step iterative methods for solving non linear equations using Steffensen's method (NTS – 2)

NTS-2: For a given initial choice x_0, compute approximate solution x_{n+1} , by the iterative schemes,

$$y_n = x_n - \frac{[f(x_n)]^2}{f(x_n + f(x_n)) - f(x_n)}$$

$$x_{n+1} = x_n - \frac{2f(x_n)}{[f'(y_n) + f'(x_n)]}$$

$$n = 0, 1, 2, 3, \dots\dots\dots$$

Theorem 3.1. *For a given x_0 is our initial guess of the root α, NTS - 2 is the form*

$$y_N(x) = x - \frac{[f(x)]^2}{f(x + f(x)) - f(x)}$$

$$f_N(x) = x - \frac{2f(x)}{[f'(y_N(x)) + f'(x)]}$$

where f_N is called NTS - 2 function stability of the fixed point $x^* = \alpha$ of f_N.
$|f'_N(\alpha)| = 0 < 1$
since $f(\alpha) = 0$
then α is hyperbolic asymptotically stable.

Recently, Najmuddin Ahmad and Vimal Pratap Singh [1] have obtained some new three step iterative methods for solving non linear equations using Steffensen's and Halley method (NTSM - 2).

NTSM - 2: For a given initial choice x_0, find the approximate solution x_{n+1} by the iterative schemes,

$$a_n = x_n - \frac{[f(x_n)]^2}{f(x_n + f(x_n)) - f(x_n)}$$

$$b_n = a_n - \frac{2f(a_n)f'(a_n)}{2f^2(a_n) - f(a_n)f''(a_n)}$$

$$x_{n+1} = x_n - \frac{2f(x_n)}{[f'(b_n) + f'(x_n)]}$$
$$n = 0, 1, 2, 3, ...$$

Theorem 3.2. *For a given x_0 is our initial guess of the root α, NTSM - 2 is the form*

$$a_N(x) = x - \frac{[f(x)]^2}{f(x + f(x)) - f(x)}$$
$$b_N(x) = a_N(x) - \frac{2f(a_N(x))f'(a_N(x))}{2f^2(a_N(x)) - f(a_N(x))f''(a_N(x))}$$
$$f_N(x) = x - \frac{2f(x)}{[f'(b_N(x)) + f'(x)]}$$

where f_N is called NTS - 2 function stability of the fixed point $x^* = \alpha$ of f_N.
$|f'_N(\alpha)| = 0 < 1$
since $f(\alpha) = 0$
then α is hyperbolic asymptotically stable.

PCNH (Predictor - Corrector Newton - Halley method) [4]: For a given initial choice x_0, find the approximate solution x_{n+1} by the iterative schemes,

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$y_n = w_n - \frac{2f(w_n)f'(w_n)}{2f'^2(w_n) - f(w_n)f''(w_n)},$$
$$x_{n+1} = y_n - 2\frac{f(y_n)}{f'(y_n)} - \frac{f^2(y_n)f''(y_n)}{2f'(y_n)^3}$$
$$n = 0, 1, 2, 3, ...$$

Theorem 3.3. *For a given x_0 is our initial guess of the root α, PCNH is the form*

$$w_N(x) = x - \frac{f(x)}{f'(x)},$$

$$y_N(x) = w_N(x) - \frac{2f(w_N(x))f'(w_N(x))}{2f'^2(w_N(x)) - f(w_N(x))f''(w_N(x))},$$
$$f_N(x) = y_N(x) - 2\frac{f(y_N(x))}{f'(y_N(x))} - \frac{f^2(y_N(x))f''(y_N(x))}{2f'(y_N(x))^3}$$

where f_N *is called PCNH function stability of the fixed point* $x^* = \alpha$ *of* f_N.
$|f'_N(\alpha)| = 0 < 1$
since $f(\alpha) = 0$
then α *is hyperbolic asymptotically stable.*

SHM (Super – Halley Method)[16]: For a given initial choice x_0, find the approximate solution x_{n+1} by the iterative schemes,

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f''(x_n)f^2(x_n)}{2f'^3(x_n) - 2f(x_n)f'(x_n)f''(x_n)}$$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n) + f''(x_n)(y_n - x_n)},$$

$$n = 0, 1, 2, 3, ...$$

Theorem 3.4. *For a given* x_0 *is our initial guess of the root* α, *SHM is the form*

$$y_N(x) = x - \frac{f(x)}{f'(x)} - \frac{f''(x)f^2(x)}{2f'^3(x) - 2f(x)f'(x)f''(x)}$$

$$f_N(x) = y_N(x) - \frac{f(y_N(x))}{f'(x) + f''(x)(y_N(x) - x)},$$

where f_N *is called SHM function stability of the fixed point* $x^* = \alpha$ *of* f_N.
$|f'_N(\alpha)| = 0 < 1$
since $f(\alpha) = 0$
then α *is hyperbolic asymptotically stable.*

ISHM (Improvement of Super – Halley Method) [10]: For a given initial choice x_0, find the approximate solution x_{n+1} by the iterative schemes,

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f''(x_n)f^2(x_n)}{2f'^3(x_n) - 2f(x_n)f'(x_n)f''(x_n)},$$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n)} - \frac{f''(x_n)f(y_n)}{2f'^3(x_n)},$$

$$n = 0, 1, 2, 3, ...$$

Theorem 3.5. *For a given* x_0 *is our initial guess of the root* α, *ISHM is the form*

$$y_N(x) = x - \frac{f(x)}{f'(x)} - \frac{f''(x)f^2(x)}{2f'^3(x) - 2f(x)f'(x)f''(x)},$$

$$f_N(x) = y_N(x) - \frac{f(y_N)}{f'(x)} - \frac{f''(x)f(y_N(x))}{2f'^3(x)},$$

where f_N is called ISHM function stability of the fixed point $x^* = \alpha$ of f_N.
$|f'_N(\alpha)| = 0 < 1$
since $f(\alpha) = 0$
then α is hyperbolic asymptotically stable.

4. Stability analysis

let us now discuss the stability of the above theorem 3.1 and theorem 3.2.

Theorem 4.1. *let $\alpha \in I$ be a simple zero of sufficiently differential function $f : I \subseteq R \to R$ for an open interval I, if x_0 is sufficiently close to α then the theorem 3.1 is stable.*

Proof. Consider to

$$y_N(x) = x - \frac{[f(x)]^2}{f(x + f(x)) - f(x)}$$

$$f_N(x) = x - \frac{2f(x)}{[f'(y_N(x)) + f'(x)]}$$

let α be a simple zero of f and x^* is a fixed points of f. We need to check the stability of the fixed point $x^* = \alpha$

To do so evaluate $f'_N(\alpha)$

$y'_N(x) = 1 - \frac{[(f(x+f(x))-f(x))2f(x)f'(x)-(f(x))^2(f'(x+f(x))(1+f'(x))]}{[f(x+f(x))-f(x)]^2}$

$y'_N(\alpha) = 1$ since $f(\alpha) = 0$,

$f'_N(x) = 1 - 2\frac{[f'(y_N(x))+f'(x))f'(x)-f(x)(f''(y_N(x))y'(x)+f''(x))]}{[f'(y_N(x))+f'(x)]^2}$

$f'_N(\alpha) = 1 - 2\frac{f'(\alpha)}{[f'(y_N(\alpha))+f'(\alpha)]}$

$y_N(\alpha) = \alpha$

now

$f'_N(\alpha) = 0$

$|f'_N(\alpha)| < 1$

then α is hyperbolic asymptotically stable.

This show that theorem 3.1 is stable.

Theorem 4.2. *let $\alpha \in I$ be a simple zero of sufficiently differential function $f : I \subseteq R \to R$ for an open interval I, if x_0 is sufficiently close to α then the theorem 3.2 is stable.*

Proof. Consider to

$$a_N(x) = x - \frac{[f(x)]^2}{f(x + f(x)) - f(x)}$$

$$b_N(x) = a_N(x) - \frac{2f(a_N(x))f'(a_N(x))}{2f^2(a_N(x)) - f(a_N(x))f''(a_N(x))}$$

$$f_N(x) = x - \frac{2f(x)}{[f'(b_N(x)) + f'(x)]}$$

let α be a simple zero of f and x^* is a fixed points of f. We need to check the stability of the fixed point $x^* = \alpha$
To do so evaluate $f'_N(\alpha)$
$a_N(\alpha) = \alpha - \frac{[f(\alpha)]^2}{f(\alpha+f(\alpha))-f(\alpha)}$
$a_N(\alpha) = \alpha$, since $f(\alpha) = 0$
$b_N(\alpha) = \alpha$
$f'_N(\alpha) = 1 - 2\frac{f'(\alpha)}{[f'(b_N(\alpha))+f'(\alpha)]}$
$b_N(\alpha) = \alpha$
now
$f'_N(\alpha) = 0$
$|f'_N(\alpha)| < 1$ then α is hyperbolic asymptotically stable.
This show that theorem 3.2 is stable.

5. Numerical Results

We present some example to illustrate the stability of NTS - 2 and NTSM - 2, see table 2 - 11. All computations are performed using MATLAB. The following examples are used for numerical testing.

Table 1 : Test functions and their roots

$Functions$	$Roots$
$f_1(x) = cos(x) - \sqrt(x) + 1$	1.390589830578210
$f_2(x) = xtan(x) + 1$	2.798386045783890
$f_3(x) = sin(x) - 1 + x$	0.510973429388569
$f_4(x) = cos(x) - x$	0.739085633215161
$f_5(x) = 2sin(x) - x$	1.895494267033980
$f_6(x) = e^x - sin(x) - 3x$	0.360421702960324
$f_7(x) = xtan(x) - 1.28$	3.492857169655640
$f_8(x) = x^2 + 4sin(x)$	-1.933753762827020
$f_9(x) = xsin(x) + cos(x)$	2.798386045783890
$f_{10}(x) = e^x - 3x$	0.619061286735945

Analysis of the Stability: Table 2 : example 1

$f_1(x)$	$PCNH$	SHM	$ISHM$	$NTS-2$	$NTSM-2$
$x_0=1$	1.39058983057821	1.39058983057821	1.39058983057821	1.39058983057821	1.39058983057821
$x_0=2$	1.39058983057821	1.39058983057821	1.39058983057821	1.39058983057821	1.39058983057821
$x_0=2.5$	1.39058983057821	1.39058983057821	1.39058983057821	1.39058983057821	1.39058983057821
$x_0=3$	$NFTR$	1.39058983057821	$NFTR$	1.39058983057821	1.39058983057821

Table 3 : example 2

$f_2(x)$	$PCNH$	SHM	$ISHM$	$NTS-2$	$NTSM-2$
$x_0=1$	2.79838604578389	$NFTR$	$NFTR$	2.79838604578389	2.79838604578389
$x_0=2$	2.79838604578389	2.79838604578389	2.79838604578389	2.79838604578389	2.79838604578389
$x_0=3$	2.79838604578389	2.79838604578389	2.79838604578389	2.79838604578389	2.79838604578389
$x_0=4.5$	$NFTR$	$NFTR$	$NFTR$	2.79838604578389	2.79838604578389

Table 4 : example 3

$f_3(x)$	$PCNH$	SHM	$ISHM$	$NTS-2$	$NTSM-2$
$x_0=1$	0.510973429388569	0.510973429388569	0.510973429388569	0.510973429388569	0.510973429388569
$x_0=2$	0.510973429388569	0.510973429388569	0.510973429388569	0.510973429388569	0.510973429388569
$x_0=4$	0.510973429388569	0.510973429388569	$NFTR$	0.510973429388569	0.510973429388569
$x_0=5$	$NFTR$	$NFTR$	0.510973429388569	0.510973429388569	0.510973429388569

Table 5 : example 4

$f_4(x)$	$PCNH$	SHM	$ISHM$	$NTS-2$	$NTSM-2$
$x_0=-2$	0.739085633215161	0.739085633215161	$NFTR$	0.739085633215161	0.739085633215161
$x_0=-1$	$NFTR$	0.739085633215161	$NFTR$	0.739085633215161	0.739085633215161
$x_0=0$	0.739085633215161	0.739085633215161	0.739085633215161	0.739085633215161	0.739085633215161
$x_0=1$	0.739085633215161	0.739085633215161	0.739085633215161	0.739085633215161	0.739085633215161

Table 6 : example 5

$f_5(x)$	$PCNH$	SHM	$ISHM$	$NTS-2$	$NTSM-2$
$x_0=0.9$	$NFTR$	$NFTR$	$NFTR$	1.89549426703398	1.89549426703398
$x_0=1$	$NFTR$	$NFTR$	$NFTR$	1.89549426703398	1.89549426703398
$x_0=2$	1.89549426703398	1.89549426703398	1.89549426703398	1.89549426703398	1.89549426703398
$x_0=2.9$	1.89549426703398	1.89549426703398	1.89549426703398	1.89549426703398	1.89549426703398

Table 7 : example 6

$f_6(x)$	$PCNH$	SHM	$ISHM$	$NTS-2$	$NTSM-2$
$x_0=-2.5$	$NFTR$	0.360421702960324	0.360421702960324	0.360421702960324	0.360421702960324
$x_0=-2$	0.360421702960324	0.360421702960324	0.360421702960324	0.360421702960324	0.360421702960324
$x_0=0$	0.360421702960324	0.360421702960324	0.360421702960324	0.360421702960324	0.360421702960324
$x_0=1$	0.360421702960324	0.360421702960324	$NFTR$	0.360421702960324	0.360421702960324

Table 8 : example 7

$f_7(x)$	$PCNH$	SHM	$ISHM$	$NTS-2$	$NTSM-2$
$x_0=1$	$NFTR$	$NFTR$	$NFTR$	$NFTR$	$NFTR$
$x_0=1.7$	$NFTR$	3.49285716965564	3.49285716965564	3.49285716965564	3.49285716965564
$x_0=2$	3.49285716965564	3.49285716965564	$NFTR$	3.49285716965564	3.49285716965564
$x_0=3$	3.49285716965564	3.49285716965564	3.49285716965564	3.49285716965564	3.49285716965564

Table 9 : example 8

$f_8(x)$	$PCNH$	SHM	$ISHM$	$NTS-2$	$NTSM-2$
$x_0=-1$	$NFTR$	$NFTR$	$NFTR$	-1.93375376282702	-1.93375376282702
$x_0=-2$	-1.93375376282702	-1.93375376282702	-1.93375376282702	-1.93375376282702	-1.93375376282702
$x_0=-3$	-1.93375376282702	-1.93375376282702	-1.93375376282702	-1.93375376282702	-1.93375376282702
$x_0=1$	$NFTR$	$NFTR$	$NFTR$	$NFTR$	$NFTR$

Table 10 : example 9

$f_9(x)$	$PCNH$	SHM	$ISHM$	$NTS-2$	$NTSM-2$
$x_0=1.1$	$NFTR$	$NFTR$	$NFTR$	2.79838604578389	2.79838604578389
$x_0=1.5$	$NFTR$	$NFTR$	$NFTR$	2.79838604578389	2.79838604578389
$x_0=2$	2.79838604578389	2.79838604578389	2.79838604578389	2.79838604578389	2.79838604578389
$x_0=3$	2.79838604578389	2.79838604578389	2.79838604578389	2.79838604578389	2.79838604578389

Table 11 : example 10

$f_{10}(x)$	$PCNH$	SHM	$ISHM$	$NTS-2$	$NTSM-2$
$x_0=-1$	0.619061286735945	0.619061286735945	0.619061286735945	0.619061286735945	0.619061286735945
$x_0=0$	0.619061286735945	0.619061286735945	0.619061286735945	0.619061286735945	0.619061286735945
$x_0=1$	0.619061286735945	0.619061286735945	0.619061286735945	0.619061286735945	0.619061286735945
$x_0=1.2$	$NFTR$	$NFTR$	0.619061286735945	0.619061286735945	0.619061286735945

$^*NFTR =$ Not found this root

6. Conclusion

With the comparative study of criteria for stability our methods NTS - 2 and NTSM -2 are more stable. This method based on fixed point method. Numerical tests show that the our method NTS - 2 and NTSM - 2 are comparably more stable than the other existing methods and provides better result.

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