South East Asian J. of Mathematics and Mathematical Sciences
Vol. 17, No. 1 (2021), pp. 425-432

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

BEST PROXIMITY POINT THEOREMS FOR $\alpha-$ GERAGHTY CONTRACTION TYPE MAPS IN METRIC SPACES

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(**Received:** *May 30, 2020* **Accepted:** *Nov. 18, 2020* **Published:** *Apr. 30, 2021)*

Abstract: In this paper we introduce a notion of α-Greaghty contraction type maps in the setting of a metric space. We also establish a Best Proximity point theorem for such maps using p-property.

Keywords and Phrases: Best Proximity point, α-Geraghty contraction type map, metric spaces, p-property, triangular α-admissible, generalized α-Geraghty contraction.

2020 Mathematics Subject Classification: 47H10.

1. Introduction and Preliminaries

The Banach contraction principle [4], which is a useful tool in the study of many branches of mathematics and mathematical sciences, is one of the earlier and fundamental results in fixed point theory. Because of its importance in nonlinear analysis, a number of authors have improved, generalized and extended the basic result either by defining a new contractive mappings in the context of complete metric space or by investigating the existing contractive mappings in various abstract spaces.

In particular, Geraghty [8] obtained a generalized of Banach contraction principle in the setting of complete metric space by considering a auxiliary function.

Later Amini Harandi and Emami [2] characterised the result of Geraghty in the context of partially ordered complete metric space. Cabellero et al. [5] discussed the existence of the best proximity of Geraghty contraction.

Recently, Samet et al [10] obtained remarkable fixed point results by defining the notion of $\alpha - \psi$-contractive mappings.Karapinar and Samet [13], introduced the concept of generalized $\alpha - \psi$-contractive mappings, which was inspired by the notion of $\alpha - \psi$-contractive mappings. Later Seong et al. [15] defined the concept of $\alpha-$Geraghty contraction type maps in the setting of a metric space. Moreover they proved the existence and uniqueness of a fixed point theory of such maps in the context of complete metric space.

Very recently Chandok, S. C et al [6] introduced the recent fixed point results in ordered metric as well as ordered metric spaces and established a much shorter and nice proofs.

Dosenovi, T. M et al [7] have considered various contractive conditions in b-metric spaces and Abbas, M. et al [1] have established a common fixed points of Suzuki type $(\alpha - \Psi)$-multivalued operators on b-metric spaces. They have discussed about the Limit shadowing property, well posedness and ULam-Hyers stability of solution of fixed point problem and developed results on existence of solution of differential inclusions involving Suzuki type multivalued mappings on b-metric spaces. Further in future we can extend the result for Best proximity point on Geraghty contractions using $b-$metric spaces.

Definition 1.1. *Let $T : X \to X$ be a map and $\alpha : X \times X \to R$ be function. Then T is said to be α-admissible if $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$.*

Definition 1.2. *An α-admissible map T is said to be triangular α-admissible if $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ implies that $\alpha(x, y) \geq 1$*

Lemma 1.3. *Let $T : X \times X$ be a triangular α-admissible map. Assume that there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$.*

Define a sequence x_n by $x_{n+1} = Tx_n$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in N$ with $n < m$.
We define by $\mathscr{F}$ the faculty of all functions $\beta : [0, \infty) \to [0, \infty)$ which satisfies the condition $lim_{n\to\infty}\beta(t_n) = 1$ implies $lim_{n\to\infty}t_n = 1$.
By using such maps Geraghty [8] observed the following interesting results.

Theorem 1.4. *Let (X, d) be a metric space, and let $T : X \to X$ be a map. Suppose that there exists $\beta \in \mathscr{F}$ such that for all $x, y \in X, d(Tx, Ty) \leq \beta(d(x, y))d(x, y)$. Then T has a unique fixed point $x^* \in X$, and $T^n x^*$ converges to x^* for each $x \in X$.*

Definition 1.5. *Let (X, d) be a metric space, and let $\alpha : X \times X \to R$ be a function. A map $T : X \to X$ is called a generalized α-Geraghty contraction type map if there exists $\beta \in \mathscr{F}$ such that for all $x, y \in X, \alpha(x, y)d(Tx, Ty) \leq \beta(M(x, y))M(x, y)$, where $M(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty)\}$.*

Remark 1.6. *Since the functions belonging to $\mathscr{F}$ are strictly smaller than one. It implies that $d(Tx, Ty) < M(x, y)$ for any $x, y \in X$ with $x \neq y$.*

Definition 1.7. $A_0 = \{x \in A; d(x, y) = d(A, B), for some y \in B\}$.
$B_0 = \{y \in A; d(x, y) = d(A, B), for some x \in A\}$.
where $d(A, B) = inf\, \{d(x, y) : x \in A, y \in B\}$.

Definition 1.8. *Let (A, B) be a pair of nonempty subsets of metric spaces (X, d) with $A_0 \neq \phi$. Then the pair (A, B) is said to have p-property if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$ $d(x_1, y_1) = d(A, B)$ and $d(x_2, y_2) = d(A, B)$ implies that $d(x_1, y_1) = d(x_2, y_2)$.*

It is easy to see that for any nonempty subset A of X, the pair (A, A) has the p-property. Also it has been shown that in [14], that any pair (A, B) of nonempty closed convex subsets of a Hilbert space H satisfies the p-property. It is shown in [14], that strict convexity is equivalent to p-property.

Definition 1.9. *Let $T : X \to X$ be a map and $\alpha : X \times X \to R$ be a function. Then T is said to be α-orbital admissible if $\alpha(x, Tx) \geq 1$ implies that $\alpha(Tx, T^2x) \geq 1$.*

Definition 1.10. *Let (X, d) be a metric space. Let A and B be a nonempty closed subsets of a complete metric space (X, d) and let $\alpha : X \times X \to R$ be a function. A mapping $T : A \to B$ is said to be generalized α-Geraghty contraction if there exists $\beta \in \mathscr{F}$ such that*

$$\alpha(x, y)d(Tx, Ty) \leq \beta(M(x, y))M(x, y), \quad \text{for any } x, y \in A \tag{1.1}$$

where $M(x, y) - maxd(x, y), d(x, Tx) - d(A, B), d(y, Ty) - d(A, B)$

2. Main Results

Theorem 2.1. *Let (A, B) be a pair of nonempty subsets of a metric space (X, d) satisfying the p-property such that $A_0 \neq \phi$ and let $\alpha : X \times X \to R$ be a function, $T : A \to B$ be a map satisfying $T(A_0) \subset B_0$.*
Suppose the following conditions are satisfied:
1) T is a generalized α-Geraghty type map.
2) T is a triangular α-admissible.
3) There exists $x_1 \in A$ such that $\alpha(x_1, Tx_1) \geq 1$.
4) T is continuous.
Then there exists a unique $x^* \in A$ such that $d(x^*, Tx^*) = d(A, B)$.
Proof. Choose $x_0 \in A$
Since $Tx_0 \in T(A_0) \subseteq B_0$.
there exists $x_1 \in A_0$ such that
$d(x_1, Tx_0) = d(A, B)$.

Since $Tx_1 \in T(A_0) \subseteq B_0$.
we determine $x_2 \in A_0$ such that
$d(x_1, Tx_0) = d(A, B)$
We define a sequence $(x_n) \subset A_0$ satisfying
$d(x_{n+1}, Tx_n) = d(A, B)$.
Suppose that $x_{n_0} = x_{n_0+1}$ for some $n_0 \in N$
Then it is clear x_{n_0} is a best proximity point of T.
By the p-property,
we assume $x_n \neq x_{n+1}$ for each $n \in N$ $d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1})$
where $x_n \neq x_{n+1}$ for all $n \in N$
By lemma 1.3, we have

$$\alpha(x_n.x_{n+1}) \geq 1 \quad \text{for all} \quad n \in N \tag{2.1}$$

By (1.1) we have

$$d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1}) \leq \alpha(x_n, x_{n+1}) d(Tx_n, Tx_{n+1}) \times \\ \times \beta(M(x_n, x_{n+1})) M(x_n, x_{n+1}) \quad \text{for all} \quad n \in N \tag{2.2}$$

where
$M(x_n, x_{n+1}) = max\,\{d(x_n, x_{n+1}), d(x_n, Tx_n) - d(A, B), d(x_{n+1}, Tx_{n+1}) - d(A, B)\}$

$$\leq max\,\{d(x_n, x_{n+1}), d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) - d(A, B), d(x_{n+1}, x_{n+2}) \\ + d(x_{n+2}, Tx_{n+1}) - d(A, B) \\ \leq max\,\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}) \tag{2.3}$$

$d(x_{n+1}, x_{n+2}) \leq \beta(M(x_n, x_{n+1})) M d(x_n, x_{n+1})$
$\leq \beta(max\,\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}) max\,\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}$
If $d(x_{n+1}, x_{n+2}) \geq d(x_n, x_{n+1})$
$d(x_{n+1}, x_{n+2}) \leq \beta(d(x_{n+1}, x_{n+2})) d(x_{n+1}, x_{n+2})$ from (2.3)
$< d(x_{n+1}, x_{n+2})$ since $\beta(t_n) = 1$
which is contradiction
Therefore $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$ for all $n \in N$
Hence sequence $\{d(x_n, x_{n+1})\}$ is nonnegative and non increasing.
consequently there exists $r \geq 0$ such that
$d(x_n, Tx_n) - d(A, B) \leq d(x_n, x_{n+1})$ and
$d(x_{n+1}, Tx_{n+1}) - d(A, B) \leq d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$
we have $M(x_n.x_{n+1}) = d(x_n, x_{n+1})$

Hence from (2.2) we have
$\frac{d(x_{n+1},x_{n+2})}{M(x_n,x_{n+1})} \leq \beta(M(x_n,x_{n+1})) < 1$
Since the sequence is non increasing and continuous
$lim_{n\to+\infty}\beta(M(x_n,x_{n+1})) = 1$
Owing to the fact, $\beta \in \mathscr{F}$ we have

$$lim_{n\to+\infty}M(x_n,x_{n+1}) = 0 \tag{2.4}$$

Hence we conclude that

$$r = lim_{n\to+\infty}d(x_n,x_{n+1}) = 0 \tag{2.5}$$

we observe that
$M(x_m,x_n) = max\left\{d(x_m,x_n), d(x_m,Tx_m) - d(A,B), d(x_n,Tx_n) - d(A,B)\right\}$
$= max\left\{d(x_m,x_n), d(x_m,x_{m+1}) + d(x_{m+1},Tx_m) - d(A,B), d(x_n,x_{n+1})\right.$
$\left.+d(x_{n+1},Tx_n) - d(A,B)\right\}$
$\leq max\left\{d(x_m,x_n), d(x_m,x_{m+1}), d(x_n,x_{n+1})\right\}$
Since $d(x_n,x_{n+1}) \to 0$.
we have

$$lim_{m,n\to+\infty}SupM(x_m,x_n) = lim_{m,n\to\infty}Supd(x_m,x_n) \tag{2.6}$$

we assert that (x_n) is a Cauchy sequence.
By using the triangular inequality and since
$d(x_{n+1},x_{n+2}) = d(Tx_n,Tx_{n+1})$ and
$d(x_{n+1},x_{m+1}) = d(Tx_n,Tx_m)$

$$\begin{aligned} d(x_m,x_n) &\leq d(x_n,x_{n+1}) + d(Tx_n,Tx_m) + d(x_{m+1},x_n) \\ &= d(x_n,x_{n+1}) + d(Tx_n,Tx_m) + d(x_{m+1},x_n) \end{aligned} \tag{2.7}$$

Combining (1.1) and (2.7) we have

$$\begin{aligned} d(x_m,x_n) &= d(x_n,x_{n+1}) + d(Tx_n,Tx_m) + d(x_{m+1},x_n) \\ &\leq d(x_n,x_{n+1}) + d(x_m,x_n)d(Tx_n,Tx_m) + d(x_{m+1},x_n) \\ &\leq d(x_n,x_{n+1}) + \beta(M(x_m,x_n))M(x_m,x_n) + d(x_{m+1},x_n) \end{aligned} \tag{2.8}$$

Together with (2.5), (2.6) and (2.7) we have
$lim_{m,n\to+\infty}Supd(x_n,x_{n+1}) \leq lim_{m,n\to+\infty}Sup\beta(M(x_n,x_m))$
$lim_{m,n\to+\infty}SupM(x_n,x_m)lim_{m,n\to+\infty}Sup\beta(M((x_n,x_m))lim_{m,n\to+\infty}Supd(x_n,x_m)$
which implies that $lim_{m,n\to+\infty}SupM(x_n,x_m) = 0$

and hence $lim_{m,n\to+\infty} Supd(x_n, x_m) = 0$
Therefore (x_n) is a cauchy sequence.
Since A is a closed subset of the complete metric space $(X, d), (x_n) \to x^*$ for some $x^* \in A$.
since T is continuous
we have $Tx_n \to Tx^*$
This implies that $d(x_{n+1}, Tx_n) = d(x^*, Tx^*)$
Taking into account, we deduce that
$d(x^*, Tx^*) = d(A, B)$
For the uniqueness
Suppose that x_1 and x_2 are best proximity points of T with $x_1 \neq x_2$
This means that $d(x_i, Tx_i) = d(A, B)$ for $i = 1, 2$
Using the p-property we have
$d(x_1, x_2) = d(Tx_1, Tx_2)$
and using the fact that T is a generalized α-Geraghty type map, we have

$$\begin{aligned} d(x_1, x_2) &= d(Tx_1, Tx_2) \\ &\leq \alpha(d(x_1, x_2))d(Tx_1, Tx_2) \\ &\leq \beta(M(x_1, x_2))M(x_1, x_2) \end{aligned}$$

we have

$$\begin{aligned} d(x_1, x_2) &\leq \beta(M(x_1, x_2))M(x_1, x_2) \\ &\leq \beta(d(x_1, x_2))d(x_1, x_2) \\ &< d(x_1, x_2) \end{aligned}$$

which is contraction
This completes the proof.

Example 2.2. Let $X = [0, \infty]$ and Let $d(x, y) = |x - y|$, for all $x \in A, y \in B$
Let $\beta(t) = \frac{1}{1+t}$ for all $t \geq 0$.
Then it is clear that $\beta \in \mathscr{F}$.
We define a mapping from $T : A \to B$ by

$$Tx = \begin{cases} \frac{1}{2x} & (0 \leq x \leq 1) \\ 2x & (x > 1) \end{cases}$$

and a function $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & (0 \leq x, y \leq 1) \\ 0 & \text{otherwise} \end{cases}$$

condition (3) of the theorem 2.1 is satisfied with $x_1 = 1$.
condition (4) of the theorem 2.1 is satisfied with $x_n = T^n x_1 = \frac{1}{n}$.
Let $x, y \in X$ such that $\alpha(x, y) \geq 1$.
Then $x, y \in [0, 1]$ and so $Tx \in [0, 1] Ty \in [0, 1]$ and $\alpha(Tx, Ty) = 1$.
Hence T is α-admissible and hence condition (2) is satisfied.
we show that the condition (1) of the theorem 2.1 is satisfied.
If $0 \leq x, y \leq 1$, then $\alpha(x, y) = 1$ and
we have

$$\begin{aligned}
&\beta(d(x,y))d(x,y) - \alpha(x,y)d(Tx,Ty) \\
&= \beta(d(x,y))d(x,y) - d(Tx,Ty) \\
&= \frac{|x-y|}{1+|x-y|} - \frac{1}{2}|x-y| \\
&= \frac{|x-y|(1-|x-y|)}{2(1+|x-y|)} \\
&\geq 0
\end{aligned}$$

hence $0 \leq x, y \leq 1$
$\alpha(x, y)d(Tx, Ty) \leq \beta(d(x, y))d(x, y) - d(A, B)$
If $0 \leq x \leq 1$ and $y > 1$ then $\alpha(x, y) = 0$ and we have
$\alpha(x, y)d(Tx, Ty) \leq \beta(d(x, y))d(x, y) - d(A, B)$
Thus all the hypothesis of the theorem 2.1 are satisfied and T has a best proximity point.

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